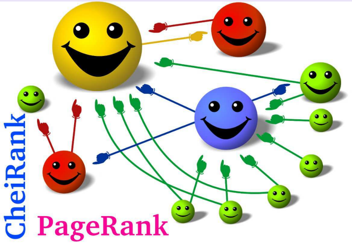
Ulam networks and fractal Weyl law

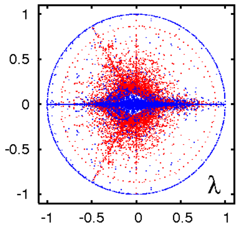
Dima Shepelyansky
www.quantware.ups-tlse.fr/dima

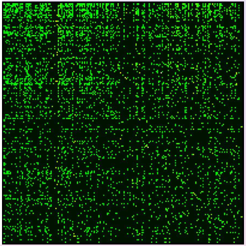
Hermann Weyl (1855 - 1955), Stanislaw Ulam (1909 - 1984),
Edward Teller (1908 - 2003), Eugene Wigner (1902 - 1995) => Budapest born

(1906) Markov vs Wigner (1955)

Collaboration: L.Ermann, K.Frahm, B.Georgeot, O.Zhirov + A.Chepelianskii
V.Kandiah, Y.-H.Eom support $\rightarrow$ EC FET Open grant NADINE







1945: Nuclear physics $\rightarrow$ Wigner (1955)$\rightarrow$ Random Matrix Theory
1991: WWW, small world social networks $\rightarrow$ Markov (1906) $\rightarrow$ Google matrix

Despite the importance of large-scale search engines on the web, very little academic research has been done on them.

S.Brin and L.Page, Comp. Networks ISDN Systems **30**, 107 (1998)

Fractal Weyl law

invented for open quantum systems, quantum chaotic scattering:
the number of Gamow eigenstates N_γ, that have escape rates γ in a finite bandwidth $0 \le \gamma \le \gamma_b$, scales as

$$N_\gamma \propto \hbar^{-\nu} \propto N^\nu, \quad \nu = d/2$$

where d is a fractal dimension of a strange invariant set formed by obits non-escaping in the future and in the past (N is matrix size)

References:
J.Sjostrand, Duke Math. J. **60**, 1 (1990)
M.Zworski, Not. Am. Math. Soc. **46**, 319 (1999)
W.T.Lu, S.Sridhar and M.Zworski, Phys. Rev. Lett. **91**, 154101 (2003)
S.Nonnenmacher and M.Zworski, Commun. Math. Phys. **269**, 311 (2007)

Resonances in quantum chaotic scattering:
three disks, quantum maps with absorption

Perron-Frobenius operators, Ulam method for dynamical maps, Ulam networks, dynamical maps, strange attractors

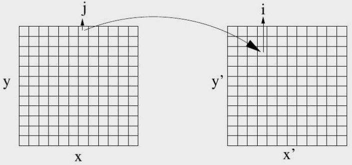
Linux kernel network $d = 1.3$, $N \le 285509$;
Phys. Rev. up to 2009 $d \approx 1$, $N = 460422$

Ulam networks

Ulam conjecture (method) for discrete approximant of Perron-Frobenius operator of dynamical systems

Discretized phase-space:
Adjacency matrix $\mathbf{A} = P(j \to i)$

$N = N_x \times N_y$ cells.



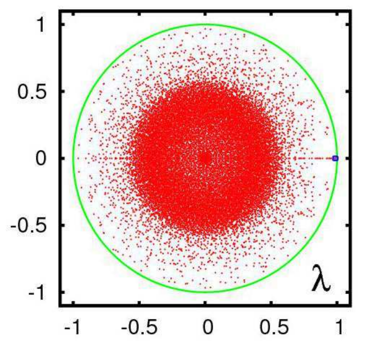
N_c: traj. from cell j
N_i: traj. to call i

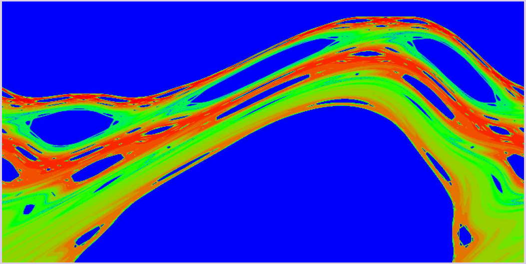
$$\begin{cases} \mathbf{A}_{i,j} = N_i/N_c \\ \sum_i \mathbf{A}_{i,j} = 1 \quad \text{(closed systems)} \end{cases}$$

S.M.Ulam, *A Collection of mathematical problems*, Interscience, **8**, 73 N.Y. (1960)
A rigorous prove for hyperbolic maps:
T.-Y.Li J.Approx. Theory **17**, 177 (1976)
Related works:
Z. Kovacs and T. Tel, Phys. Rev. A 40, 4641 (1989)
M.Blank, G.Keller, and C.Liverani, Nonlinearity **15**, 1905 (2002)
D.Terhesiu and G.Froyland, Nonlinearity **21**, 1953 (2008)

Links to Markov chains: $\infty\infty\infty\infty\infty\infty\infty\infty\infty\infty$
Contre-example: Hamiltonian systems with invariant curves, e.g. the Chirikov standard map: noise, induced by coarse-graining, destroys the KAM curves and gives homogeneous ergodic eigenvector at $\lambda = 1$

Ulam method for the Chirikov standard map





$\bar{y} = y + K \sin x$, $\bar{x} = x + \bar{y}$ $(mod 2\pi)$; $K = 0.971635...$

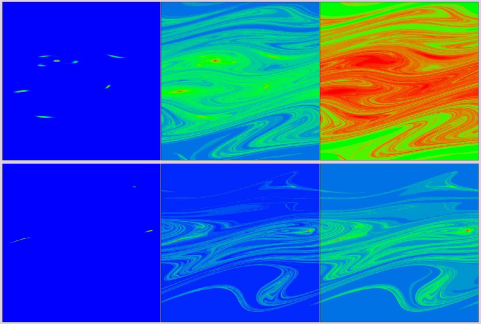
Left: spectrum $G\psi = \lambda\psi$, $M \times M/2$ cells; $M = 280$, $N_d = 16609$, exact and Arnoldi method for matrix diagonalization; generalized Ulam method of one trajectory.

Right: modulus of eigenstate of $\lambda_2 = 0.99878...$, $M = 1600$, $N_d = 494964$. Here $K = K_G$

(Frahm, DS (2010))

Google matrix of dynamical attractors

Weak point of AB model => large gap, no sensitivity to α



PageRank p_j for the Google matrix generated by the Chirikov typical map at $T = 10$, $k = 0.22$, $\eta = 0.99$ (set $T10$, top row) and $T = 20$, $k = 0.3$, $\eta = 0.97$ (set $T20$, bottom row) with $\alpha = 1, 0.95, 0.85$ (left to right). The phase space region $0 \leq x < 2\pi; -\pi \leq p < \pi$ is divided on $N = 3.6 \cdot 10^5$ cells.

Chirikov typical map (1969) with dissipation

$\bar{p} = \eta p + k \sin(x + \theta_t)$, $\bar{x} = x + \bar{p}$

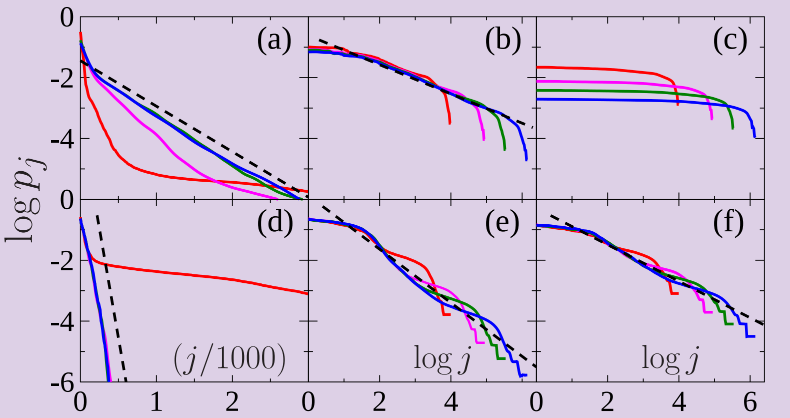
$\theta_t = \theta_{t+T}$ are random phases periodically repeated after T iterations,

chaos border $k_c \approx 2.5/T^{3/2}$, Kolmogorov-Sinai entropy $h \approx 0.29k^{2/3}$;

grid of $N = N_x \times N_p$ cells with $N_c \sim 10^4$ trajectories which generates links (transition probabilities) from one cell to another; effective noise of cell size;

maximum $N = 22500; 1.44 \cdot 10^6$

PageRank distribution

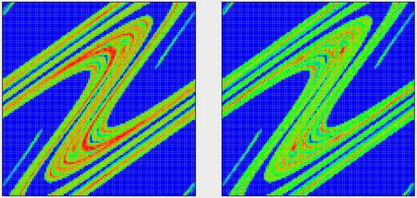


Differential distribution of number of nodes with PageRank distribution p_j for $N = 10^4$, $9 \cdot 10^4$, $3.6 \cdot 10^5$ and $1.44 \cdot 10^6$ curves, the dashed straight lines show fits $p_j \sim 1/j^\beta$ with β: 0.48 (b), 0.88 (e), 0.60 (f). Dashed lines in panels (a),(d) show an exponential Boltzmann decay (see text, lines are shifted in j for clarity). In panels (a),(d) the curves at large N become superimposed. Panel order as in color Fig. above.

Ulam method for dissipative systems

Scattering

$$\begin{cases} \bar{y} &= y + K\sin(x + y/2) \\ \bar{x} &= x + (y + \bar{y})/2 \pmod{2\pi} \end{cases}$$

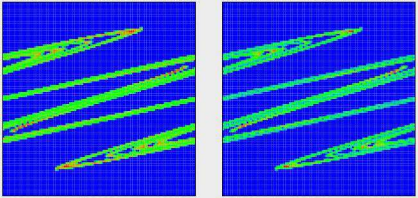


$N = 110 \times 110,\ K = 7, a = 2$

$\lambda_1 = 0.756 \qquad \lambda_3 = -0.01 + i0.513$

Dissipation

$$\begin{cases} \bar{y} &= \eta y + K\sin x \\ \bar{x} &= x + \bar{y} \pmod{2\pi} \end{cases}$$

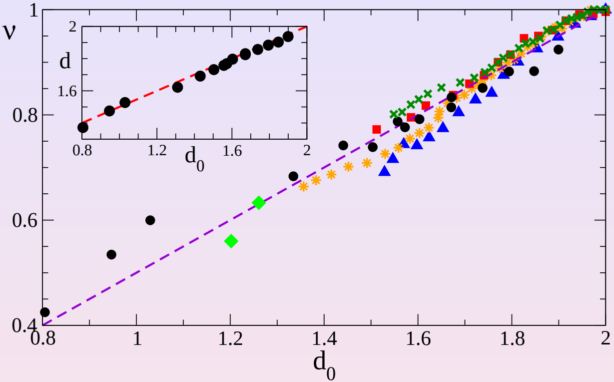


$N = 110 \times 110,\ K = 7, \eta = 0.3$

$\lambda_1 = 1 \qquad \lambda_3 = -0.258 + i0.445$

(Ermann, DS (2010))

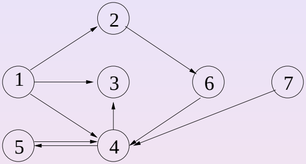
Fractal Weyl law for Ulam networks



Fractal Weyl law for three different models with dimension d_0 of invariant set. The fractal Weyl exponent ν is shown as a function of fractal dimension d_0 of the strange repeller in model 1 and strange attractor in model 2 and Henon map; dashed line shows the theory dependence $\nu = d_0/2$. Inset shows relation between the fractal dimension d of trajectories nonescaping in future and the fractal inv-set dimension d_0 for model 1; dashed line is $d = d_0/2 + 1$. (Ermann, DS (2010))

How Google works

Markov chains (1906) and Directed networks



Weighted adjacency matrix

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For a directed network with N nodes the adjacency matrix $\mathbf{A}$ is defined as $A_{ij} = 1$ if there is a link from node j to node i and $A_{ij} = 0$ otherwise. The weighted adjacency matrix is

$$S_{ij} = A_{ij} / \sum_k A_{kj}$$

In addition the elements of columns with only zeros elements are replaced by $1/N$.

Google Matrix and Computation of PageRank

$\mathbf{P} = \mathbf{SP} \Rightarrow \mathbf{P}$= stationary vector of $\mathbf{S}$; can be computed by iteration of $\mathbf{S}$.

To remove convergence problems:

- Replace columns of 0 (dangling nodes) by $\frac{1}{N}$:

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & \frac{1}{7} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{7} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{7} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{7} & 0 & 1 & 1 & 1 \\ 0 & 0 & \frac{1}{7} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{7} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{7} & 0 & 0 & 0 & 0 \end{pmatrix}; \mathbf{S}^* = \begin{pmatrix} \frac{1}{7} & 1 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{1}{7} \\ \frac{1}{7} & 0 & 0 & 0 & 0 & 1 & \frac{1}{7} \\ \frac{1}{7} & 0 & 0 & 0 & 0 & 0 & \frac{1}{7} \\ \frac{1}{7} & 0 & \frac{1}{2} & 0 & 1 & 0 & \frac{1}{7} \\ \frac{1}{7} & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{7} \\ \frac{1}{7} & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{7} \\ \frac{1}{7} & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{7} \end{pmatrix}.$$

- To remove degeneracies of $\lambda = 1$, replace $\mathbf{S}$ by **Google matrix**

  $\mathbf{G} = \alpha\mathbf{S} + (1-\alpha)\frac{\mathbf{E}}{N}$; $\quad GP = \lambda P \quad$ => Perron-Frobenius operator

- α models a random surfer with a random jump after approximately 6 clicks (usually $\alpha = 0.85$); PageRank vector => P at $\lambda = 1$ ($\sum_j P_j = 1$).

- CheiRank vector P^*: $G^* = \alpha\mathbf{S}^* + (1-\alpha)\frac{\mathbf{E}}{N}$, $G^*P^* = P^*$
  ($\mathbf{S}^*$ with inverted link directions)
  Fogaras (2003) ... Chepelianskii arXiv:1003.5455 (2010) ...

Real directed networks

Real networks are characterized by:

- **small world property**: average distance between 2 nodes $\sim \log N$
- **scale-free property**: distribution of the number of ingoing or outgoing links $\rho(k) \sim k^{-\nu}$

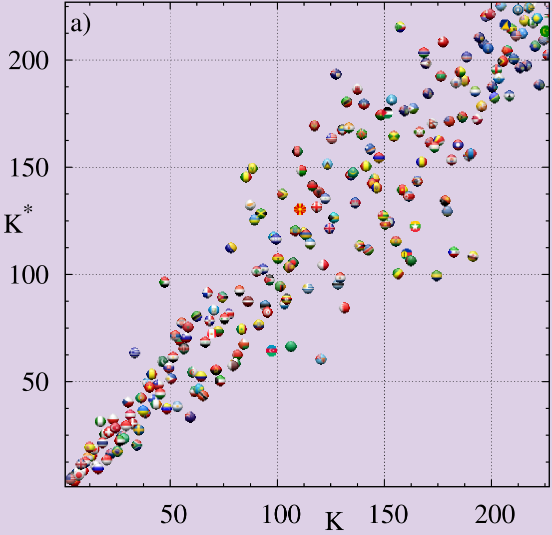
PageRank vector for large WWW:

- $P(K) \sim 1/K^{\beta}$, where K is the ordered rank index
- number of nodes N_n with PageRank P scales as $N_n \sim 1/P^{\nu}$ with numerical values $\nu = 1 + 1/\beta \approx 2.1$ and $\beta \approx 0.9$.
- PageRank $P(K)$ on average is proportional to the number of ingoing links
- CheiRank $P^*(K^*) \sim 1/K^{*\beta}$ on average is proportional to the number of outgoing links ($\nu \approx 2.7; \beta = 1/(\nu - 1) \approx 0.6$)
- WWW at present: $\sim 10^{11}$ web pages

Donato *et al.* EPJB **38**, 239 (2004)

Ranking of World Trade

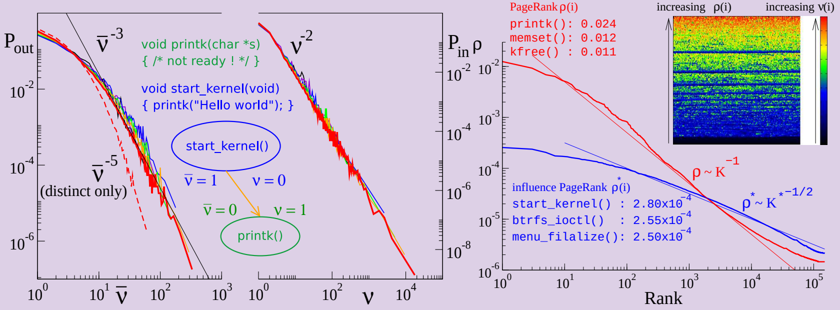
UN COMTRADE database 2008: All commodities



Ermann, DS arxiv:1103.5027 (2011)

Linux Kernel Network

Procedure call network for Linux

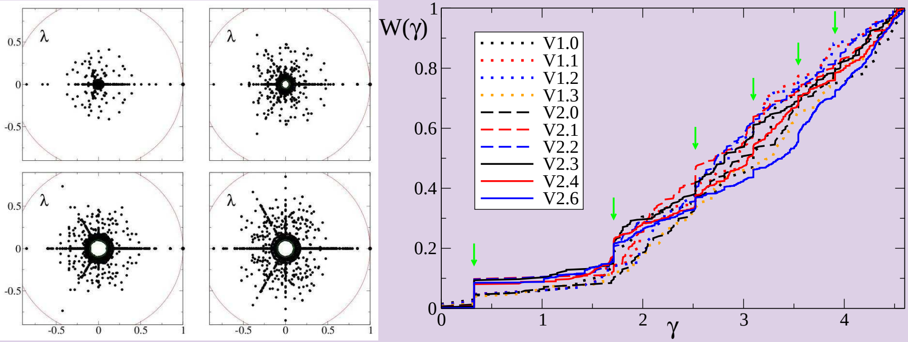


Links distribution (left); PageRank and inverse PageRank (CheiRank) distribution (right) for Linux versions up to 2.6.32 with $N = 285509$ ($\rho \sim 1/j^{\beta}, \beta = 1/(\nu - 1)$).

(Chepelianskii arxiv:1003.5455)

Fractal Weyl law for Linux Network

Sjöstrand Duke Math J. 60, 1 (1990), Zworski *et al.* PRL 91, 154101 (2003) $\rightarrow$ quantum chaotic scattering;
Ermann, DS EPJB 75, 299 (2010)$\rightarrow$ Perron-Frobenius operators

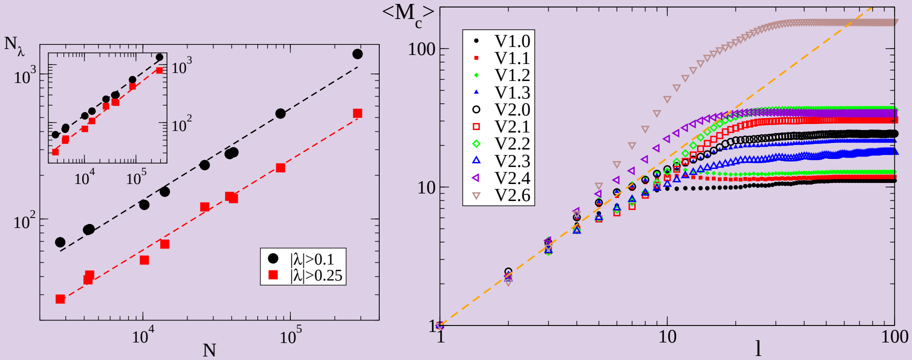


Spectrum of Google matrix (left); integrated density of states for relaxation rate $\gamma = -2\ln|\lambda|$ (right) for Linux versions, $\alpha = 0.85$.

(Ermann, Chepelianskii, DS (2011))

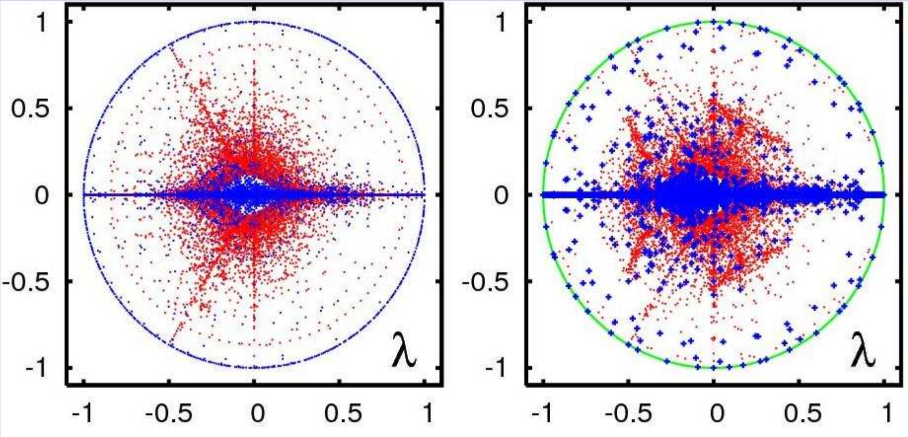
Fractal Weyl law for Linux Network

Number of states $N_\lambda \sim N^\nu$, $\nu = d/2$ $(N \sim 1/\hbar^{d/2})$



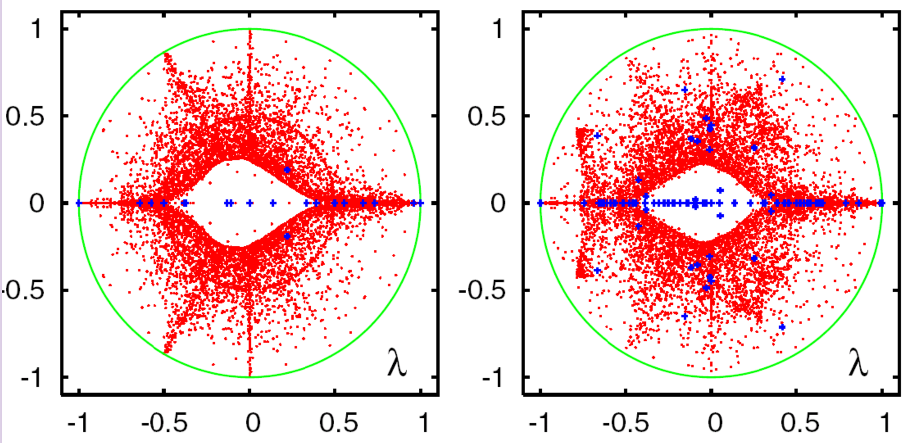
Number of states N_λ with $|\lambda| > 0.1; 0.25$ vs. N, lines show $N_\lambda \sim N^\nu$ with $\nu \approx 0.65$ (left); average mass $< M_c >$ (number of nodes) as a functon of network distance l, line shows the power law for fractal dimension $< M_c > \sim l^d$ with $d \approx 1.3$ (right).

Spectrum of UK University networks



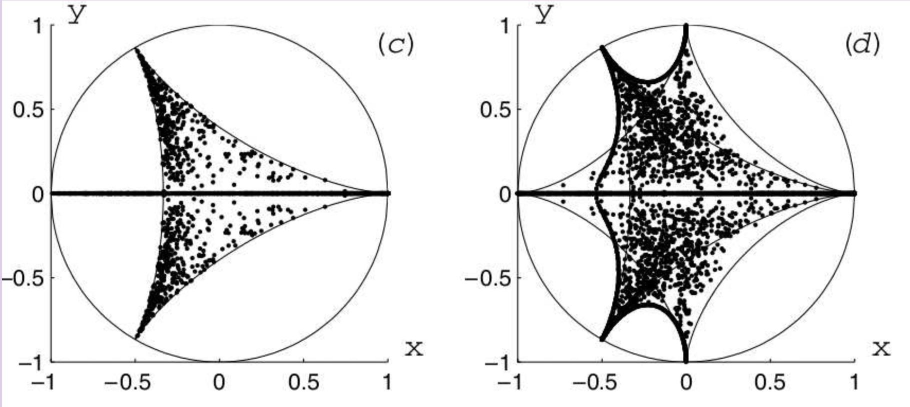
Arnoldi method: Spectrum of Google matrix for Univ. of Cambridge (left) and Oxford (right) in 2006; 20% of sub-spaces ($N \approx 200000$, $\alpha = 1$). [Frahm, Georgeot, DS arxiv:1105.1062 (2011)]

Spectrum of UK University networks



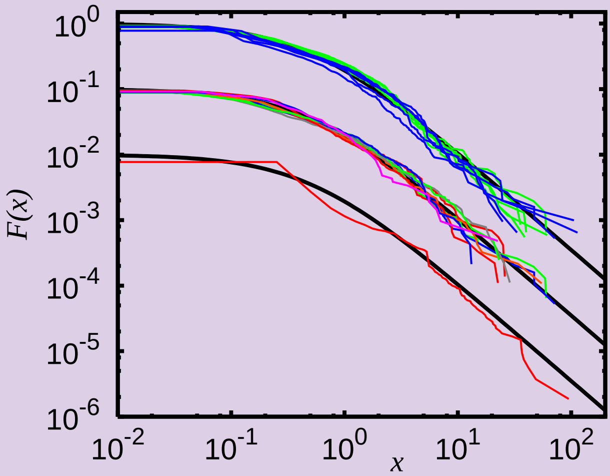
Spectrum of CheiRank Google matrix for Univ. of Cambridge (left) and Oxford (right) in 2006 ($N \approx 200000$, $\alpha = 1$) [Frahm, Georgeot, DS arxiv:1105.1062 (2011)]

Spectrum of random orthostochastic matrices



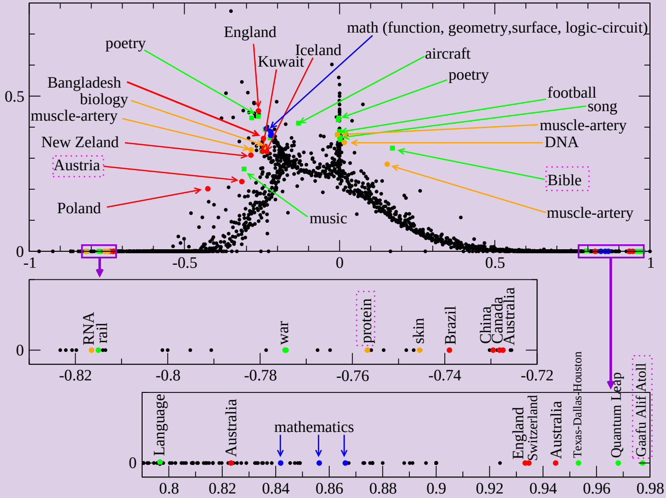
Spectrum $N = 3$ (left), 4 (right) [K.Zyczkowski *et al.* J.Phys. A **36**, 3425 (2003)]

Invariant subspaces size distribution



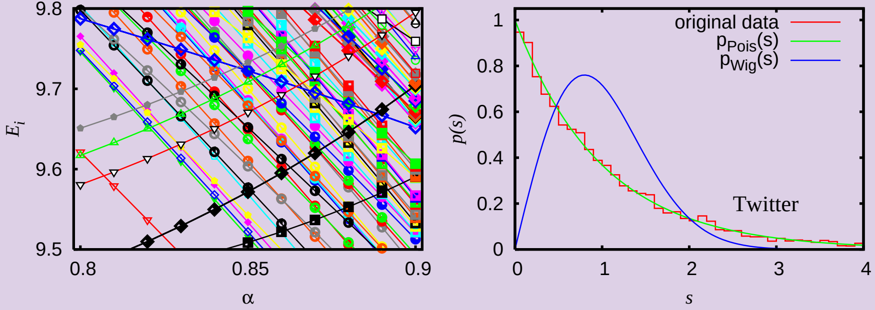
$F(x)$ integrated number of invariant subspaces with size larger that d/d_0; $x = d/d_0$, d_0 is average size of subspaces (top: Cambridge, Oxford 2002-2006; middle: all others; bottom: Wikipedia CheiRank). Curve: $F(x) = 1/(1+2x)^{3/2}$.

Wikipedia spectrum and eigenstates



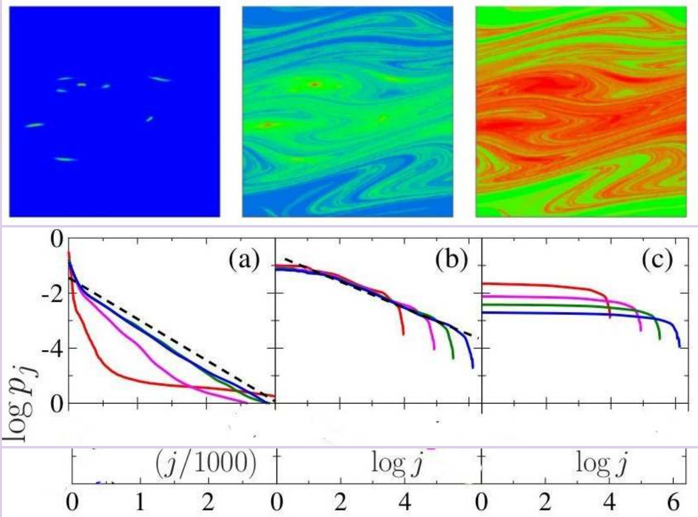
Spectrum S of EN Wikipedia, Aug 2009, $N = 3282257$. Eigenvalues-communities are labeled by most repeated words following word counting of first 1000 nodes. (Ermann, Frahm, DS 2013)

Poisson statistics of Twitter network 2009



Left panel: Dependence of certain top PageRank levels $E_i = -\ln(P_i)$ on the damping factor α for entire Twitter $N \approx 4.1 \times 10^7$. Data points on curves with one color correponds to the same node i. *Right panel:* Histogramm of unfolded level spacing statistics for Twitter. The Poisson distribution $p_{\mathrm{Pois}}(s) = \exp(-s)$ and the Wigner surmise $p_{\mathrm{Wig}}(s) = \frac{\pi}{2}\, s \exp(-\frac{\pi}{4}\, s^2)$ are also shown for comparison. (Frahm, DS 2014)

Anderson delocalization of PageRank ?



Ulam network of dynamical map $\alpha = 1; 0.95; 0.85$

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Network analysis and applications
Session I, June 21 - July 5, 2014

School Organizers:
András Benczúr (Budapest), Andreas Kaltenbrunner (Barcelona), Dima Shepelyansky (Toulouse)

Scope: In past ten years, modern societies developed enormous communication and social networks. New characterization tools of complex networks provide possibilities for information retrieval in social networks, communication, economy, gene, protein and other networks. The interdisciplinary approaches based on complex networks and Markov chains allow to obtain advanced results in such diverse fields as physics, computer science and bioinformatics. The school will present lectures of world leading experts in these fields.

http://www.quantware.ups-tlse.fr/ecoleluchon2014/

Selected References:

Fractal Weyl law and Ulam networks:
R1. L.Ermann and D.L.Shepelyansky, "Ulam method and fractal Weyl law for Perron-Frobenius operators", Eur. Phys. J. B v.75, p.299-304 (2010)
R2. L.Ermann, A.D.Chepelianskii and D.L.Shepelyansky, "Fractal Weyl law for Linux Kernel Architecture", Eur. Phys. J. B v.79, p.115-120 (2011)
PageRank-CheiRank:
R3. A.O.Zhirov, O.V.Zhirov and D.L.Shepelyansky, *Two-dimensional ranking of Wikipedia articles*, Eur. Phys. J. B **77**, 523 (2010)
R4. L.Ermann and D.L.Shepelyansky, "Google matrix of the world trade network", Acta Physica Polonica A v.120(6A), pp. A158-A171 (2011)
Spectrum of Google matrix:
R5. K.M.Frahm, B.Georgeot and D.L.Shepelyansky, "Universal emergence of PageRank", J. Phys, A: Math. Theor. v.44, p.465101 (2011)
R6. L.Ermann, K.M.Frahm and D.L.Shepelyansky, "Spectral properties of Google matrix of Wikipedia and other networks", Eur. Phys. J. B v.86, p.193 (2013)
R7. K.M.Frahm, Y.-H.Eom and D.L.Shepelyansky, "Google matrix of the citation network of Physical Review", arXiv:1310.5624 [physics.soc-ph]
See pdf-s and more at: http://www.quantware.ups-tlse.fr/dima/subjgoogle.html