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Degree-degree correlations in directed networks with heavy-tailed degrees

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Introduction

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- Undesirable behavior for heavy-tailed degrees, Litvak, van der Hofstad [2013]

Scatter plots

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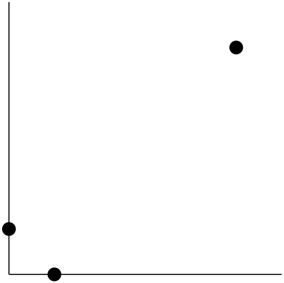

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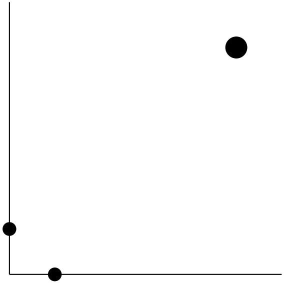

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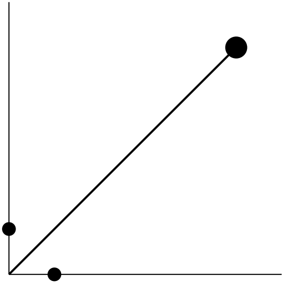

Correlation: $+1$

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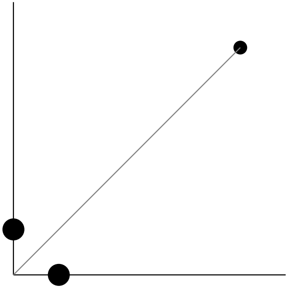

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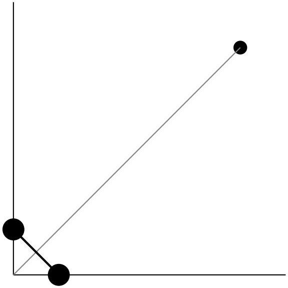

Correlation: -1

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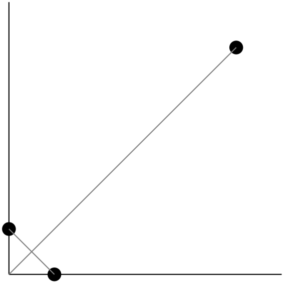

Correlation: ± 1?

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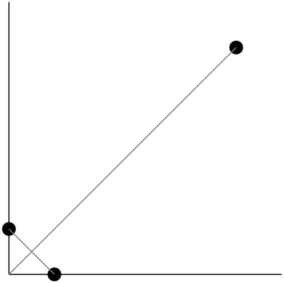

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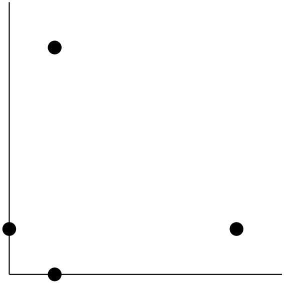

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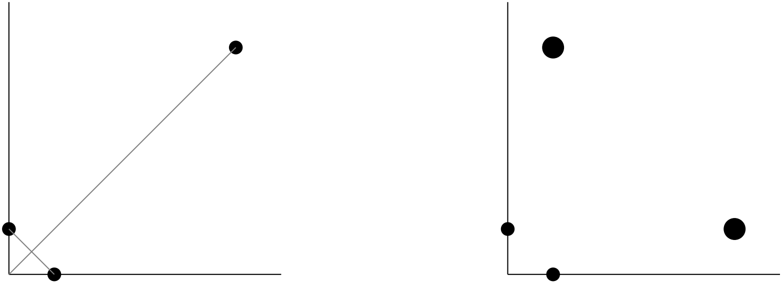

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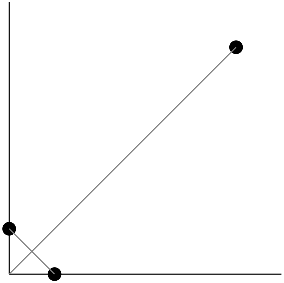

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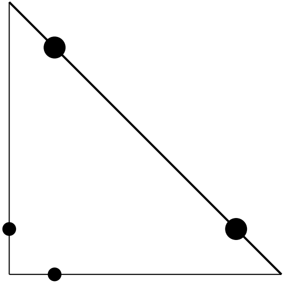

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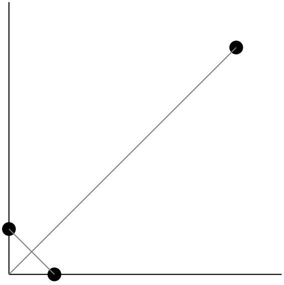

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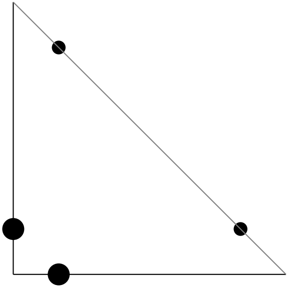

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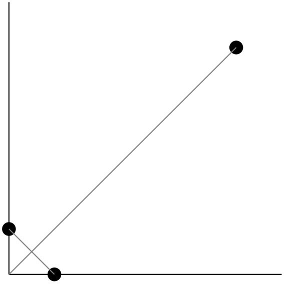

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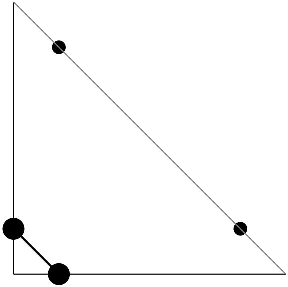

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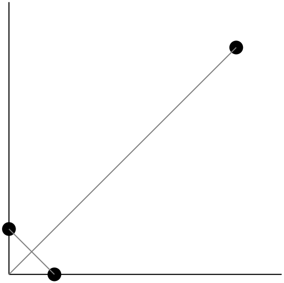

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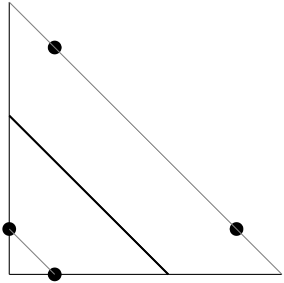

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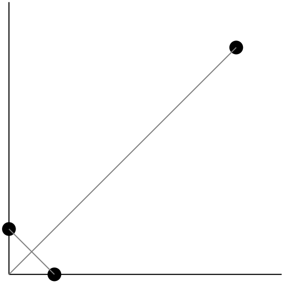

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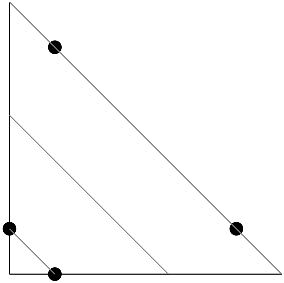

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Degree-degree correlation & Notations

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Given a directed graph $G = (V, E)$.

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$$e \in E$$

$$\longrightarrow$$

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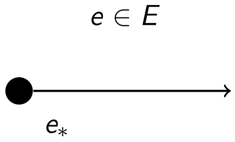

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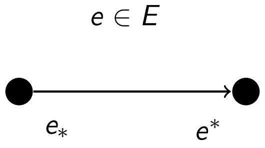

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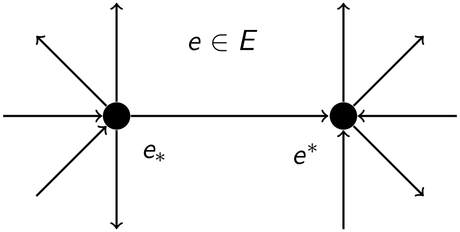

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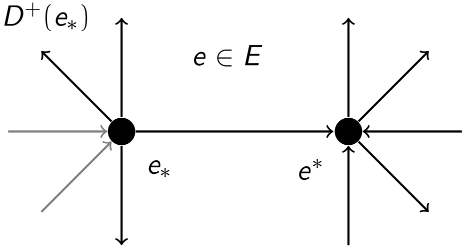

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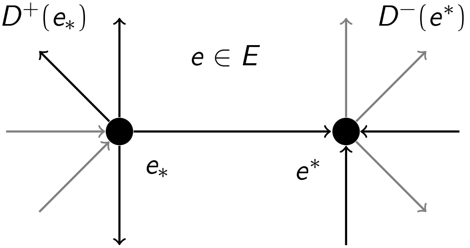

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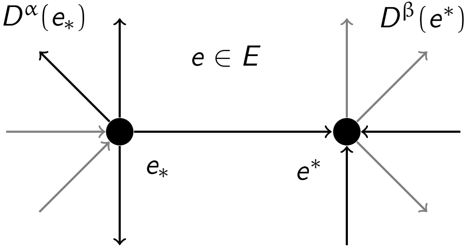
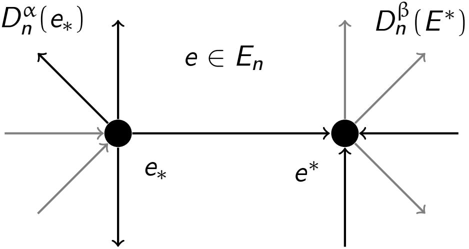


Index degree type by $\alpha, \beta \in \{+, -\}$.

Degree-degree correlation & Notations

Given a sequence $\{G_n\}_{n\in\mathbb{N}}$ of directed graphs. Denote by $G_n = (V_n, E_n)$ an element of this sequence.



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Four correlation types

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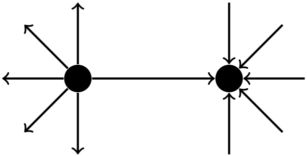
Out/In



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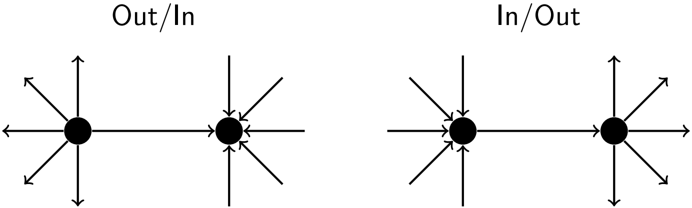

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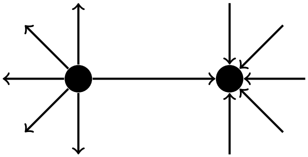

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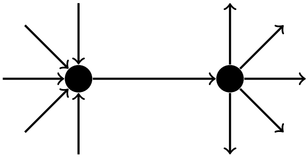

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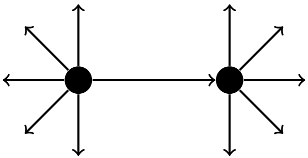

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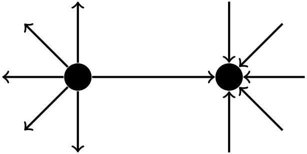

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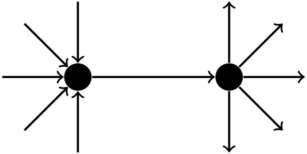

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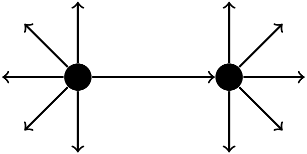

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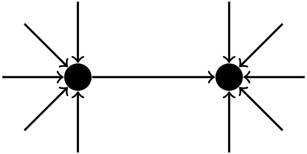

General formula edges

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$$r_{\alpha}^{\beta}(G) = \frac{1}{\sigma_{\alpha}(G)\sigma^{\beta}(G)}\frac{1}{|E|}\sum_{e \in E} D^{\alpha}(e_*)D^{\beta}(e^*) - \hat{r}_{\alpha}^{\beta}(G)$$

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Sequences of 'heavy-tailed' graphs

The space $\mathcal{G}_{\gamma_+,\gamma_-}$

Denote by $\mathcal{G}_{\gamma_+,\gamma_-}$ the space of all sequences $\{G_n\}_{n\in\mathbb{N}}$ of directed graphs with the following properties:

G1 $|V_n| = n$.

G2 There exists and $N \in \mathbb{N}$ such that for all $n \geq N$ there exist $v, w \in V_n$ with $D_n^\alpha(v)$, $D_n^\alpha(w) > 0$ and $D_n^\alpha(v) \neq D_n^\alpha(w)$, for all $\alpha \in \{+, -\}$.

G3 For all $p, q \in \mathbb{R}_{>0}$,

$$\sum_{v\in V_n} D_n^+(v)^p D_n^-(v)^q = \Theta(n^{p/\gamma_+ \vee q/\gamma_- \vee 1}).$$

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G4 For all $p, q \in \mathbb{R}_{>0}$, if $p < \gamma_+$ and $q < \gamma_-$ then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{v\in V_n} D_n^+(v)^p D_n^-(v)^q := d(p,q) \in (0,\infty).$$

Where the limits are such that for all $a, b \in \mathbb{N}$, $k, m > 1$ with $1/k + 1/m = 1$, $a + p < \gamma_+$ and $b + q < \gamma_-$ we have,

$$d(a,b)^{\frac{1}{m}} d(p,q)^{\frac{1}{k}} > d\left(\frac{a}{m} + \frac{p}{k}, \frac{b}{m} + \frac{q}{k}\right).$$

Convergence of Pearson's correlation coefficients

Theorem

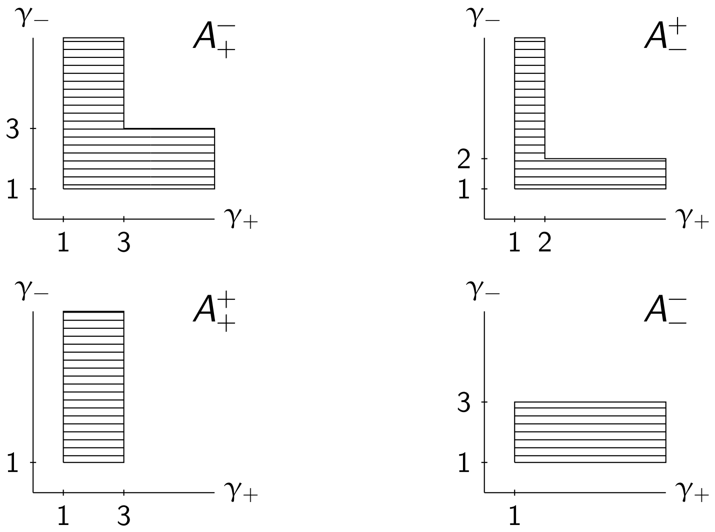
Let $\alpha, \beta \in \{+, -\}$, then there exists an area $A_\alpha^\beta \subset \mathbb{R}^2$ such that for $(\gamma_+, \gamma_-) \in A_\alpha^\beta$ and $\{G_n\}_{n\in\mathbb{N}} \in \mathcal{G}_{\gamma_+,\gamma_-}$

$$\lim_{n\to\infty} \hat{r}_\alpha^\beta(G_n) = 0$$

and hence

$$\lim_{n\to\infty} r_\alpha^\beta(G_n) \geq 0.$$

Area's of convergence



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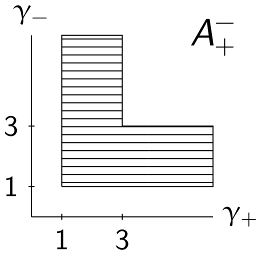

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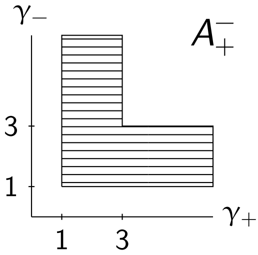

$$\hat{r}_+^- = \frac{1}{\sigma_+(G_n)\sigma^-(G_n)} \frac{1}{|E_n|^2} \sum_{v \in V_n} D_n^+(v)^2 \sum_{v \in V_n} D_n^-(v)^2$$

$$\sigma_+(G_n) = \sqrt{\frac{1}{|E_n|} \sum_{v \in V_n} D_n^+(v)^3 - \frac{1}{|E_n|^2} \left(\sum_{v \in V_n} D_n^+(v)^2\right)^2}$$

$$\sigma^-(G_n) = \sqrt{\frac{1}{|E_n|} \sum_{v \in V_n} D_n^-(v)^3 - \frac{1}{|E_n|^2} \left(\sum_{v \in V_n} D_n^-(v)^2\right)^2}$$

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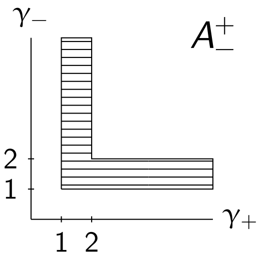

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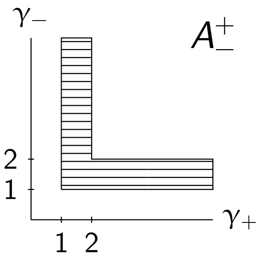

$$\hat{r}_-^+ = \frac{1}{\sigma_-(G_n)\sigma^+(G_n)} \frac{1}{|E_n|^2} \sum_{v \in V_n} D_n^-(v) D_n^+(v) \sum_{v \in V_n} D_n^-(v) D_n^+(v)$$

$$\sigma_-(G_n) = \sqrt{\frac{1}{|E_n|} \sum_{v \in V_n} D_n^+(v) D_n^-(v)^2 - \frac{1}{|E_n|^2} \left(\sum_{v \in V_n} D_n^+(v) D_n^-(v) \right)^2}$$

$$\sigma^-(G_n) = \sqrt{\frac{1}{|E_n|} \sum_{v \in V_n} D_n^-(v) D_n^+(v)^2 - \frac{1}{|E_n|^2} \left(\sum_{v \in V_n} D_n^-(v) D_n^-(v) \right)^2}$$

Introduction
Degree-degree correlations
Pearson's correlation coefficients
Rank correlations
Example
Wikipedia
Further research

Spearman's rho

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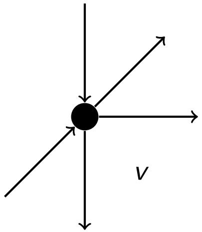

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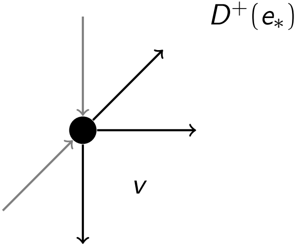

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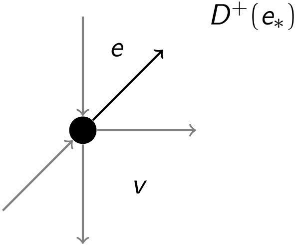

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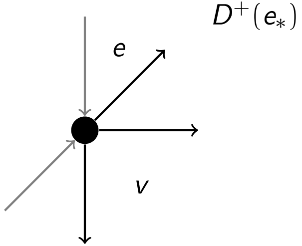

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$$\frac{3|E|(|E|+1)^2}{|E|^3 - |E|} = 3 + \frac{6(|E|+1)}{|E|^2 - 1} \geq 3$$

Taking average ranking

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$(X_i, Y_i), (X_j, Y_j)$ concordant if

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or

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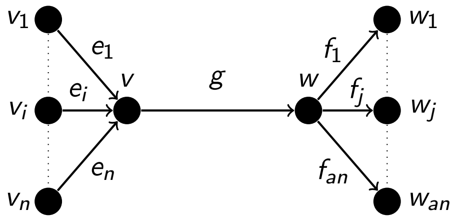
$$\tau_\alpha^\beta(G) = \frac{2\left(\mathcal{C}_\alpha^\beta - \mathcal{D}_\alpha^\beta\right)}{|E|(|E|-1)}$$

Two In/Out bridge graphs

Two In/Out bridge graphs

$G(n, a)$



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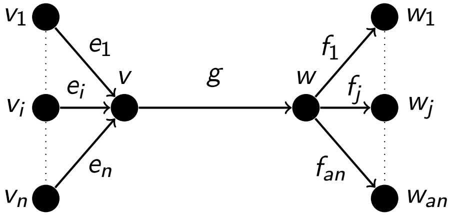

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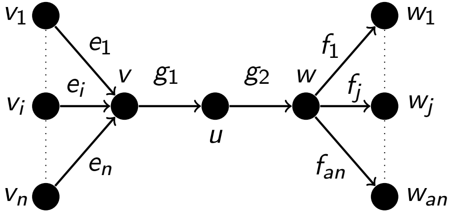

Two In/Out bridge graphs, scatterplots

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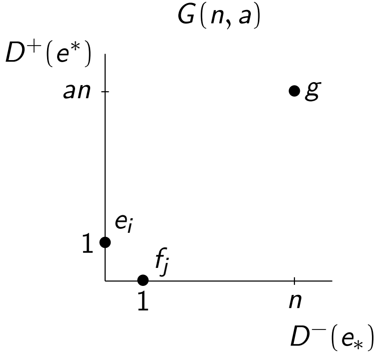

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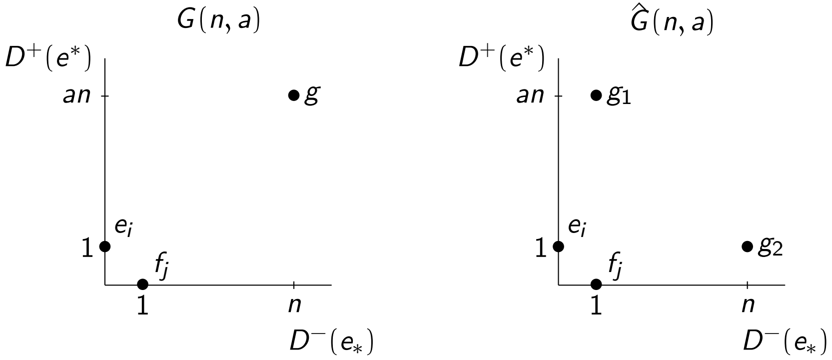

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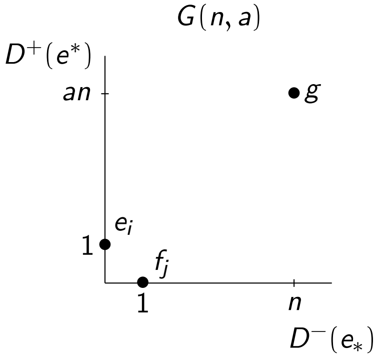

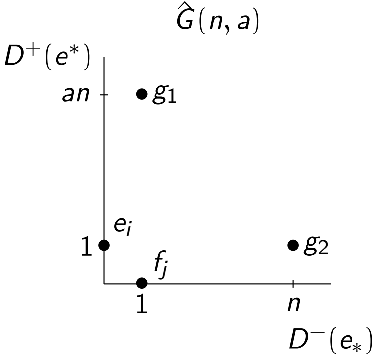

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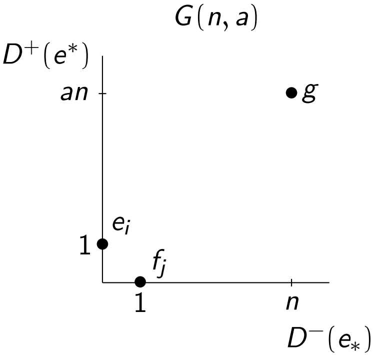

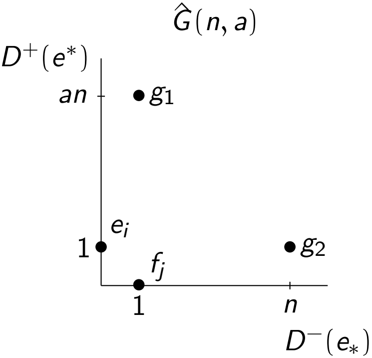

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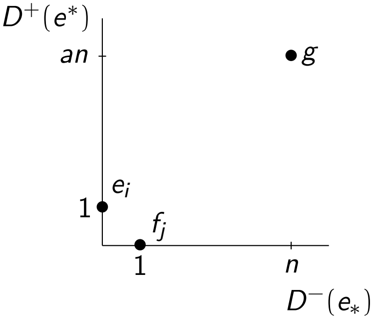

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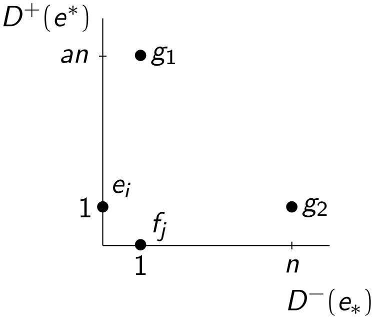

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Degree-degree correlations in Wikipedia

Graph	α/β	Pearson	Spearman		Kendall
			Average	Uniform	
DE wiki	+/-	-0.0552	-0.1435	-0.1434	-0.0986
	-/+	0.0154	0.0484	0.0481	0.0326
	+/+	-0.0323	-0.0640	-0.0640	-0.0446
	-/-	-0.0123	0.0120	0.0119	0.0074
EN wiki	+/-	-0.0557	-0.1999	-0.1999	-0.1364
	-/+	-0.0007	0.0240	0.0239	0.0163
	+/+	-0.0713	-0.0855	-0.0855	-0.0581
	-/-	-0.0074	-0.0666	-0.0664	-0.0457
NL wiki	+/-	-0.0585	-0.3018	-0.3017	-0.2089
	-/+	0.0100	0.0730	0.0727	0.0504
	+/+	-0.0628	0.0016	0.0016	0.0015
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Table: Results on the wikipedia graphs obtained from the `http://law.di.unimi.it/` database

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Pearson's correlation coefficients
Rank correlations
Example
Wikipedia
Further research

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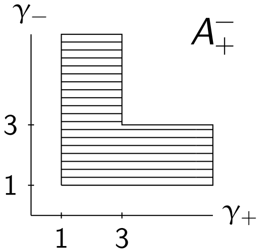
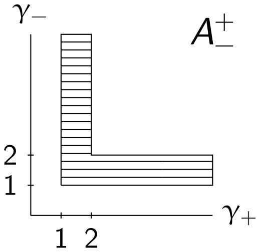
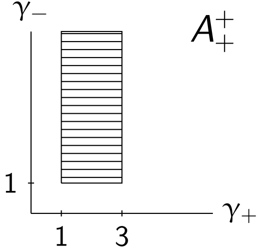

γ_-
A_+^-
3
1
γ_+
1
3


γ_-
A_-^+
2
1
γ_+
1
2

Thank you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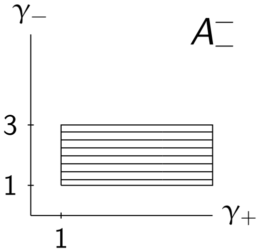
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