UNIVERSITY OF TWENTE.

Analysis of centrality measures based on network structure

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Publications

- N. Litvak, and R. van der Hofstad, "Uncovering disassortativity in large scale-free networks." *Phys.Rev.E* v.87, p.022801, 2013
- N. Litvak, and R. van der Hofstad, "Degree-degree correlations in random graphs with heavy-tailed degrees." Accepted in *Internet Mathematics*, 2013
- P. van der Hoorn and N. Litvak, "Degree-degree correlations in directed networks with heavy-tailed degrees." arXiv:1310.6528, 2013
- K. Avrachenkov, N. Litvak, V. Medyanikov, and M. Sokol, "Alpha current flow betweenness centrality." Accepted In: *WAW2013*, Harvard University, 2013
- K. Avrachenkov, N. Litvak, M. Sokol, and D.Towsley, "Quick detection of nodes with large degrees." *WAW2012*, Halifax, NS, Canada, pp. 54-65, 2013
- L. Ostroumova, K. Avrachenkov and N. Litvak. "Quick detection of popular entities in large directed networks." Submitted

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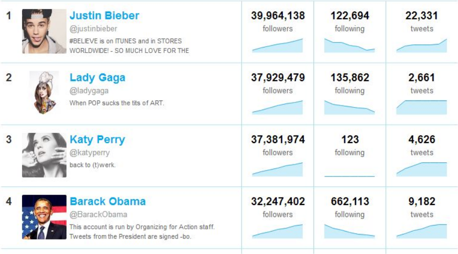

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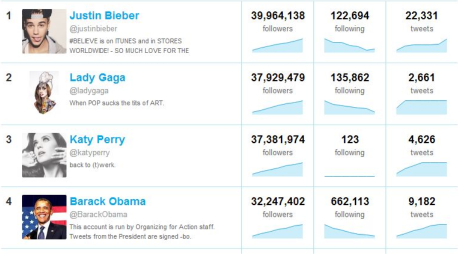

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- Hubs play a crucial role in the analysis of networks

Formal view on the hubs

Let D be a degree of a random node. Regular varying distribution:

$$P(D > x) = L(x)x^{-\gamma} \quad (1)$$

$L(x)$ is slowly varying, i.e. $\lim_{t\to\infty} L(tx)/L(t) = 1$, $x \geqslant 0$

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Extreme value theory. Let $F_1 \geqslant F_2 \geqslant \cdots \geqslant F_N$ be the order statistics of the i.i.d. r.v.'s $D_1, D_2, \ldots, D_N$ as in (1). Then there are (a_N), (b_N) such that for finite k

$$\left(\frac{F_1 - b_N}{a_N}, \cdots \frac{F_k - b_N}{a_N}\right) \xrightarrow{d} \left(\frac{E_1^{-\delta} - 1}{\delta}, \cdots, \frac{\left(\sum_{i=1}^{k} E_i\right)^{-\delta} - 1}{\delta}\right),$$

where $\delta = 1/\gamma$ and E_i'are i.i.d. exponential(1) r.v.'s.

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Example. $P(D > x) = Cx^{-\gamma}$, then $a_N = \delta C^{\delta} N^{\delta}$, $b_N = C^{\delta} N^{\delta}$.
The largest degrees are 'of the order' $N^{1/\gamma}$.

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Goal: Find top-k most popular entities in social (directed) networks (nodes with highest in/out-degrees, largest interest groups, largest user categories), using the minimal number of API requests.

Problem formulation

- Consider a bi-partite graph (V, W, E)
- V and W are sets of entities, $|V| = M$, $|W| = N$.
- A directed edge $(v, w) \in E$ represents a relation between $v \in V$ and $w \in W$.
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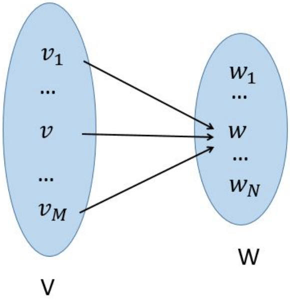

Example. $V = W$ is a set of Twitter users, (v, w) means that v follows w.

Example. V is a set of users, W is a set of interest groups, (v, w) means that user v is a member of an interest group w.

Algorithm for finding top-k most popular entities

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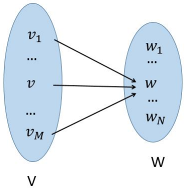
1. Choose a set $A \subset V$ of n_1 nodes sampled from V at random.
2. For each $v \in A$ retrieve the id's of nodes in W that have an edge from v.
3. Compute S_w – the number of edges of $w \in W$ from A.
4. Retrieve the actual degrees for the n_2 nodes w with the largest values of S_w.
5. Return the identified top-k list of most popular entities in W.



In total, we use $n = n_1 + n_2$ requests to API (Step 2 and Step 4).

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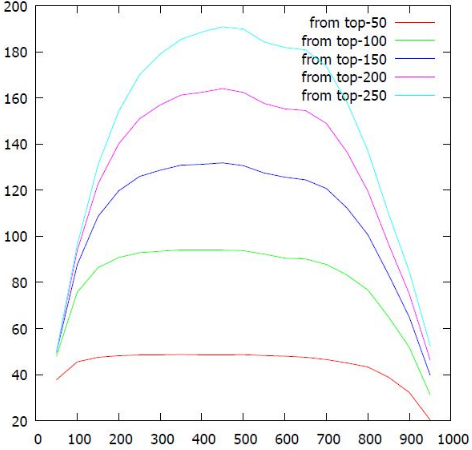
- Make a guess: We use 1000 requests to API. For which k can we identify a top-k list of most followed Twitter users with 90% precision?

Results

$N = 500M$, $n = 1000$



Interest groups VKontakte

- Popular social network in Russian, more than 200M users.

Rank	Number of participants	Topic
1	4,35M	humor
2	4,1M	humor
3	3,76M	movies
4	3,69M	humor
5	3,59M	humor
6	3,58M	facts
7	3,36M	cookery
8	3,31M	humor
9	3,14M	humor
10	3,14M	movies
100	1,65M	success

- With $n_1 = 700$, $n_2 = 300$, our algorithm identifies on average 73.2 from the top-100 interest groups (averaged over 25 experiments). The standard deviation is 4.6.

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- Popular groups are easier to find than popular users!

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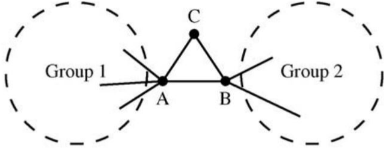

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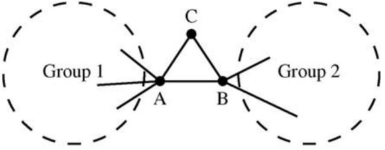

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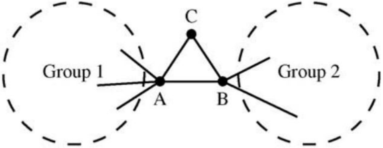

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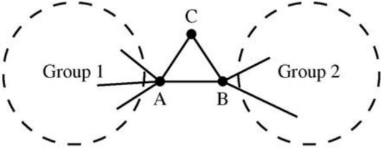

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- Graph is an electrical network, edges are unit resistances, current is induced to s, t is connected to the ground
- The CF-betweenness of edge $e \in E$ is the amount of current through e, averaged over source-destination pairs (s, t)

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Formal definition

- A unit of current is supplied to a source node $s \in V$
- A destination node $t \in V$ connected to the ground
- $\varphi_v^{(s,t)}$ is the absolute potential of node $v \in V$
- $\varphi_t^{(s,t)} = \varphi_{n+1}^{(s,t)} = 0$
- $\varphi^{(s,t)} = [\varphi_1^{(s,t)}, \ldots, \varphi_{n-1}^{(s,t)}]^T$
- Kirchhoff's current law:

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- α-CF betweenness: $x_e^{\alpha} = \dfrac{1}{n(n-1)} \displaystyle\sum_{s,t \in V, s \neq t} x_e^{(s,t)}, \quad e \in E.$

Analysis and computation

Theorem

The voltage drop along the edge (v, w) is given by

$$\varphi_v^{(s,t)} - \varphi_w^{(s,t)} = (c_{s,v} - c_{s,w}) + \frac{c_{s,t}}{c_{t,t}}(c_{t,w} - c_{t,v}),$$

where $C = (c_{v,w}) = [D - \alpha A]^{-1}$.

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$$\varphi_v^{(s,t)} = (1-\alpha)^{-1}\tilde{\pi}_{s,t}(v)d_s^{-1}$$

Datasets

	$\|V\|$	$\|E\|$	$\langle deg(v) \rangle$	$diam(G)$	$C_{\text{clustering}}$	$\langle d(u,v) \rangle$
Dolphins network	62	159	5.13	8	0.259	3.357
VKontakte AMCP	2092	14816	14.16	14	0.338	4.598
Watts-Strogatz	1000	6000	12.00	6	0.422	3.713
Enron	36692	183831	10.02	11	0.4970	$\approx$ 4.8

- The small graphs are used to compare CF and α-CF betweenness
- On the Enron graph, only α-CF betweenness can be computed

Correlations between centrality measures

Kendall tau for centrality measures in the social graph VKontakte AMCP:

	D	PR	Cl	B/w	CF	(0.8)	tr(0.8)	(0.98)
Degree	1.000	0.655	0.679	0.521	0.545	0.659	0.668	0.599
PageRank	0.655	1.000	0.375	0.662	0.717	0.833	0.811	0.766
Closeness	0.679	0.375	1.000	0.382	0.356	0.424	0.445	0.395
Between.	0.521	0.662	0.382	1.000	0.761	0.760	0.749	0.778
CF	0.545	0.717	0.356	0.761	1.000	0.812	0.833	0.917
αCF(0.8)	0.659	0.833	0.424	0.760	0.812	1.000	0.938	0.878
αCF-tr(0.8)	0.668	0.811	0.445	0.749	0.833	0.938	1.000	0.903
αCF(0.98)	0.599	0.766	0.395	0.778	0.917	0.878	0.903	1.000

Influence on the network connectivity

Inverse average distance: $< d^{-1} >= \frac{1}{n(n-1)} \sum_{u,v \in V, u \neq v} \frac{1}{d(u,v)}$

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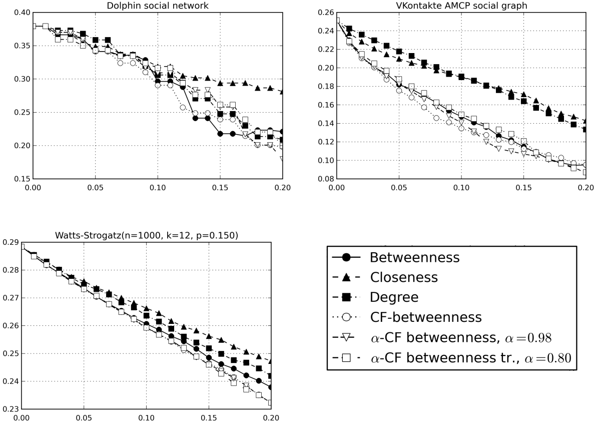
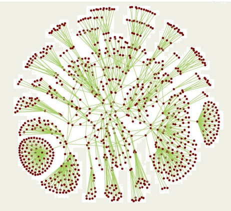
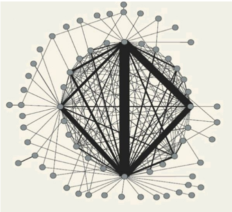


Correlations in power law networks

- **We study the dependencies between degrees of neighboring nodes in graphs with power law degree distribution**

Example: Internet and network of bank transactions





Assortativity coefficient

- $G = (V, E)$ undirected graph of n nodes, E'– directed edges
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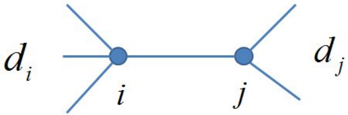
- Newman (2002): assortativity measure $\rho(G)$

$$\rho(G) = \frac{\frac{1}{|E'|}\sum_{(v,w)\in E'} D_v D_w - \left(\frac{1}{|E'|}\sum_{(v,w)\in E'} \frac{1}{2}(D_v + D_w)\right)^2}{\frac{1}{|E'|}\sum_{(v,w)\in E'} \frac{1}{2}(D_v^2 + D_w^2) - \left(\frac{1}{|E'|}\sum_{(v,w)\in E'} \frac{1}{2}(D_v + D_w)\right)^2}$$

- Statistical estimation of the Pearson's correlation coefficient between degrees on two ends of a random edge



Assortative and disassortative graphs

- Newman(2003)

	network	type	size n	assortativity r	error σ_r	ref.
social	physics coauthorship	undirected	52 909	0.363	0.002	a
	biology coauthorship	undirected	1 520 251	0.127	0.0004	a
	mathematics coauthorship	undirected	253 339	0.120	0.002	b
	film actor collaborations	undirected	449 913	0.208	0.0002	c
	company directors	undirected	7 673	0.276	0.004	d
	student relationships	undirected	573	−0.029	0.037	e
	email address books	directed	16 881	0.092	0.004	f
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	Internet	undirected	10 697	−0.189	0.002	h
	World-Wide Web	directed	269 504	−0.067	0.0002	i
	software dependencies	directed	3 162	−0.016	0.020	j
biological	protein interactions	undirected	2 115	−0.156	0.010	k
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- Note: large networks are never strongly disassortative... Dorogovtsev et al. (2010), Raschke et al. (2010)

Convergence of $\rho(G)$ to a non-negtive value

Theorem

Let $(G_n)_{n\geqslant 1}$ be a sequence of graphs of size n satisfying that there exist $\gamma \in (1,3)$ and $0 < c < C < \infty$ such that $cn \leqslant |E| \leqslant Cn$, $cn^{1/\gamma} \leqslant \max_{v\in V_n} D_v \leqslant Cn^{1/\gamma}$ and $cn^{(2/\gamma)\vee 1} \leqslant \sum_{v\in V_n} D_v^2 \leqslant Cn^{(2/\gamma)\vee 1}$. Then, any limit point of the Pearson's correlation coefficient $\rho(G_n)$ is non-negative.

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Alternative: rank correlations

- $G = (V, E)$, E – set of edges, E' – set of directed edges
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- Ties are resolved at random by adding independent *Uniform*$(0, 1)$ random variables (Mesfioui and Tajar, 2005)

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- Factor $|E'|$ cancels, no influence of high dispersion

Convergence criteria in random graphs

$(G_n)_{n\geqslant 1}$ be a sequence of random graphs of size n, $G_n = (V_n, E_n)$.
(X_n, Y_n) degrees on both sides of a uniform directed edge $e \in E_n'$.

Theorem

If every bounded continuous $h\colon \mathbb{R}^2 \to \mathbb{R}$

$$\mathbb{E}_n[h(X_n, Y_n)] \xrightarrow{\mathbb{P}} \mathbb{E}[h(X, Y)],$$

where the r.h.s. is non-random, then

$$\rho^{\mathrm{rank}}(G_n) \xrightarrow{\mathbb{P}} \rho^{\mathrm{rank}} = 12 \cdot \mathrm{Cov}(F_X(X), F_X(Y)),$$

If, in addition, $\mathbb{E}_n[X_n^2] \xrightarrow{\mathbb{P}} \mathbb{E}[X^2] < \infty$, and $\mathrm{Var}(X) > 0$, then

$$\rho(G_n) \xrightarrow{\mathbb{P}} \rho = \frac{\mathrm{Cov}(X, Y)}{\mathrm{Var}(X)}.$$

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Theorem

Let $(G_t^{(m)})_{t\geqslant 1}$ be the PAM. Then

$$\rho^{\mathrm{rank}}(G_t^{(m)}) \xrightarrow{\mathbb{P}} \rho^{\mathrm{rank}},$$

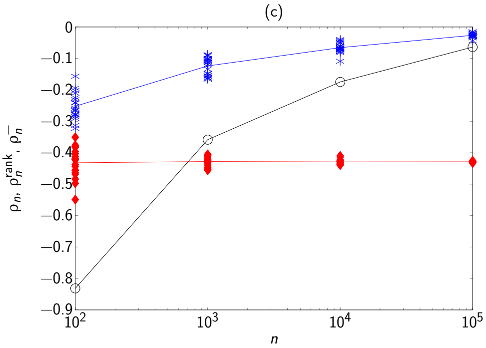
$$\rho(G_t^{(m)}) \xrightarrow{\mathbb{P}} \begin{cases} 0 & \text{if } \delta \leqslant m, \\ \rho & \text{if } \delta > m, \end{cases}$$

where, abbreviating $a = \delta/m$,

$$\rho = \frac{(m-1)(a-1)[2(1+m)+a(1+3m)]}{(1+m)[2(1+m)+a(5+7m)+a^2(1+7m)]}.$$

Preferential Attachment (PA) graph

$\rho(G_n)$ (blue), $\rho^{rank}(G_n)$ (red), and mean $\rho^{-}(G_n)$ (black) in 20 simulations for different n



Web and social networks

Dataset	Description	# nodes	max d	$\rho(G_n)$	$\rho(G_n)^{\mathrm{rank}}$	$\rho^{-}(G_n)$
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eu-2005	.eu web crawl	862,664	68,963	-0.0562	-0.2525	-0.0670
uk@100,000	.uk web crawl	100,000	55,252	-0.6536	-0.5676	-1.117
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- More on correlations in directed networks: talk of Pim

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- Optimization of the web crawler BUbiNG (jointly with Milano)