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Monte Carlo methods and mathematical analysis of directed networks

Nelly Litvak
P2: University of Twente, The Netherlands

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P2: University of Twente, The Netherlands

Nelly Litvak, Pim van der Hoorn

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Overview:

- Monte Carlo algorithms for networks
- Statistical methods for graphs
- Local and global centralities in directed random graphs

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  - Routing via large degree nodes
  - Proxy for various centrality measures
  - Node clustering and classification
  - Epidemic processes on networks
  - Finding most popular entities (e.g. interest groups)

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  - Node clustering and classification
  - Epidemic processes on networks
  - Finding most popular entities (e.g. interest groups)
  - Many companies maintain network statistics (*twittercounter.com*, *followerwonk.com*, *twitaholic.com*, *www.insidefacebook.com*, *yavkontakte.ru*)

Top-k most popular entities in directed networks

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- The network can be accessed only via API, with limited access.
- Randomized algorithms: Find a 'good enough' answer with a small answer of API requests.
- A lot of attention in the literature.

Two-stage algorithm

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- **Stage 1:** Use n_1 API requests to retrieve id's of the followees of n_1 random users
- **Stage 2:** Use n_2 API requests to check *real* degrees of the n_2 users with largest number of followers among the n_1 random users from Stage 1.
- **Result:** Return the identified top-k list of most popular users.

In total, we use $n = n_1 + n_2$ requests to API

Results on Twitter

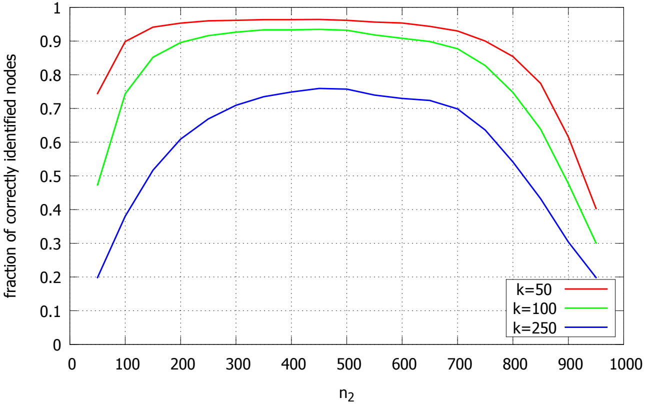

Figure : The fraction of correctly identified top-k most followed Twitter users as a function of n_2, with $n = 1000$.

Known algorithms

- Random-walk based. Cooper, Radzik, Siantos (2012)
  Transitions probabilities along undirected edges (x, y) are proportional to $(d(x)d(y))^b$, where $d(x)$ is the degree of a vertex x and $b > 0$ is some parameter.

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- Random Walk Avrachenkov, L, Sokol, Towsley (2012)
  Random walk with uniform jumps. In an undirected graphs the stationary distribution is a linear function of degrees.
- Crawl-AI and Crawl-GAI. Kumar, Lang, Marlow, Tomkins (2008) At every step all nodes have their *apparent in-degrees* S_j, $j = 1, \dots, N$: the number of discovered edges pointing to this node. Designed for WWW crawl.

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- HighestDegree. Borgs, Brautbar, Chayes, Khanna, Lucier (2012) Retrieve a random node, check in-degrees of its out-neighbors. Proceed while resources are available.

Comparison of the algorithms

Table : Percentage of correctly identified nodes from top-100 in Twitter averaged over 30 experiments, $n = 1000$

Algorithm	mean	standard deviation
Two-stage algorithm	92.6	4.7
Random walk (strict)	0.43	0.63
Random walk (relaxed)	8.7	2.4
Crawl-GAI	4.1	5.9
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Advantages of the two-stage algorithm:

- does not waste resources
- obtains *exact* degrees of the n_2 'most promising' nodes

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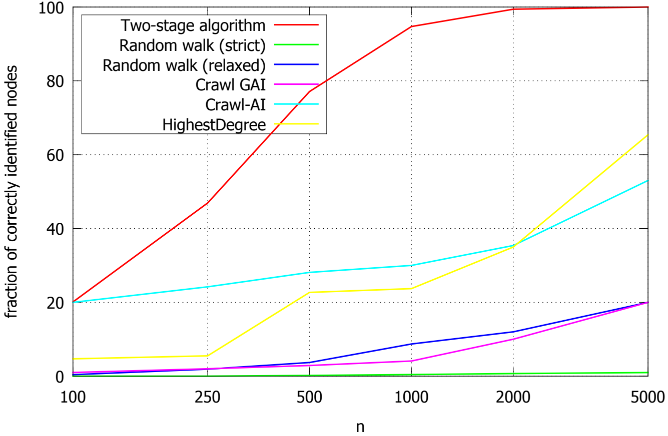

Figure : The fraction of correctly identified top-100 most followed Twitter users as a function of n averaged over 10 experiments.

Performance prediction

$G = (V, E)$ – directed graph, $|V| = N$

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- $S_{i_1} \geqslant S_{i_2} \geqslant \ldots \geqslant S_{i_N}$ be the order statistics of $S_1, \ldots, S_N$.
- Performance measure:

$$E[\text{fraction of correctly identified top-}k\text{ entities}]$$

$$= \frac{1}{k}\sum_{j=1}^{k} P(j \in \{i_1, \ldots, i_{n_2}\}). \qquad (1)$$

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- Computation of $P(j \in \{i_1, \ldots, i_{n_2}\})$ is not feasible even if degrees are known

Poisson prediction

- $P(j \in \{i_1, \ldots, i_{n_2}\})$
  $= P(S_j > S_{i_{n_2}}) + P(S_j = S_{i_{n_2}}, j \in \{i_1, \ldots, i_{n_2}\})$
- **Example.** Twitter graph, take $n_1 = n_2 = 500$. Then the average number of nodes i with $S_i = 1$ among the top-l nodes is

$$\sum_{i=1}^{l} P(S_i = 1) = \sum_{i=1}^{l} 500 \, \frac{F_i}{5 \cdot 10^8} \left(1 - \frac{F_i}{5 \cdot 10^8}\right)^{499},$$

  which is 2540.6 for $l = 10,000$ and it is 57.4 for $l = n_2 = 500$. Hence, typically, $[S_{i_{500}} = 1]$. The event $[i \in \{i_1, \ldots, i_{n_2}\}]$ occurs only for a small fraction of nodes i with $[S_i = 1]$.

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- Approximation:
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- Assume F_j and F_{n_2} are known, then approximate $S_j \sim Poisson(n_1 F_j / N)$

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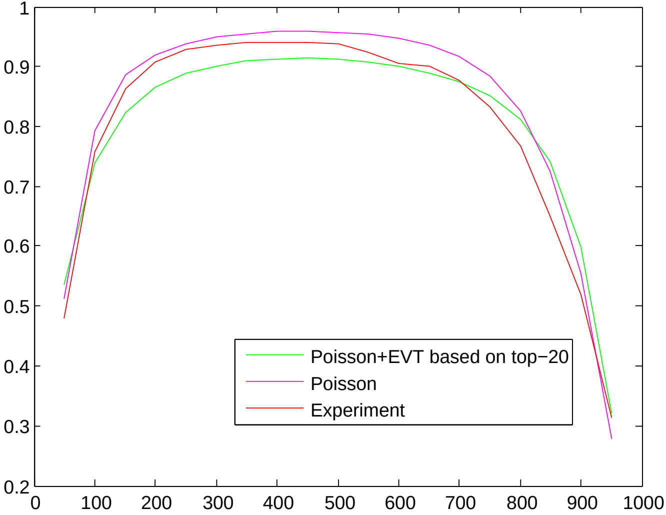
- Estimator for high degrees: Dekkers et al. (1989)
$\hat{f}_j = \hat{F}_m \left(\frac{m}{j-1} \right)^{1/\hat{\gamma}}, \qquad j > 1, j << N.$
- Use $S_j \sim Poisson(n_1 \hat{f}_j / N)$

Performance predictions on the Twitter graph



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- $n = O(n_1)$ (SLLN)
- Assume that $k = o(n)$ as $n \to \infty$, then the maximizer of the probability $P(k \in \{i_1, \ldots, i_{n_2}\})$ is

$$n_2 = (3\gamma k^{\gamma} n)^{\frac{1}{\gamma+1}} (1 + o(1)).$$

Sublinear complexity

$|V| = N$

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- For any fixed $\varepsilon, \delta > 0$, our algorithm finds the fraction $1 - \varepsilon$ of top-k nodes with probability $1 - \delta$ in

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- K.Avrachenkov, N.Litvak, L.Ostroumova-Prokhorenkova and E.Suyargulova, **Quick detection of high-degree entities in large directed networks**, IEEE International Conference on Data Mining (ICDM 2014), (arXiv:1410.0571v2[cs.SI]) [M10-WP1.4]

Directed random graphs

- Null-models for statistical analysis of real networks
- Theoretical characterization of centralities in networks
- In the literature, attention is mainly on undirected networks and their geometric properties (degree distributions, distances, component sizes etc.)
- We analyze centralities and statistical estimators in directed random graphs

Directed Configuration Model

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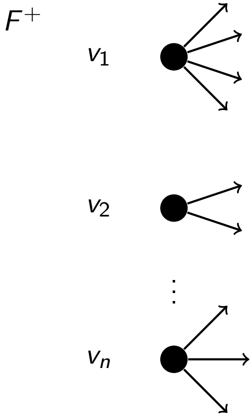

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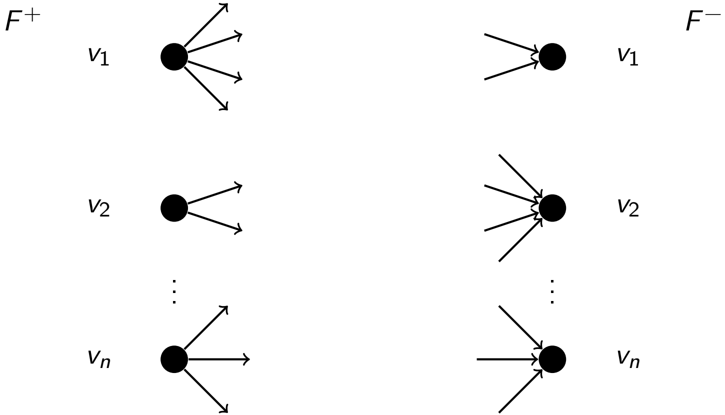

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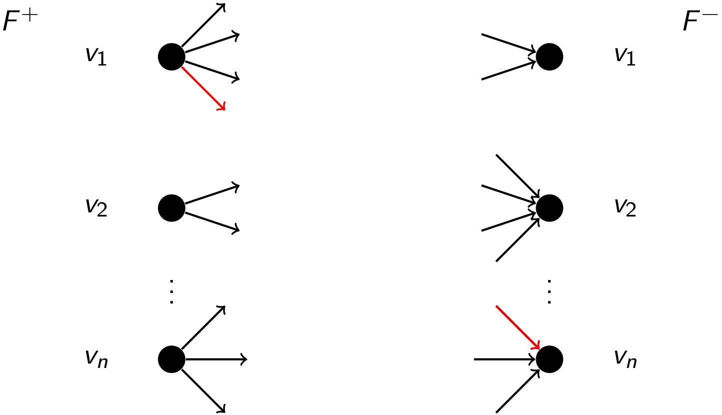

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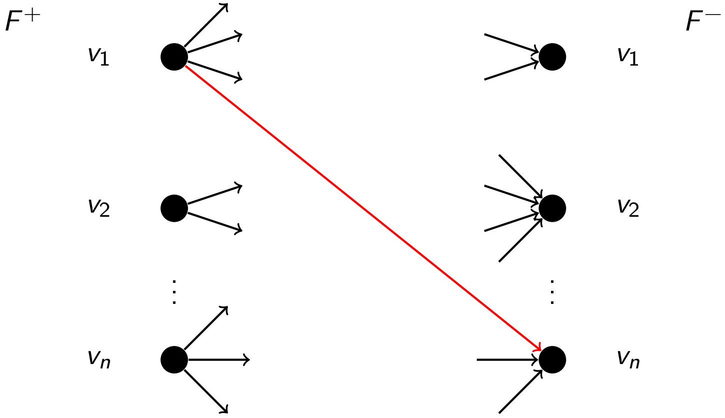

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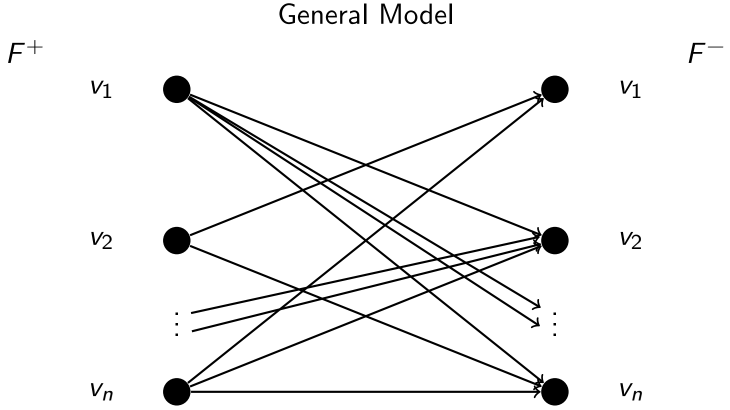

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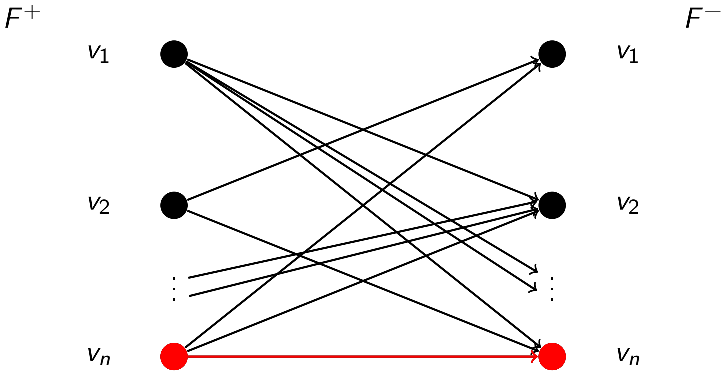

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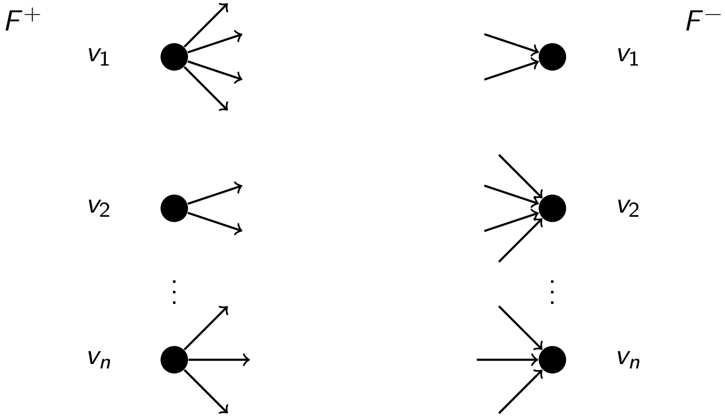

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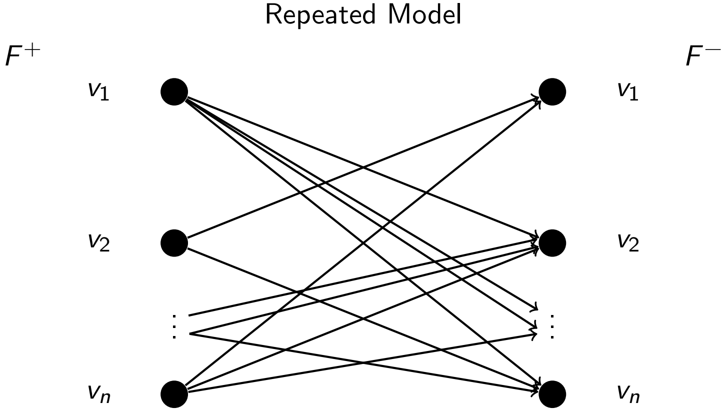

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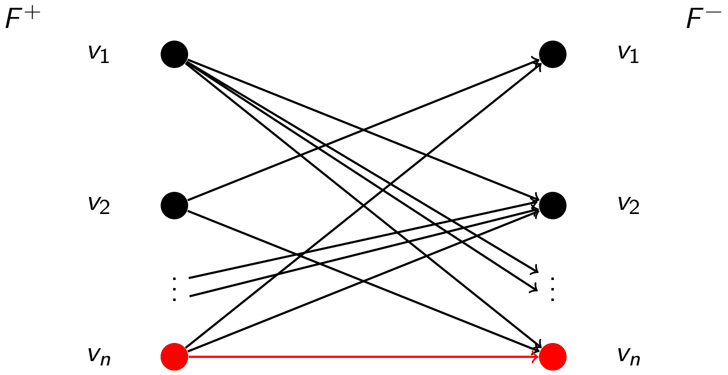

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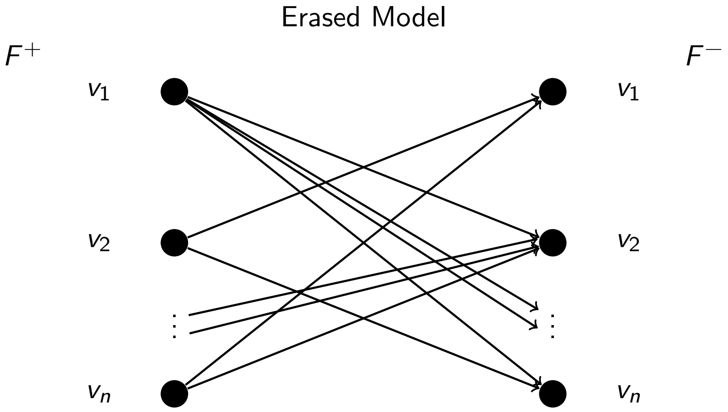

Heavy-tailed degree distributions

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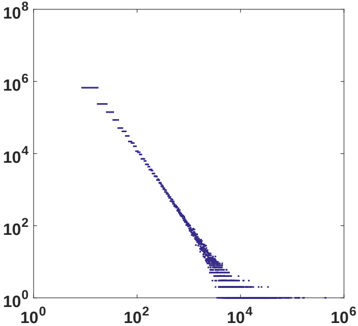
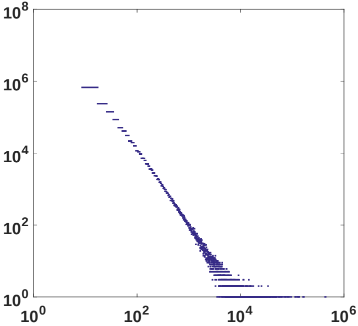


Loglog plot distribution in-degrees of English Wikipedia (data from U.Milan)

Heavy-tailed degree distributions



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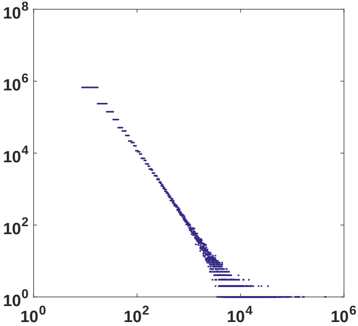

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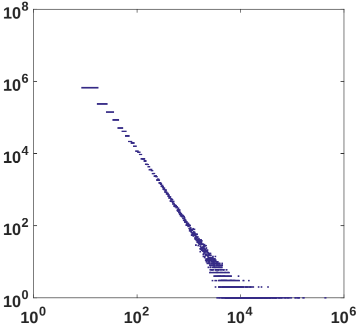

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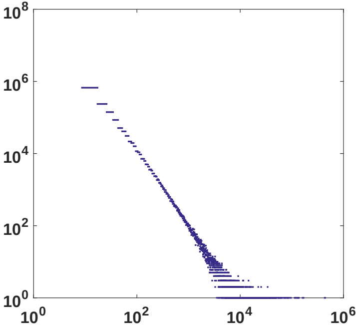

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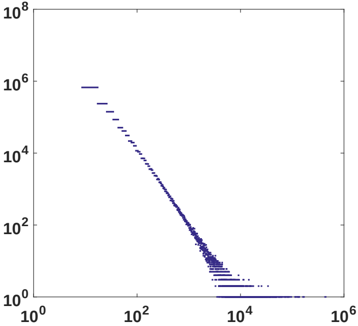

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$$1 < \gamma \leqslant 2 \quad \Rightarrow \quad \mathbb{E}[D] < \infty \quad \mathbb{E}\left[D^2\right] = \infty$$

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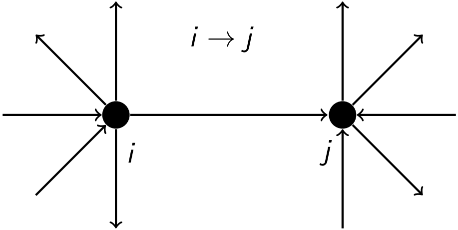

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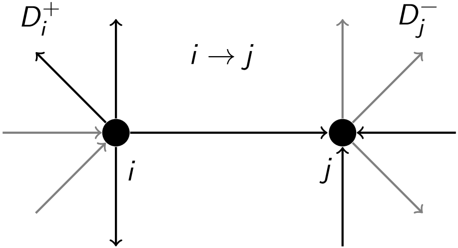

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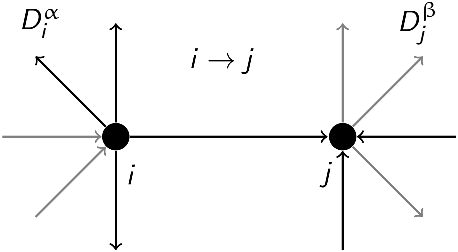

Index degree type by $\alpha, \beta \in \{+, -\}$.

Four types of degree-degree correlation

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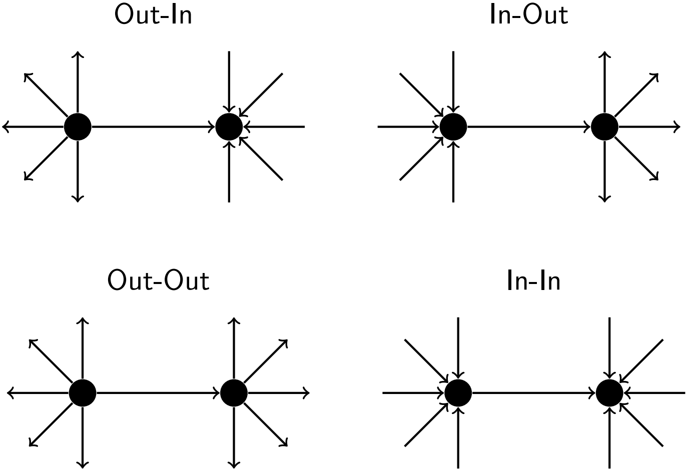

Degree-degree correlations in practice

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- Information flow neural networks.
- Stability of P2P networks under attack.
- Epidemics on networks.
- Network Observability.
- Opinion dynamics based on social influence.
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We have E joint measurements $\{D_i^{\alpha}, D_j^{\beta}\}_{i \to j}$

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Newman 2003

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Theorem 1 (vdHoorn and L 2014)

Let $\alpha, \beta \in \{+, -\}$. Then there exists an area $A_\alpha^\beta \subset \mathbb{R}^2$ such that if $\{G_n\}_{n\in\mathbb{N}}$ is a sequence of graphs with scale-free degree distributions where the tail-exponents $(\gamma_+, \gamma_-) \in A_\alpha^\beta$,

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$$1 < \gamma_\pm \leqslant 2 \in A_\alpha^\beta, \text{ for all } \alpha, \beta \in \{+, -\}$$

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Rank the degrees in descending order

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$$\rho_\alpha^\beta(G_n) := r(R^\alpha, R^\beta)$$

Statistical consistency Spearman's rho

Theorem 2 (vdHoorn and L 2014)

Let $\{G_n\}_{n\in\mathbb{N}}$ be a sequence of random graphs, $\alpha, \beta \in \{+,-\}$ and suppose there exist integer valued random variables $\mathcal{D}^\alpha$ and $\mathcal{D}^\beta$ such that

$$p_\alpha^\beta(k,\ell) \xrightarrow{\mathbb{P}} \mathbb{P}\left(\mathcal{D}^\alpha = k, \mathcal{D}^\beta = \ell\right) \quad \text{as } n \to \infty.$$

Then, as $n \to \infty$,

$$\rho_\alpha^\beta(G_n) \xrightarrow{\mathbb{P}} \rho\left(\mathcal{D}^\alpha, \mathcal{D}^\beta\right)$$

Spearman's rho in the Erased Configuration Model

- Simple graph: multiple edges and loops are removed
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Let $\{G_n\}_{n\in\mathbb{N}}$ be a sequence of graphs of size n, generated by either the Repeated or Erased Configuration Model and $\alpha, \beta \in \{+, -\}$. Then, as $n \to \infty$,

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- Use Theorem 2

$$p_\alpha^\beta(k, \ell) \xrightarrow{\mathbb{P}} \mathbb{P}\left(\mathcal{D}^\alpha = k, \mathcal{D}^\beta = \ell\right) = \mathbb{P}\left(\mathcal{D}^\alpha = k\right)\mathbb{P}\left(\mathcal{D}^\beta = \ell\right)$$

- ECM is a null-model for degree-degree correlations

Erased model in practice

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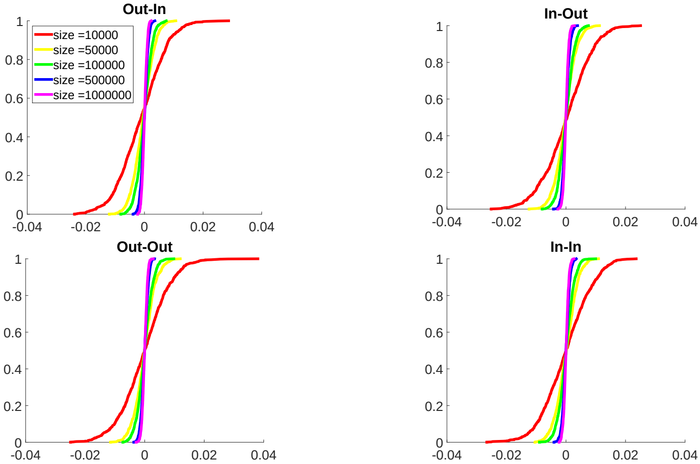

Figure : Empirical cdf of $\rho_{\alpha}^{\beta}(G_n)$ for ECM graphs with $\gamma_{\pm} = 2.1$

Erased model in practice

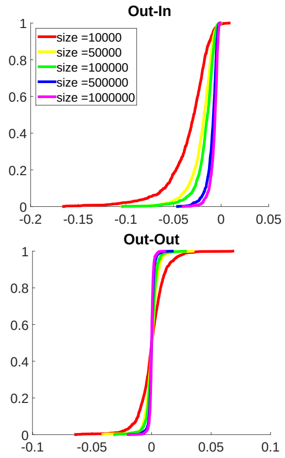

Figure : Empirical cdf of $\rho_\alpha^\beta(G_n)$ for ECM graphs with $\gamma_\pm = 1.5$

Why is Out-In different?

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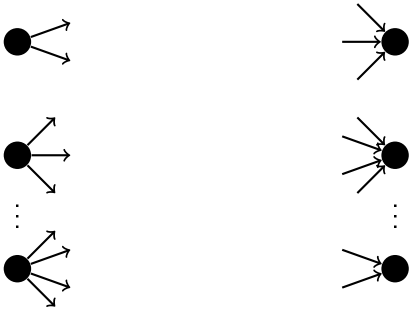

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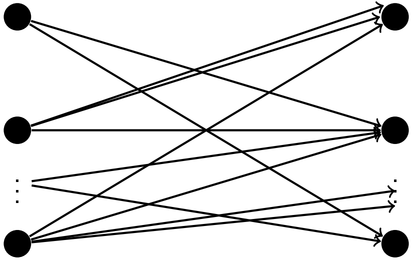

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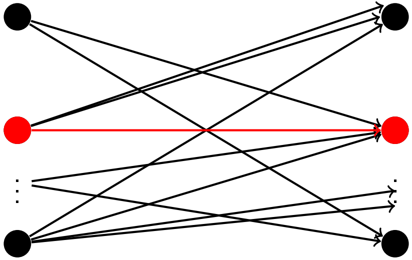

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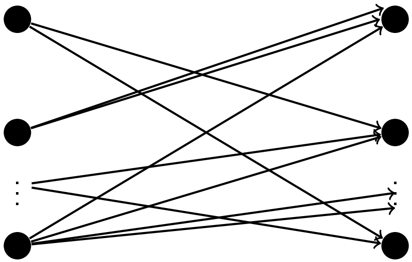

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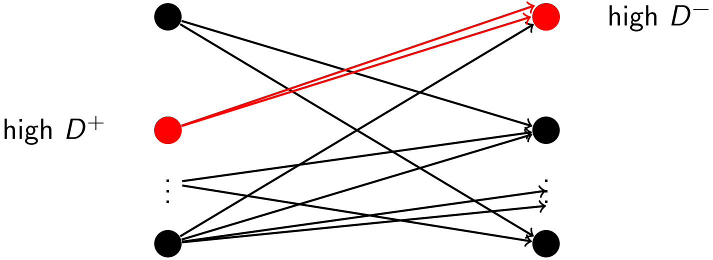

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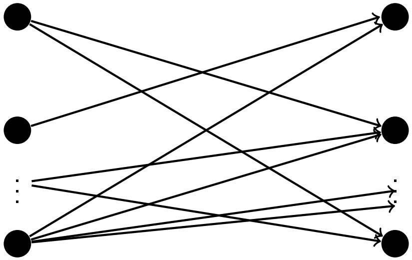

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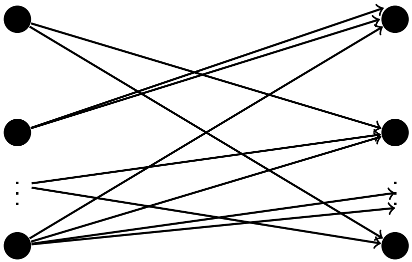

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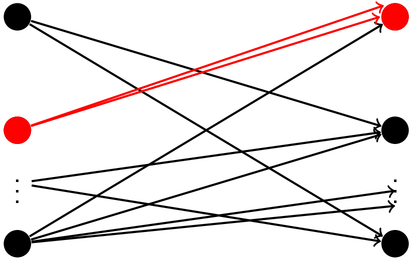

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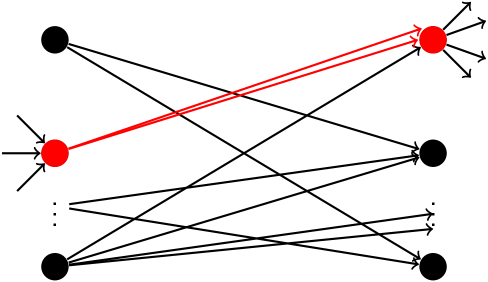

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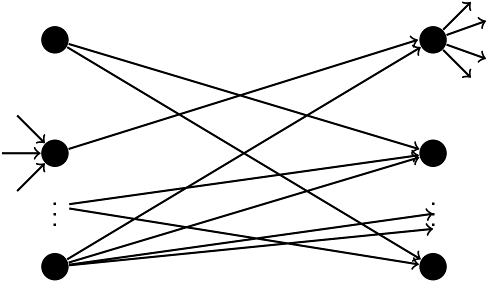

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$$\frac{1}{E}\sum_{i,j=1}^{n}\mathbb{E}_n\left[E_{ij}^c\right] \leqslant 1-\frac{n^2}{E}+\frac{1}{E}\sum_{i,j=1}^{n}\exp\left\{\frac{D_i^+ D_j^-}{E}\right\}$$

CLT for heavy-tailed distributions and Tauberian theorem

A second upper bound

$$\frac{1}{E}\sum_{i,j=1}^{n}\mathbb{E}_n\left[E_{ij}^c\right] \leqslant \frac{n^2}{E}\left(\frac{1}{n^2}\sum_{i,j=1}^{n}\frac{D_i^+ D_j^-}{E} - 1 + \frac{1}{n^2}\sum_{i,j=1}^{n}\exp\left\{\frac{D_i^+ D_j^-}{E}\right\}\right)$$

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$$\frac{1}{E}\sum_{i,j=1}^{n}\mathbb{E}_n\left[E_{ij}^c\right] \leqslant O\left(n^{\frac{1}{\gamma_+\wedge\gamma_-}-1}\right) + O\left(n^{1-(\gamma_+\wedge\gamma_-)}\right)$$

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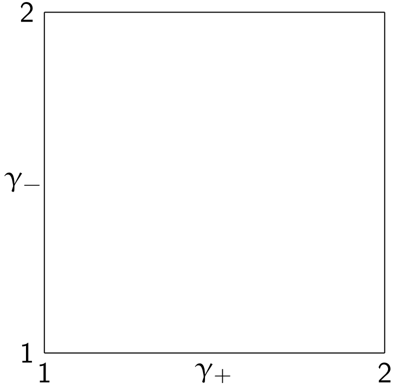
Phase transitions for $\rho^-_+(G_n)$

Phase transitions for $\rho_{+}^{-}(G_n)$



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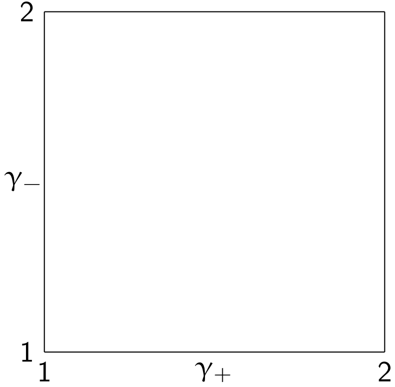

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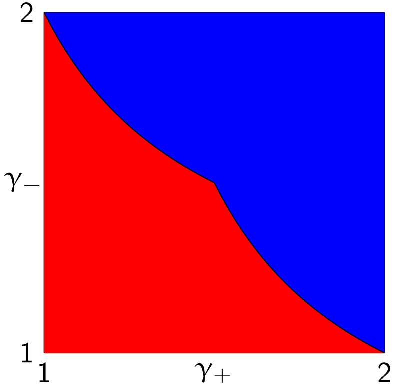

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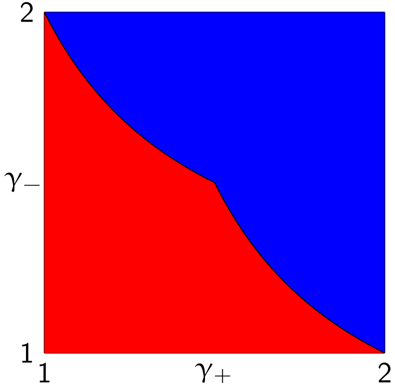

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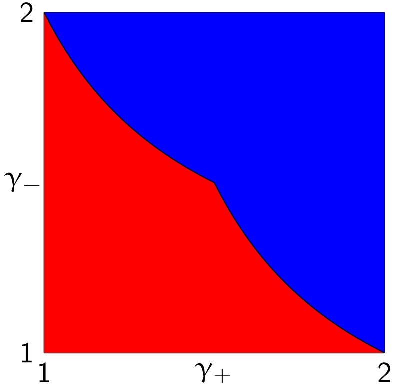

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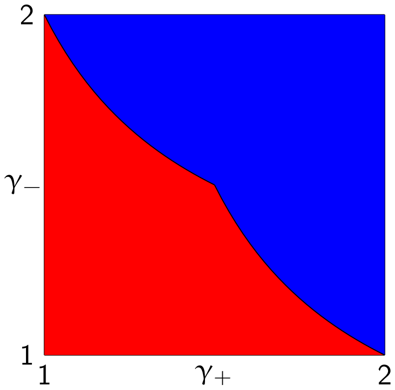

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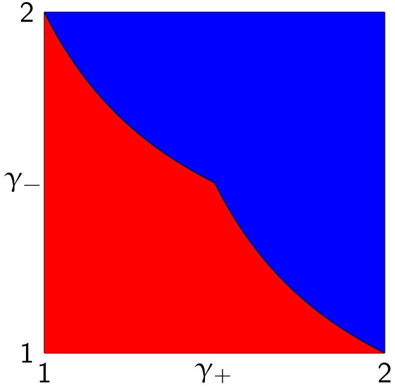

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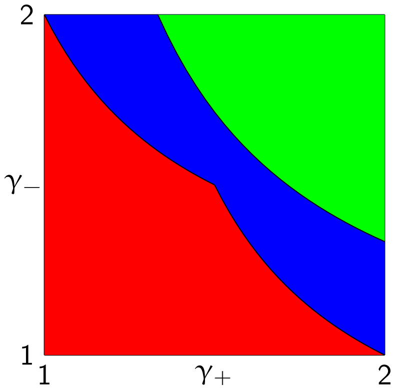

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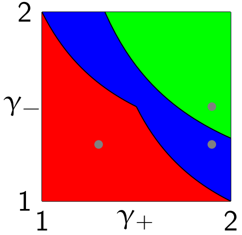

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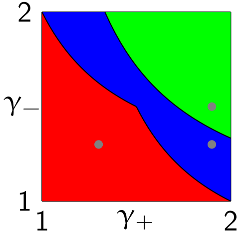

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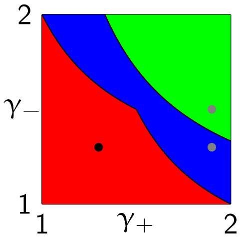

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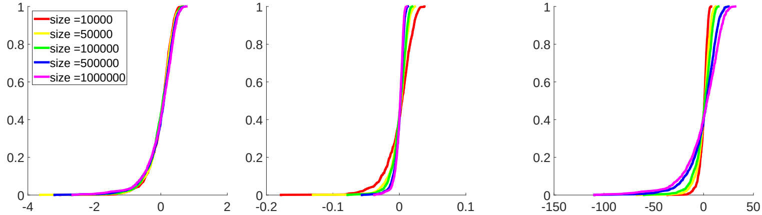

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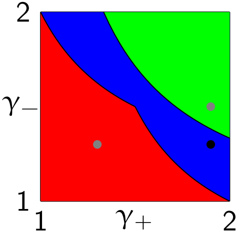

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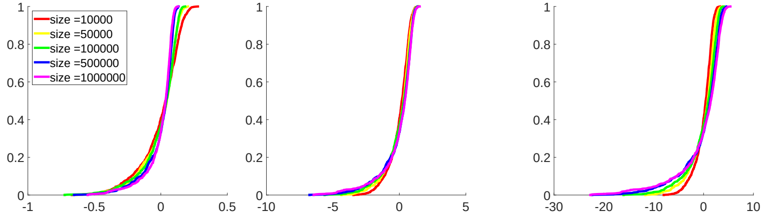

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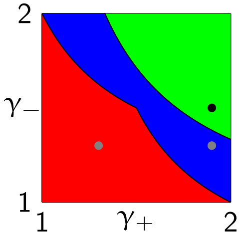

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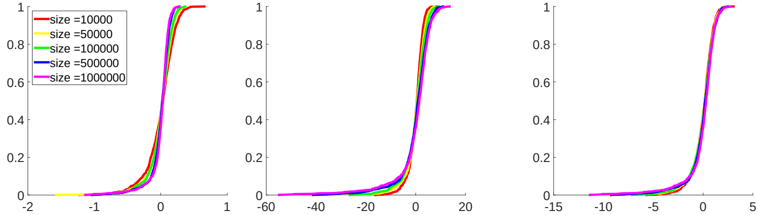

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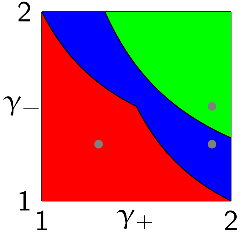

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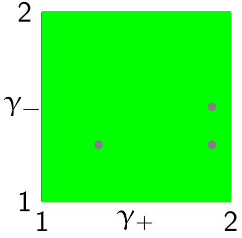
Scaling of $\rho^+_-(G_n)$ in practice



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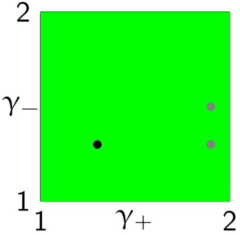

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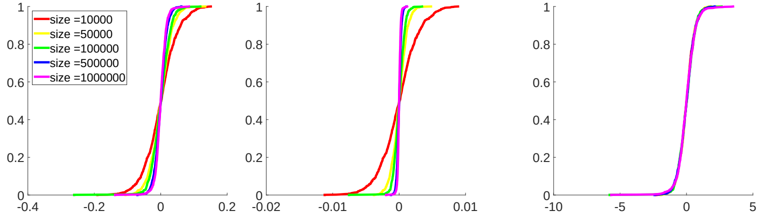

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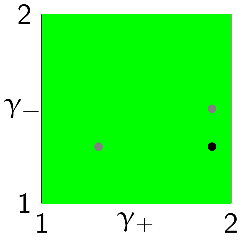

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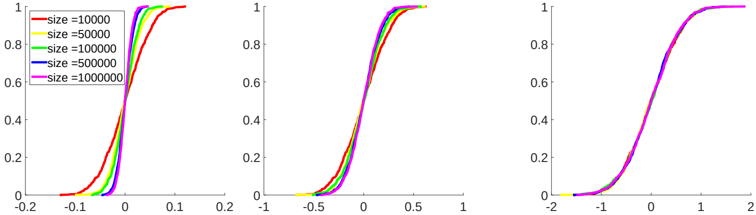

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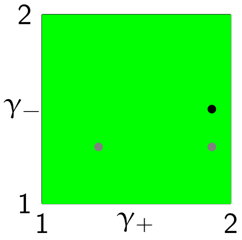

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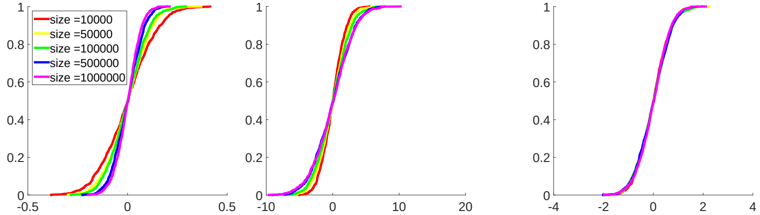

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- P. van der Hoorn and N. Litvak, **Convergence of rank based degree-degree correlations in random directed networks**, Moscow Journal of Combinatorics and Number Theory (2015) (arXiv:1407.7662[math.PR], 2014) [M13-WP4.3]
- P. van der Hoorn and N. Litvak, **Phase transitions for scaling of structural correlations in directed networks**, (arXiv:1504.01535[physics.soc-ph], 2015 [M13- WP4.3]

PageRank in Directed Configuration Model (DCM)

- PageRank R_i of page $i = 1, \dots, n$ is defined as a stationary distribution of a random walk with jumps:

$$R_i = \sum_{j \to i} \frac{c}{d_j} R_j + (1 - c) q_i, \quad i = 1, \dots, n$$

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- N.Chen, N.Litvak and M.Olvera-Cravioto, **Ranking algorithms on directed configuration networks**, (arXiv:1409.7443v2[math.PR], 2014) [M7-WP5.2]

Bi-directed degree sequence

- Directed graph on n nodes $V = \{v_1, \ldots, v_n\}$.
- Extended bi-degree sequence $(\mathbf{N}_n, \mathbf{D}_n, \mathbf{C}_n, \mathbf{Q}_n) = \{(N_i, D_i, C_i, Q_i) : 1 \leqslant i \leqslant n\}$

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- **Assumption 1.** Existence of certain limits in the spirit of the weak law of large numbers, including $\frac{1}{n}\sum_{i=1}^{n} D_i^2$ to be bounded in probability (finite variance of the out-degrees).
- **Assumption 2.** In a sequence of random graphs of growing size, the empirical probabilities $P(D_i = k)$ converge to certain distributions.

PageRank in the DCM

- $M = M(n) \in \mathbb{R}^{n \times n}$ is related to the adjacency matrix of the graph:

$$M_{i,j} = \begin{cases} s_{ij} C_i, & \text{if there are } s_{ij} \text{ edges from } i \text{ to } j, \\ 0, & \text{otherwise.} \end{cases}$$

- $Q \in \mathbb{R}^n$ is a personalization vector
- We are interested in the distribution of one coordinate, $R_1^{(n)}$, of the vector $\mathbf{R}^{(n)} \in \mathbb{R}^n$ defined by

$$\mathbf{R}^{(n)} = \mathbf{R}^{(n)} M + Q$$

Original and size-biased distribution

- Given the extended bi-degree sequence $(\mathbf{N}_n, \mathbf{D}_n, \mathbf{C}_n, \mathbf{Q}_n)$:
- Empirical distribution for the root node's parameters:

$$F_n^*(m, q) := \frac{1}{n} \sum_{k=1}^{n} 1(N_k \leqslant m, Q_k \leqslant q),$$

converges to $F^*(m, q) := P(\mathcal{N}_0 \leqslant m, \mathcal{Q}_0 \leqslant q)$

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- Empirical distribution for a node that has a out-link to any arbitrary node (size-biased by out-degree)

$$F_n(m, q, x) := \sum_{k=1}^{n} 1(N_k \leqslant m, Q_k \leqslant q, C_k \leqslant x) \frac{D_k}{L_n}$$

converges to $F(m, q, x) := P(\mathcal{N} \leqslant m, \mathcal{Q} \leqslant q) P(\mathcal{C} \leqslant x)$.

Main result

$$\mathcal{R} \stackrel{\mathcal{D}}{=} \sum_{j=1}^{\mathcal{N}} \mathcal{C}_j \mathcal{R}_j + \mathcal{Q},$$

- Let $\mathcal{R}$ denote the *endogenous* solution to the SFPE above.
- The *endogenous* solution is the limit of iterations of the recursion starting, say, from $R_0 = \mathbf{1}$.
- **Main result:**

$$R_1^{(n)} \Rightarrow \mathcal{R}^*, \qquad n \to \infty,$$

where $\Rightarrow$ denotes weak convergence and $\mathcal{R}^*$ is given by

$$\mathcal{R}^* := \sum_{j=1}^{\mathcal{N}_0} \mathcal{C}_j \mathcal{R}_j + \mathcal{Q}_0,$$

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- **3. Convergence to a weighted branching process.** Show that the rank of the root node of the TBT converges weakly to the stated limit. Chen and Olvera-Cravioto (2014)

Matrix iterations

$$\mathbf{R}^{(n,0)} = B,$$
$$\mathbf{R}^{(n,1)} = \mathbf{R}^{(n,0)}M + Q = BM + Q,$$
$$\dots$$
$$\mathbf{R}^{(n,k)} = \sum_{i=0}^{k-1} QM^i + BM^k, \quad k \geqslant 1.$$

Under event $B_n = \left\{\max_{1\leqslant i\leqslant n}|C_i|D_i \leqslant c,\ \frac{1}{n}\sum_{i=1}^{n}|Q_i| \leqslant H\right\}$

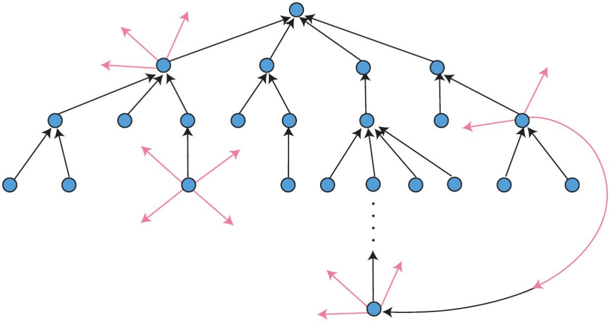
$$\left|\left|\mathbf{R}^{(n,k)} - \mathbf{R}^{(n,\infty)}\right|\right|_1 \leqslant \|\mathbf{r}_0\|_1 c^k + \sum_{i=0}^{\infty}\|\mathbf{Q}\|_1 c^{k+i} = |r_0|nc^k + \|\mathbf{Q}\|_1 \frac{c^k}{1-c}.$$

All nodes are symmetric! Markov inequality:

$$P\left(\left|R_1^{(n,\infty)} - R_1^{(n,k)}\right| > x_n^{-1}\right| B_n\Big) = O\left(x_n c^k\right)$$

Coupling with branching tree

- We start with random node (node 1) and explore its neighbours, labeling the stubs that we have already seen
- τ – the number of generations of WBP completed before coupling breaks



Coupling with branching tree

Lemma (Chen, L, Olvera-Cravioto 2014)

Suppose $(\mathbf{N}_n, \mathbf{D}_n, \mathbf{C}_n, \mathbf{Q}_n)$ *satisfies WLLN,* $\mu = E(\mathcal{N}\mathcal{D})/E(\mathcal{D})$*. Then,*

- *for any* $1 \leqslant k \leqslant h \log n$ *with* $0 < h < 1/(2\log\mu)$*, if* $\mu > 1$*,*
- *for any* $1 \leqslant k \leqslant n^b$ *with* $b < 1/2$*, if* $\mu \leqslant 1$*,*

we have

$$P\left(\tau \leqslant k \middle| \Omega_n\right) = \begin{cases} O\left((n/\mu^{2k})^{-1/2}\right), & \mu > 1, \\ O\left((n/k^2)^{-1/2}\right), & \mu = 1, \\ O\left(n^{-1/2}\right), & \mu < 1, \end{cases}$$

as $n \to \infty$*.*

Remark: μ corresponds to the average number of offspring of a node in TBT.

Numerical results-1

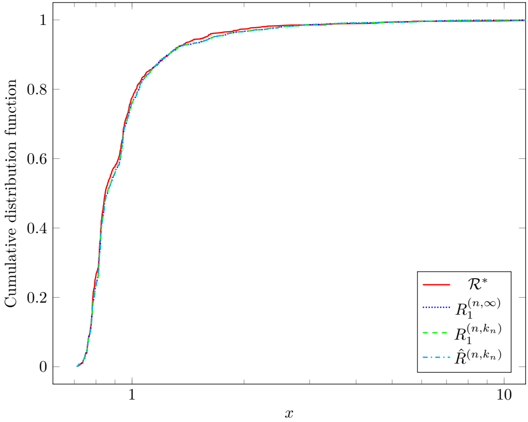


Figure : The empirical CDFs of 1000 samples of $\mathcal{R}^*$, $R_1^{(n,\infty)}$, $R_1^{(n,k_n)}$ and $\hat{R}^{(n,k_n)}$ for $n = 10000$ and $k_n = 9$.

Numerical results-2

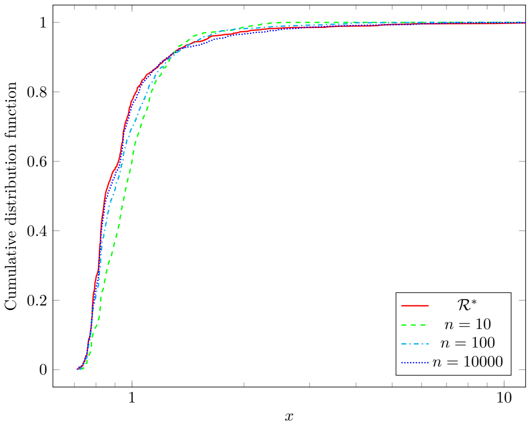


Figure : The empirical CDFs of 1000 samples of $\mathcal{R}^*$ and $R_1^{(n,\infty)}$ for $n = 10$, 100 and 10000.

Wiki graph

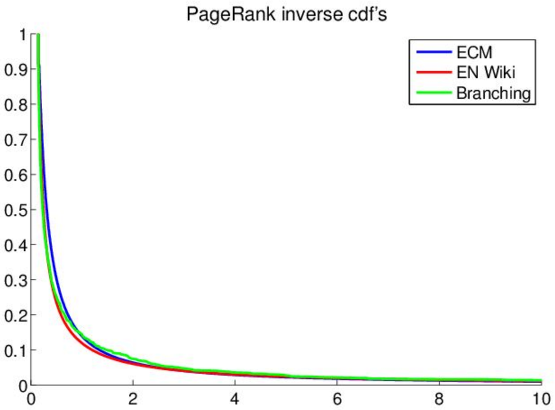


Figure : The empirical distribution of PageRank in English Wikipedia graph and its theoretical prediction. Dataset from U.Milan

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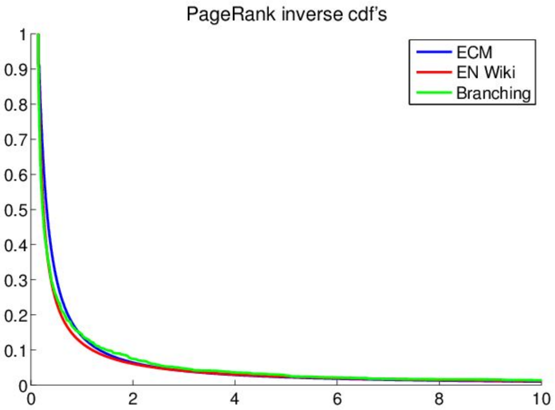

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- Extension to dynamic centralities (jointly with MTA SZTAKI)