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Towards two-dimensional search engines

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Abstract. We study the statistical properties of various directed networks using ranking of their nodes based on the dominant vectors of the Google matrix known as PageRank and CheiRank. On average PageRank orders nodes proportionally to a number of ingoing links, while CheiRank orders nodes proportionally to a number of outgoing links. In this way the ranking of nodes becomes two-dimensional that paves the way for development of two-dimensional search engines of new type. Information flow properties on PageRank-CheiRank plane are analyzed for networks of British, French and Italian Universities, Wikipedia, Linux Kernel, gene regulation and other networks. Methods of spam links control are also analyzed.

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1 Introduction

During the last decade the modern society developed enormously large communication networks. The well known example is the World Wide Web (WWW) which starts to approach to 10^{11} webpage [1]. The sizes of social networks like Facebook [2] and VKONTAKTE [3] also become enormously large reaching 600 and 100 millions user pages respectively. The information retrieval from such huge data bases becomes the foundation and main challenge for search engines [4,5]. The fundamental basis of Google search engine is the PageRank algorithm [6], which ranks all websites according to the probability of the PageRank vector of the Google matrix (see e.g. detailed description at [7], historical surveys of PageRank are given at [8, 9]).

The Google matrix G of a directed network with N nodes is given by

$$G_{ij} = \alpha S_{ij} + (1-\alpha)/N \ , \qquad (1)$$

where the matrix S is obtained by normalizing to unity all columns of the adjacency matrix $A_{i,j}$, and replacing columns with zero elements by $1/N$. An element A_{ij} of the adjacency matrix is equal to unity if a node j points to node i and zero otherwise. The damping parameter α in the WWW context describes the probability $(1-\alpha)$ to jump to any node for a random surfer. The value $\alpha = 0.85$ gives a good classification for WWW [7] and thus we also use this value here. A few examples of Google matrix for various directed networks are shown in Fig. 1. The matrix G belongs to the class of Perron-Frobenius operators [7], its largest eigenvalue is $\lambda = 1$ and other eigenvalues have $|\lambda| \leq \alpha$. The right eigenvector at $\lambda = 1$ gives the probability $P(i)$ to find a random surfer at site i and is called the PageRank. Once the PageRank is found, all nodes can be sorted by decreasing probabilities $P(i)$. The node rank is then given by index $K(i)$ which reflects the relevance of the node i. The PageRank dependence on K is well described by a power law $P(K) \propto 1/K^{\beta_{in}}$ with $\beta_{in} \approx 0.9$. This is consistent with the relation $\beta_{in} = 1/(\mu_{in} - 1)$ corresponding to the average proportionality of PageRank probability $P(i)$ to its in-degree distribution $w_{in}(k) \propto 1/k^{\mu_{in}}$ where $k(i)$ is a number of ingoing links for a node i [10,7]. For the WWW it is established that for the ingoing links $\mu_{in} \approx 2.1$ (with $\beta_{in} \approx 0.9$) while for out-degree distribution w_{out} of outgoing links a power law has the exponent $\mu_{out} \approx 2.7$ [11,12]. Similar values of these exponents are found for the WWW British university networks [13], the procedure call network (PCN) of Linux Kernel software introduced in [14] and for Wikipedia hyperlink citation network of English articles (see e.g. [15]).

The PageRank gives at the top the most known and popular nodes. However, an example of the Linux PCN studied in [14] shows that in this case the PageRank puts at the top certain procedures which are not very important from the software view point (e.g. *printk*). As a result it was proposed [14] to use in addition another ranking taking the network with inverse link directions in the adjacency matrix corresponding to $A_{ij} \rightarrow A^T = A_{ji}$ and constructing from it an additional Google matrix G^* according to relation (1) at the same α. The eigenvector of G^* with eigenvalue $\lambda = 1$ gives then a new inverse PageRank $P^*(i)$ with ranking index $K^*(i)$. This ranking was named CheiRank [15] to mark that it allows to *chercher*

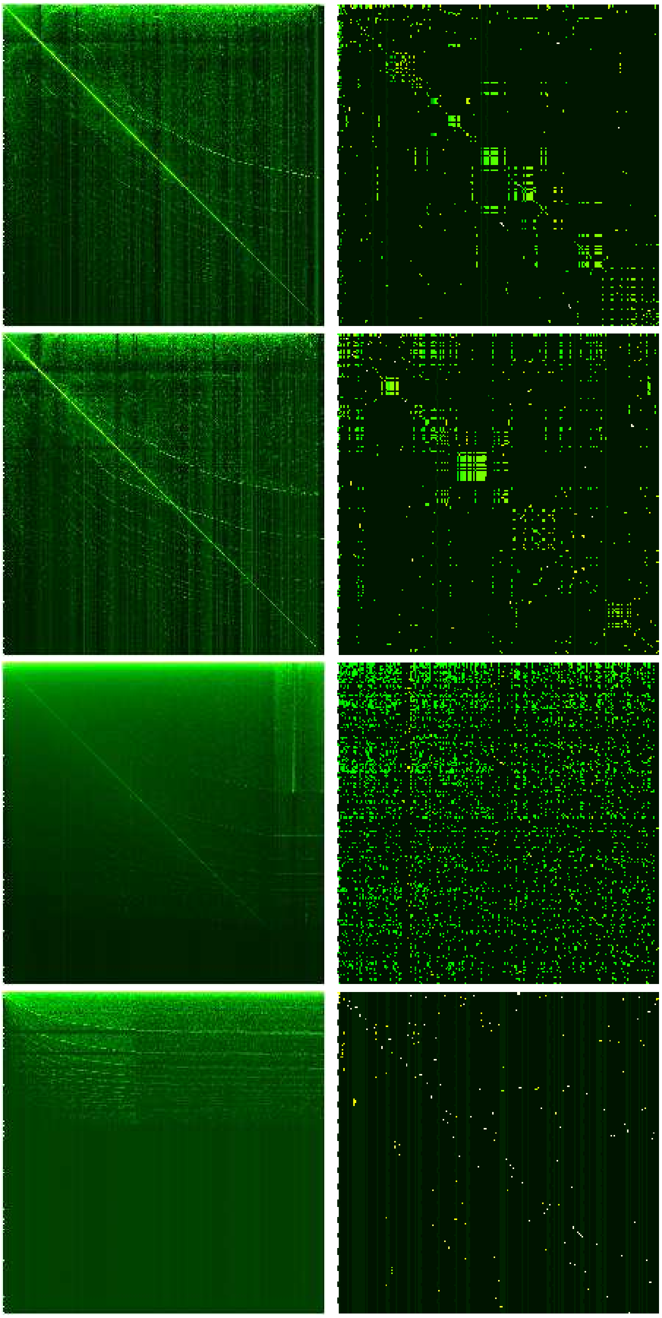

Fig. 1. Left column: coarse-grained density of Google matrix elements $G_{i,j}$ written in the PageRank basis $K(i)$ with indexes $j \to K(i)$ (in x-axis) and $i \to K'(i)$ (in a usual matrix representation with $K = K' = 1$ on top left corner); the coarse graining is done on 500×500 square cells for the networks of University of Cambridge 2006, University of Oxford 2006, Wikipedia English articles, PCN of Linux Kernel V2.6 (from top to bottom). Right column shows the first 200×200 matrix elements of G matrix without coarse graining with the same order of panels as in the left column. Here $\alpha = 0.85$. Color shows the density of matrix elements changing from black for minimum value $((1-\alpha)/N$ to white for maximum value via green and yellow (density is coarse-grained in left column).

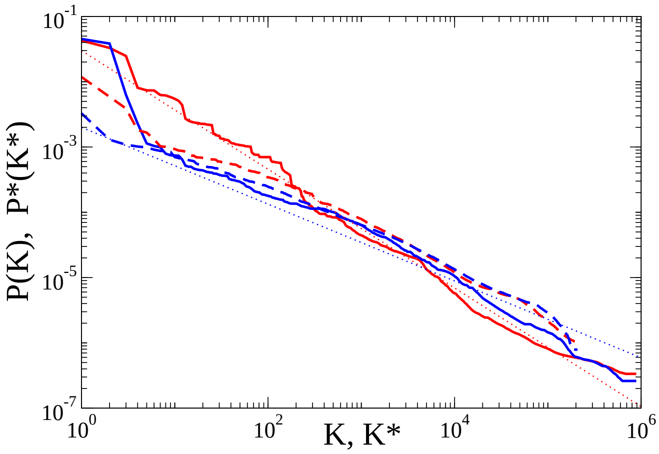

Fig. 2. Dependence of probabilities of PageRank $P(K)$ (red curve) and CheiRank $P^*(K^*)$ (blue curve) on corresponding ranks K and K^* for the network of University of Cambridge in 2006 (dashed curve) and in 2011 (full curve). The power law dependencies with the exponents $\beta \approx 0.91; 0.59$, corresponding to the relation $\beta = 1/(\mu-1)$ with $\mu = 2.1; 2.7$ respectively, are shown by dotted straight lines.

l'information in a new way. Indeed, for the Linux PCN the CheiRank gives at the top more interesting and important procedures compared to the PageRank [14] (e.g. *start_kernel*). While the PageRank ranks the network nodes in average proportionally to a number of ingoing links, the CheiRank ranks nodes in average proportionally to a number of outgoing links. Since each node belongs both to CheiRank and PageRank vectors the ranking of information flow on a directed network becomes **two-dimensional**.

An example of variation of PageRank probability $P(K)$ with K and CheiRank probability $P^*(K^*)$ with K^* are shown in Fig. 2 for the WWW network of University of Cambridge in years 2006 and 2011. Other examples for PCN Linux Kernel and Wikipedia can be find in [14,15]. Detailed parameters of networks which we analyze in this paper and their sources are given in Appendix.

A detailed comparative analysis of PageRank and CheiRank two-dimensional classification was done in [15] on the example of Wikipedia hyperlink citation network of English articles. It was shown that CheiRank highlights communicative property of nodes leading to a new way of two-dimensional ranking. While according to the PageRank top three countries are *1. USA, 2. UK, 3. France* the CheiRank gives *1.India, 2.Singapore, 3.Pakistan* as most communicative Wikipedia country articles. Top 100 personalities of PageRank has the following percents in 5 main category activities 58 (politics), 10 (religion), 17 (arts), 15 (science), 0 (sport). Clearly the significance of politicians is overestimated. In contrast, the CheiRank gives more balanced distribution over these categories with 15, 1, 52, 16, 16 respectively. It allows to classify information in a new way finding composers, architects, botanists, astronomers who are not well known but who, for example, discovered a lot of Australians butterflies (*George Lyell*) or many asteroids (*Nikolai Chernykh*). The 2DRanking was also applied to the brain model of neuronal network [16] and the business process management network [17] and it was shown that it gives a new useful way of information treatment in these networks. The 2DRanking in the PageRank-CheiRank plane also naturally appears for the world trade network corresponding to import and export trade flows [18]. Thus the 2DRanking based on PageRank and CheiRank paves the way to a development of 2D search engines which can become more intelligent than the present Google search based on 1D PageRank algorithm.

In this work we study the statistical properties of such a 2DRanking using examples of various real directed networks including the WWW of British, French and Italian University networks [19], Wikipedia network [15], Linux Kernel networks [14,20], gene regulation networks [21,22] and other networks. The paper is constructed as following: in Section 2 we study the properties of node density in the plane of PageRank and CheiRank, in Section 3 the correlator properties between PageRank and CheiRank vectors are analyzed for various networks, information flow on the plane of PageRank and CheiRank is analyzed in Section 4, methods of control of SPAM outgoing links are discussed in Section 5, 2DRanking applications for the gene regulation networks are considered in Section 6, discussion of results is presented in Section 7. The parameters of the networks and references on their sources are given in Appendix.

2 Node Density of 2DRanking

A few examples of the Google matrix for four directed networks are shown in Fig. 1. There is a significant similarity in the global structure of G for Universities of Cambridge and Oxford with well visible hyperbolic curves (left column) even if at small scales the matrix elements are rather different (right column) in these two networks (see Fig. 1). Such hyperbolic curves are also visible in the Google matrix of Wikipedia (left column) even if here they are less pronounced due to much larger averaging inside the cells which contain about 15 times larger number of nodes (see network parameters in Appendix). At small scale G matrix of Wikipedia is much more dense compared to the cases of Cambridge and Oxford (right column). We attribute such an increase of density of significant matrix elements to a stronger connectivity between nodes with large K in Wikipedia compared to the case of universities where the links have more hierarchical structure. Partially this increase of density can be attributed to a larger number of links per node in the case of Wikipedia but this increase by a factor 2.1 is not so strong and cannot explain all the differences of densities at small K scale. For Wikipedia there is about 20% of nodes at the bottom of the matrix where there are almost no links. For PCN of Linux Kernel this fraction becomes significantly larger with about 60% of nodes. The hyperbolic curves are still well visible for Linux PCN inside remaining 40% of nodes. On a small scale the density of matrix elements for Linux is rather small compared to the three previous cases. We attribute this to a much smaller number of links per node which is by factor 5 smaller for Linux compared to the university networks of Fig. 1 (see data in Appendix).

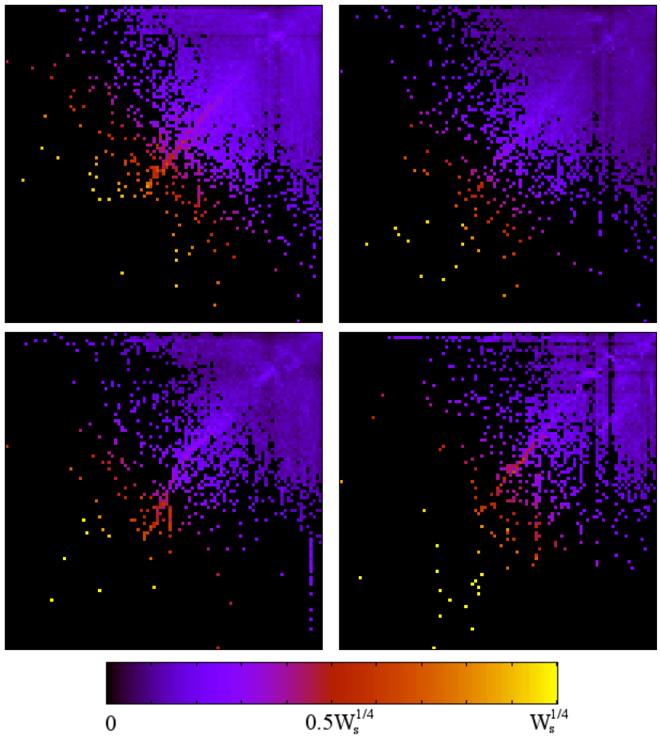

Fig. 3. Density distribution $W(K,K^*) = dN_i/dKdK^*$ for networks of four British Universities in the plane of PageRank K and CheiRank K^* indexes in log-scale $(\log_N K, \log_N K^*)$. The density is shown for 100×100 equidistant grid in $\log_N K, \log_N K^* \in [0,1]$, the density is averaged over all nodes inside each cell of the grid, the normalization condition is $\sum_{K,K^*} W(K,K^*) = 1$. Color varies from black for zero to yellow for maximum density value W_M with a saturation value of $W_s^{1/4} = 0.5 W_M^{1/4}$ so that the same color is fixed for $0.5 W_M^{1/4} \le W^{1/4} \le W_M^{1/4}$ to show in a better way low densities. The panels show networks of University of Cambridge (2006) with $N = 212710$ (top left); University of Oxford with $N = 200823$ (top right); University of Bath with $N = 73491$ (bottom left); University of East Anglia with $N = 33623$ (bottom right).

The distributions of density of nodes $W(K,K^*) = dN_i/dKdK^*$ in the plane of PageRank and CheiRank in log-scale are shown for four networks of British Universities in Fig. 3. Here dN_i is a number of nodes in a cell of size $dKdK^*$ (see detailed description in [15]). Even if the coarse-grained G matrices for Cambridge and Oxford look rather similar the density distributions in (K,K^*) plane are rather different. The density distributions for all four universities clearly show that nodes with high PageRank have low CheiRank that corresponds to zero density at low K, K^* values. At large K, K^* values there is a maximum line of density which is located not very far from the diagonal $K \approx K^*$. The presence of such a line should correspond to significant correlations between $P(K(i))$ and $P^*(K^*(i))$ vectors that will be discussed in

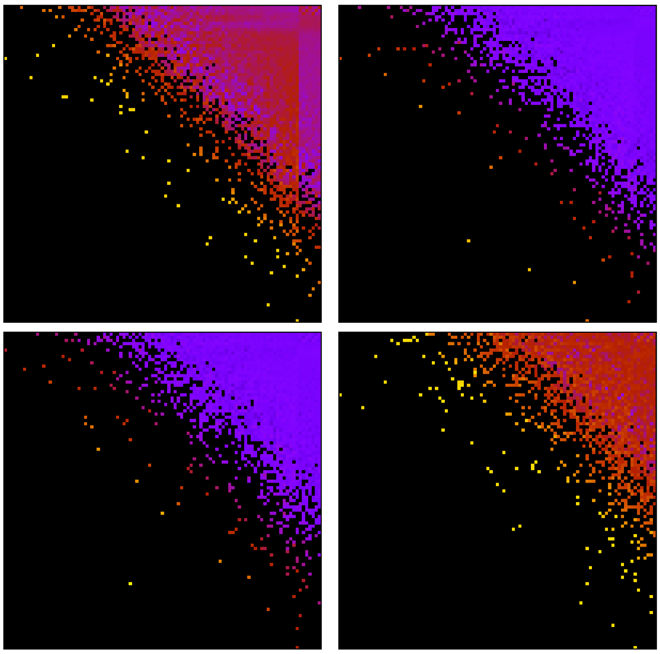

Fig. 4. Density distribution $W(K, K^*) = dN_i/dKdK^*$ of four Linux Kernel networks shown in the same frame as in Fig. 3. The panels show networks for Linux versions V2.0 with $N = 14080$ (top left); V2.3 with $N = 41117$ (top right); V2.4 with $N = 85757$ (bottom left); V2.6 with $N = 285510$ (bottom right). Color panel is the same as in Fig. 3 with a saturation value of $W_s^{1/4} = 0.2W_M^{1/4}$ so that the same color is fixed for $0.2W_M^{1/4} \le W^{1/4} \le W_M^{1/4}$ to show in a better way low densities.

more detail in next Section. The presence of correlations between $P(K(i))$ and $P^*(K^*(i))$ leads to a probability distribution with one main maximum along a diagonal at $K + K^* = const$. This is similar to the properties of density distribution for the Wikipedia network discussed in [15] (see also bottom right panel in Fig. 13 below).

The density of nodes for Linux networks is shown in Fig. 4. In these networks the density is homogeneous along lines $K + K^* = const$ that corresponds to absence of correlations between $P(K(i))$ and $P^*(K^*(i))$. Indeed, in absence of such correlations the distribution of nodes in K, K^* plane is given by the product of independent probabilities. In the log-scale format used in Fig. 4 this leads to a homogeneous density of nodes in the top right corner of $(\log_N K, \log_N K^*)$ plane as it was discussed in [15] (see right panel in Fig. 4 there). Indeed, the distributions in Fig. 4 are very homogeneous inside top-right triangle. We note that, a part of fluctuations, the distributions remain rather stable even if the size of the network is changed by factor 20 from V2.0 to V2.6 version. The physical reasons for absence of correlations between $P(i)$ and $P^*(i)$ have been explained in [14] on the basis of the concept of "separation of concerns" used in software architecture. We discuss the properties of these correlations in the next Section.

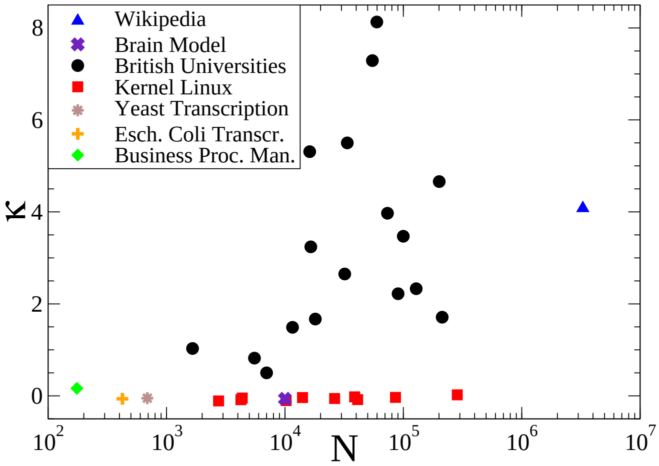

Fig. 5. Correlator κ as a function of the number of nodes N for different networks: Wikipedia network, 17 British Universities, 10 versions of Kernel Linux Kernel PCN, Escherichia Coli and Yeast Transcription Gene networks, Brain Model Network and Business Process Management Network. The parameters of networks are given in Appendix.

3 Correlations between PageRank and CheiRank

The correlations between PageRank and CheiRank can be quantitatively characterized by the correlator

$$\kappa(\tau) = N \sum_{i=1}^{N} P(K(i) + \tau) P^*(K^*(i)) - 1 \ . \qquad (2)$$

Such a correlator was introduced in [14] for $\tau = 0$ and we will use the same notation $\kappa = \kappa(\tau = 0)$.

The values of κ for networks of various size N are shown in Fig. 5. The two types of networks are well visible according to these data. The human created university and Wikipedia networks have typical values of κ in the range $1 < \kappa < 8$. Other networks like Linux PCN, Gene Transcription networks, brain model and business process management networks have $\kappa \approx 0$.

The dependence of $\kappa(\tau)$ on the correlation "time" τ is shown in Fig. 6. For the PCN of Linux there are no correlations at any τ while for the university networks we find that the correlator drops to small values with increase of $|\tau|$ (e.g. $|\tau| > 5$) even if at certain rather large values of $|\tau|$ significant values of correlator κ can reappear.

It is interesting to see what are typical values $\kappa_i = NP(K(i))P^*(K^*(i))$ of contributions in the correlator sum (2) at $\tau = 0$. The distribution of κ_i values for a few networks are shown in Fig. 7. All of them follow a power law with an exponent a =1.23 for PCN Linux, 0.70 Wikipedia and Univ. of Cambridge 0.76 (2006) and 0.66 (2011). We note that further studies are required to obtain analytically the values of the exponent a. In the later two cases the exponent and the distribution shape remains stable in time, however, in 2011 there appear few nodes with very large κ_i values which give a significant increase of the correlator from $\kappa = 1.71$ (in 2006) up to $\kappa = 30.0$ (in

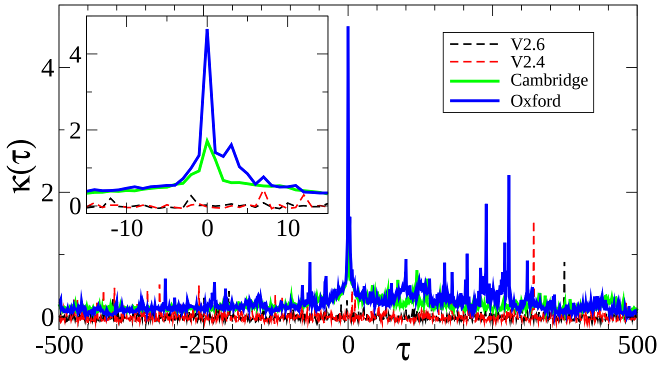

Fig. 6. Correlator $\kappa(\tau)$ for two different long and short range of τ in the main and inset panel respectively. The Kernel Linux PCN $V2.6$ and $V2.4$ are shown by dashed curves while and Universities networks of Cambridge and Oxford are shown by full curves.

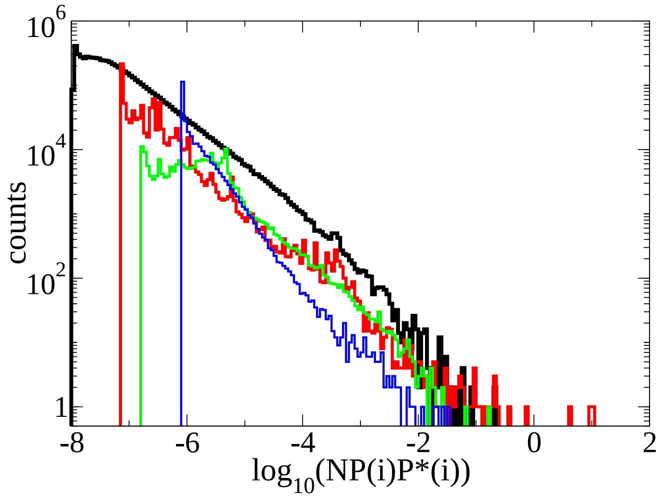

Fig. 7. Histogram of frequency appearance of correlator components $\kappa_i = NP(K(i))P^*(K^*(i))$ for networks of Wikipedia (black), University of Cambridge in 2006 (green) and in 2011 (red), and PCN of Linux Kernel V2.6 (blue). For the histogram the whole interval $10^{-8} \leq \kappa_i \leq 10^2$ is divided in 200 cells of equal size in logarithmic scale.

2011). It is possible that such a situation can appear if it is imposed that practically any page points to the main university page which may have rather high CheiRank due to many outgoing links to other departments and university divisions. We will return to a discussion of university networks collected in 2011 in Section 7.

Another way to analyze the correlations between PageRank and CheiRank is simply to count the number of nodes $\Delta(n)$ inside a square $1 \leq K(i), K^*(i) \leq n$. For a totally correlated distribution with $K(i) = K^*(i)$ we have $\Delta(n)/N = n/N$ while in absence of correlations we should have points homogeneously distributed inside a square $n \times n$ that gives $\Delta(n)/N = (n/N)^2$. The dependence of such point-count correlator $\Delta(n)$ on size n is displayed in Fig. 8 for various networks. These data clearly show that the Linux PCN is uncorrelated being close to the limiting uncorrelated dependence while Wikipedia and British

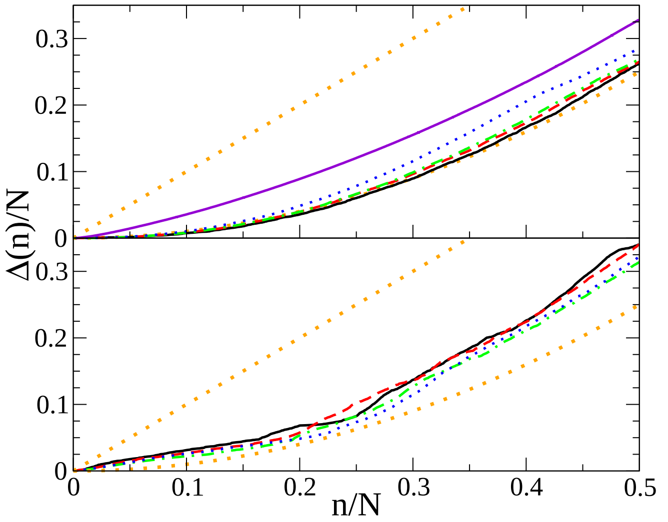

Fig. 8. Dependence of the point-count correlation function $\Delta(n)/N$ on n/N for networks of Wikipedia, British Universities, and Kernel Linux PCN. The curves in the top panel show the cases of Wikipedia (solid violet) and four versions of PCN of Linux Kernel with $V2.0$ (solid black), $V2.3$ (dashed red), $V2.4$ (dot-dashed green), and $V2.6$ (dotted blue). The curves in the bottom panel show the cases of British Universities with East Anglia (solid black), Bath (dashed red), Oxford (dot-dashed green), and Cambridge 2006 (dotted blue). Dotted orange curves represent the totally correlated case with $\Delta(n)/N = n/N$, and the totally uncorrelated one with $\Delta(n)/N = (n/N)^2$.

University networks show intermediate strength of correlations being between the two limiting functions of $\Delta(n)$.

4 Information flow of 2DRanking

According to 2DRanking all network nodes are distributed on a two-dimensional plane (K, K^*). The directed links of the network create an information flow in this plane. To visualize this flow we use the following procedure:
a) each node is represented by one point in the (K, K^*) plane;
b) the whole space is divided in equal size cells with indexes (i, i^*) with the number of nodes inside each cell being n_{i,i^*}, in Fig. 9 we use cells of equal size in usual (left column) and logarithmic (right column) scales;
c) for each node inside the cell (i, i^*), pointing to any other cell $(i', i^{*\prime})$, we compute the vector $(i' - i, i^{*\prime} - i^*)$ and average it over all nodes n_{i,i^*} inside the cell (the weight of links is not taken into account);
d) we put an arrow centered at (i, i^*) with the modulus and direction given by the average vector computed in *c)*.

Examples of such average flows for the networks of Fig. 1 are shown in Fig. 9. All flows have a fixed point attractor. The fixed point is located at rather large values $K, K^* \sim N/4$ that is due to the fact that in average nodes with maximal values $K, K^* \sim N$ point to lower values. At the same time nodes with very small $K, K^* \sim 1$ still

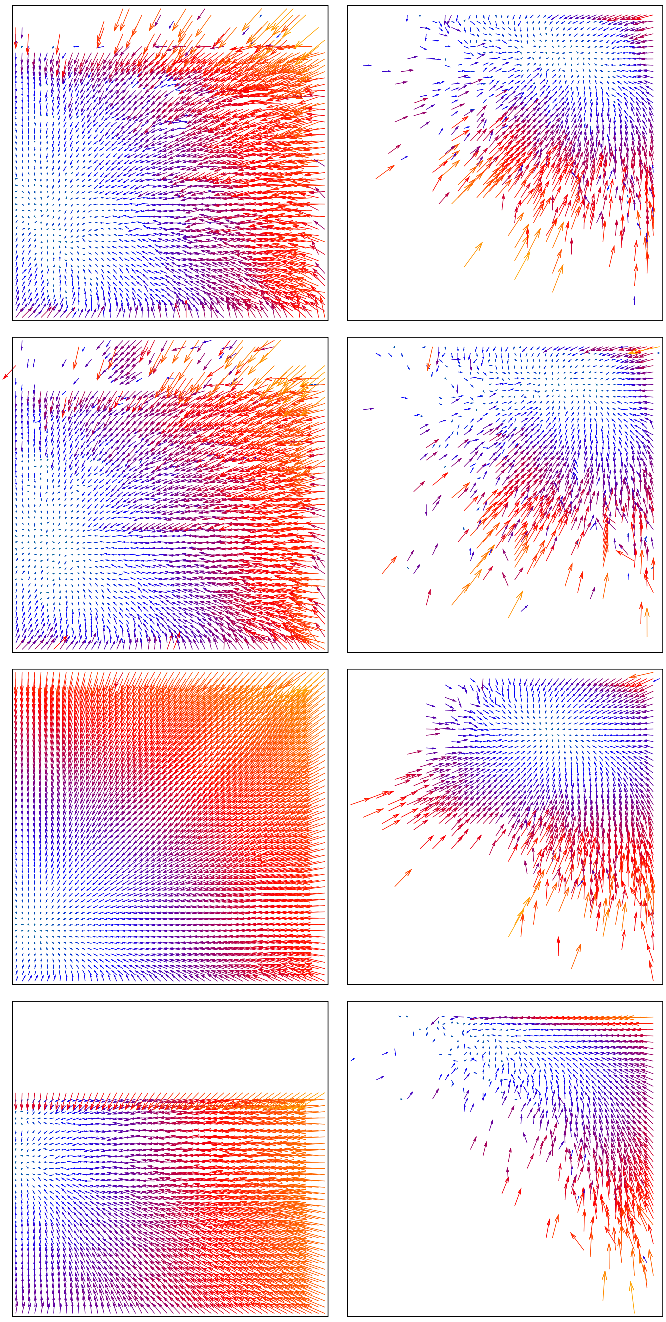

Fig. 9. Information flow on PageRank - CheiRank plane (K, K^*) generated by directed links of the networks of Fig. 1. Outgoing links flow is shown in linear scale (K, K^*) with $K, K^* \in [1, N]$ on left panels, and in logarithmic scale $(\log_N K, \log_N K^*)$ for $\log_N K, \log_N K^* \in [0, 1]$ on right panels. The flow is shown by arrows which size is proportional to the vector amplitude, which is also indicated by color [from yellow for large to blue for small amplitudes]. The rows corresponds to University of Cambridge (2006); University of Oxford (2006), Wikipedia English articles, PCN of Linux Kernel V.2.6 (from top to bottom).

point to some nodes which have larger values of K, K^* that places the fixed point at certain intermediate K, K^* values. A more detailed analysis of statistical properties of information flows on PageRank-CheiRank plane requires further studies.

5 Control of spam links

For many networks ingoing and outgoing links have their own importance and thus should be treated on equal grounds by PageRank and CheiRank as it is described above. However, for the WWW it is more easy to manipulate outgoing links which are handled by an owner of a given web page, while ingoing links are handled by other users. This requires to introduce some level of control on the outgoing links which should be taken into account for the ratings. Since it is very easy to create links to highly popular sites, we will call "spam links" links for which the destination site is much more popular than the source. A quantitative measure of popularity can be provided by the Pageranks of the sites. We do not think that spam links are frequent in control networks such as procedure calls in the Linux kernel and gene regulation. Indeed an excessive number of such links can become harmful for the network performance. However, for WWW networks spam links are probably more widespread. Some websites may try to improve their rating by carefully choosing their outgoing links. Also it is a common policy to have links back to a website's root pages to facilitate navigation. Naturally, a good rating should not be sensitive to the presence of such links. Thus it is imporant to treat spam-links appropriately in order to construct a two dimensional web-search engine.

With this aim we propose the following filter procedure for computation of CheiRank. The standard procedure described above is to invert the directions of all links of the network and then to compute the CheiRank. The filter procedure inverts a link from j to i only if $\eta P(K(j)) > P(K(i))$ where η is some positive filter parameter. After a such inversion of certain links, while other links remain unchanged, the matrix S^* and G^* are computed and the CheiRank vector $P^*(K^*(i))$ of G^* is determined in a usual way. From the definition it is clear that for $\eta = 0$ there are no inverted links and thus after filtering P^* is the same as the PageRank vector P. In the opposite limit $\eta = \infty$ all links are inverted and P^* is then the usual CheiRank discussed in previous sections. Thus intermediate values of η allow to handle the properties of CheiRank depending on a wanted strength of filtering.

The dependence of the fraction f of inverted links (defined as a ration between the number of inverted links to the total number of links) on the filter parameter η is shown for various networks in Fig. 10. There is a significant jump of f at $\eta \approx 1$ for British University networks. In fact the condition $\eta \approx 1$ corresponds approximately to the border relation $P(K) \approx P(K')$ with $K \approx K'$ that marks the diagonal of the G matrix shown in Fig. 1 which has a significant density of matrix elements. As a result for $\eta > 1$ we have a significant increase of inversion of links

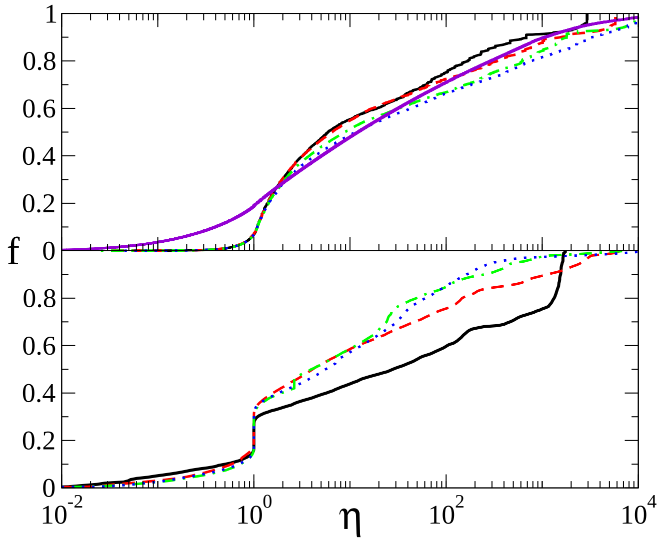

Fig. 10. Fraction f of inverted links as a function of filter parameter η for various studied networks. Top panel: Wikipedia (violet curve) and four versions of Kernel Linux PCN with $V2.0$ (solid black curve), $V2.3$ (dashed red curve), $V2.4$ (dot-dashed green curve), $V2.6$ (dotted blue curve). Bottom panel shows data for British University networks with East Anglia (solid black curve), Bath (dashed red curve), Oxford (dot-dashed green curve), Cambridge 2006 (dotted blue curve).

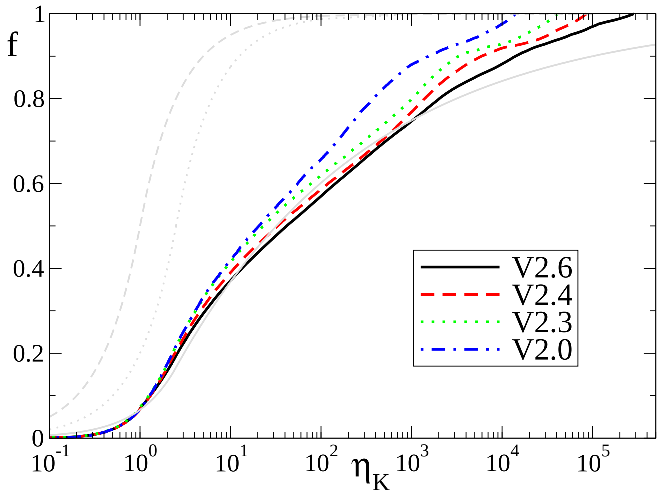

Fig. 11. Fraction f of inverted links in the (K, K') plane with the condition $K(j) < \eta_K K(i)$ shown as a function of filter parameter η_K for Linux networks versions shown by different curves. Grey curves from left to right are the theory curves with $a = 1, \nu = 0$ (dashed); $a = 0.4, \nu = 0$ (dotted); $a = 0.4, \nu = 0.8$ (full) (see text).

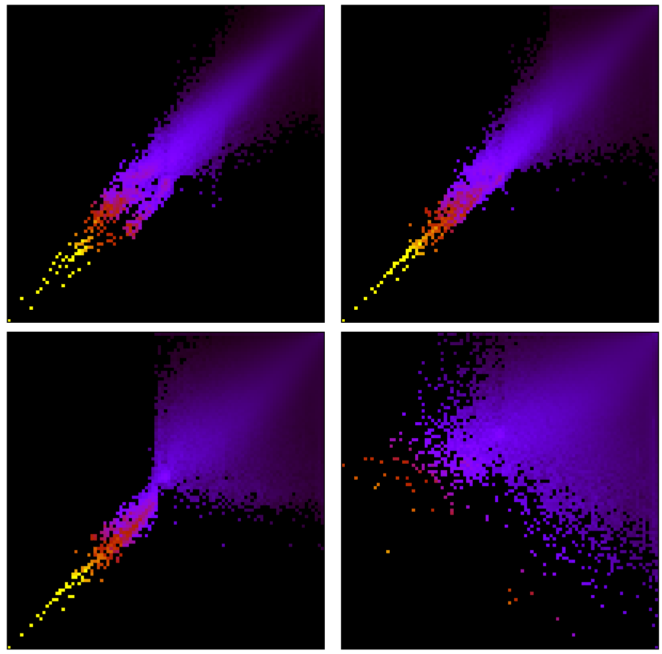

Fig. 12. Density distribution $W(K, K^*) = dN_i/dKdK^*$ for Wikipedia in the plane of PageRank and filtered CheiRank indexes, $(\log_N K, \log_N K^*)$, in a equidistant 100×100 lattice with $\log_N K, \log_N K^* \in [0, 1]$. The filter parameter is $\eta = 10$ (left-top panel), 100 (right-top panel), 1000 (left-bottom panel), 10^5 where all links are inverted (right-bottom panel). The color panel is the same as in Fig. 3 with the saturation value $W_s^{1/4} = 0.5 W_M^{1/4}$.

leading to a jump of f present in Fig. 10. The diagonal density is most pronounced for university networks so that for them the jump of f is mostly sharp.

It is also convenient to consider another condition for link inversion defined not for $P(K_i)$ but directly in the plane (K, K') defined by the condition: links are inverted only if $K(j) < \eta_K K(i)$ (where node j points to node i, $j \to i$). In a first approximation we can assume that the links are homogeneously distributed in the plane of transitions from K to K'. This density is similar to the density distribution of Google matrix elements $G_{K'K}$ shown in Fig. 1. For the homogeneous distribution the fraction f of inverted links is given by an area $\eta_K/2$ of a triangle, which hight is 1 and the basis is η_K, for $\eta_K \le 1$. In a similar way we have $f = 1 - 1/2\eta_K$ for $\eta_K \ge 1$. We can generalize this distribution assuming that there are only links with $1 \le K' \le aN$, that is approximately the case for Linux network where $a = 0.4$ (see Fig. 1 bottom row), and that inside this interval the density of links decreases as $1/(K')^\nu$. Then after computing the area we obtain the expression for the fraction of inverted links valid for $0 \le \nu < 1$:

$$f(\eta_K) = \begin{cases} \frac{1-\nu}{2-\nu}(a\eta_K) & \eta_K \le 1/a \\ 1 + \left(\frac{1-\nu}{2-\nu} - 1\right)(a\eta_K)^{\nu-1} & \eta_K > 1/a \end{cases} \quad (3)$$

The comparison of this theoretical expression with the numerical data for Linux PCN is shown in Fig. 11. It shows that the data for Linux are well described by the theory (3) with $a = 0.4$ and $\nu = 0.8$. The last value takes into account the fact that the density of links decreases with PageRank index K' as it is well visible in Fig. 1.

The variation of nodes density in the plane of PageRank and filtered CheiRank (K, K^*) for the Wikipedia network is shown in Fig. 12 with the filtering by η for $P(K)$

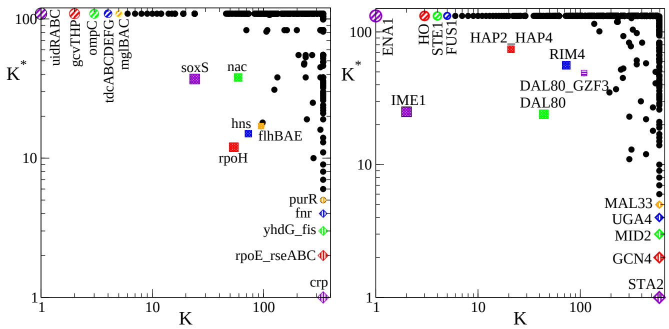

Fig. 13. Distribution of nodes in the plane of PageRank K and CheiRank K^* for Escherichia Coli, and Yeast transcription networks on left and right panels respectively (network data are taken from [22]). The top five nodes of PageRank, CheiRank and 2DRank are labeled by their corresponding node names.

and $P(K')$ values. At moderate values $\eta = 10$ the density is concentrated near diagonal, with further increase of $\eta = 100; 1000$ a broader density distribution appears at large K values which goes to smaller and smaller K until the limiting distribution without filtering is established at very large η. The top 100 Wikipedia articles obtained with filtered CheiRank at the above values of η are given at [23]. We also give there top articles in 2DRank which gives articles in order of their appearance on the borders of a square of increasing size in (K, K^*) plane (see detailed description in [15]). These data clearly show that filtering eliminates articles with many outgoing links and gives a significant modification of top CheiRank articles. Thus the described method can be efficiently used for control of spam links present at the WWW.

6 2DRanking of gene regulation networks

The method of 2DRanking described above is rather generic and can be applied to various types of directed networks. Here we apply it to gene regulation networks of Escherichia Coli and Yeast with the network links taken from [22]. Such transcription regulation networks control the expression of genes and have important biological functions [21].

The distribution of nodes in PageRank-CheiRank plane is shown in Fig. 13. The top 5 nodes in CheiRank are those which send many outgoing orders, top 5 in PageRank are those which obtain many incoming signals and the top 5 in 2DRank combine these two functions. For these networks the correlator κ is close to zero (even slightly negative) which indicates the statistical independence between outgoing and ingoing links quite similarly to the case of the PCN for the Linux Kernel. This may indicate that a slightly negative correlator κ is a generic property for the data flow network of control and regulation systems. Whether the obtained ratings can bring some insights on the functioning of gene regulation can only be assessed by experts in the field. However, we hope that such an analysis will prove to be useful for a better understanding of gene regulation networks.

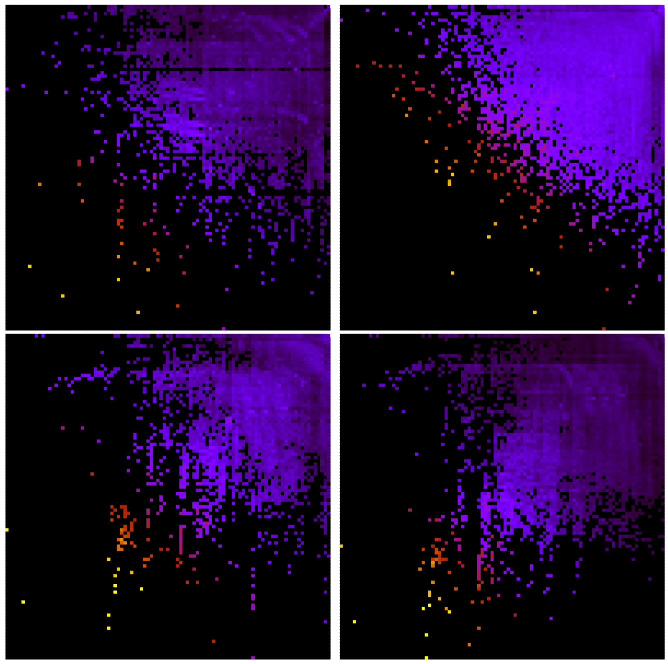

Fig. 14. Density distribution $W(K, K^*) = dN_i/dKdK^*$ shown in the same frame as in Fig. 3 for networks collected in 2011: University of Cambridge (top left), University of Bologna (top right), ENS Paris for crawling level 5 (bottom left) and 7 (bottom right). The color panel is the same as in Fig. 3 with the saturation value $W_s^{1/4} = 0.5 W_M^{1/4}$.

7 Discussion

Above we presented extensive studies of statistical properties of 2DRanking based of PageRank and CheiRank for various types of directed networks. All studied networks are of a free-scale type with an algebraic distribution of ingoing and outgoing links with a usual values of exponents. In spite of that their statistical characteristics related to PageRank and CheiRank are rather different. Some networks have high correlators between PageRank and CheiRank (e.g. Wikipedia, British Universities), while others have practically zero correlators (PCN of Linux Kernel, gene regulation networks). The distribution of nodes in PageRank-CheiRank plane also varies significantly between different types of networks. Thus 2DRanking discussed here gives more detailed classification of information flows on directed networks.

We think that 2DRanking gives new possibilities for information retrieval from large databases which are growing rapidly with time. Indeed, for example the size of the Cambridge network increased by a factor 4 from 2006 to 2011 (see Appendix and Fig. 2). At present, web robots start automatically generate new webpages. These features can be responsible for appearance of gaps in density distribution in (K, K^*) plane at large $K, K^* \sim N$ values visible for large scale university networks of Cambridge and ENS Paris in 2011 (see Fig. 14). Such an automatic generation of links can change the scale-free properties of networks. Indeed, for ENS Paris we observe appear-

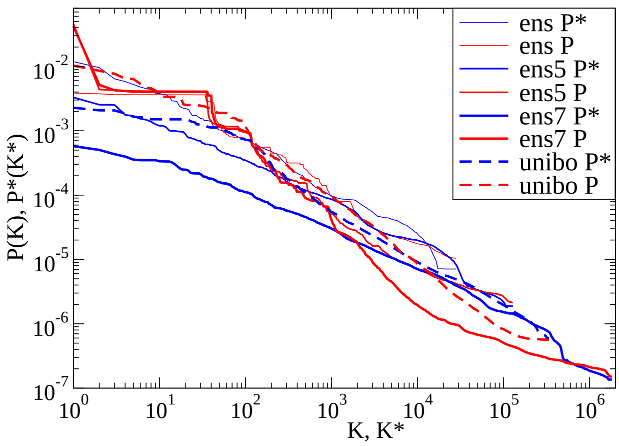

Fig. 15. Dependence of probabilities of PageRank $P(K)$ (red curve) and CheiRank $P^*(K^*)$ (blue curve) on corresponding ranks K and K^* for the networks of ENS Paris (crawling levels 3,5,7) and University of Bologna.

ance of large step in the PageRank distribution $P(K)$ shown in Fig. 15. This step for $P(K)$ remains not sensitive to the deepness of crawling which goes on a level of $3, 5$ and 7 links. However, the CheiRank distribution changes with the deepness level becoming more and more flat (see Fig. 15). Such a tendency in a modification of network statistical properties is visible in 2011 for large size university networks, while networks of moderate size, like University of Bologna 2011 (see data in Figs. 14,15), are not yet affected. A sign of ongoing changes is a significant growth of the correlator value κ which increases up to very large value (30 for Cambridge 2011 and 63 for ENS Paris). There is a danger that automatic generation of links can lead to a delocalization transition of PageRank that can destroy efficiency of information retrieval from the WWW. We note that it is known that PageRank delocalization can appear in certain models of Markov chains and Ulam networks [24]. Such a delocalization of PageRank would make the ranking of nodes inefficient due to high sensitivity of ranking to fluctuations that would create a very dangerous situation for the WWW information retrieval and ranking.

Our studies of 2DRanking pave the way to development of two-dimensional search engines which will use the advantages of both PageRank and CheiRank. While the Google search engine uses as the fundamental basis one-dimension ranking related to PageRank [7], the new two-dimensional search engine, which we call **Dvvadi** from Russian “dva (two)” and “dimension”, will use the complementary ranking abilities of both PageRank and CheiRank. We think that such **Dvvadi** engine/motor [23] will find useful applications for treatment of enormously large databases created by modern society.

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8 Appendix

In this work we used the directed networks listed below (number of nodes N, number of links N_{links} and correlator between PageRank and CheiRank κ). Additional data can be find at [23].

Linux Kernel Procedure Call Networks taken from [14] (see also [20]) with the parameters for various kernel versions:
V1.0 with $N = 2752$, $N_{links} = 5933$, $\kappa = -0.11$;
V1.1 with $N = 4247$, $N_{links} = 9710$, $\kappa = -0.083$;
V1.2 with $N = 4359$, $N_{links} = 10215$, $\kappa = -0.048$;
V1.3 with $N = 10233$, $N_{links} = 24343$, $\kappa = -0.102$;
V2.0 with $N = 14080$, $N_{links} = 34551$, $\kappa = -0.037$;
V2.1 with $N = 26268$, $N_{links} = 59230$, $\kappa = -0.058$;
V2.2 with $N = 38767$, $N_{links} = 87480$, $\kappa = -0, 022$;
V2.3 with $N = 41117$, $N_{links} = 89355$, $\kappa = -0.081$;
V2.4 with $N = 85757$, $N_{links} = 195106$, $\kappa = -0.034$;
V2.6 with $N = 285510$, $N_{links} = 588861$, $\kappa = 0.022$.

Web networks of British Universities dated by year 2006 are taken from [19]:
RGU (Abardeen) with $N = 1658$, $N_{links} = 15295$, $\kappa = 1.03$; **Uwic (Wales)** with $N = 5524$, $N_{links} = 111733$, $\kappa = 0.82$; **NTU (Nottingham)** with $N = 6999$, $N_{links} = 143358$, $\kappa = 0.50$; **Liverpool** with $N = 11590$, $N_{links} = 141447$, $\kappa = 1.49$; **Hull** with $N = 16176$, $N_{links} = 236525$, $\kappa = 5.31$; **Keele** with $N = 16530$, $N_{links} = 117944$, $\kappa = 3.24$; **UCE (Birmingham)** with $N = 18055$, $N_{links} = 351227$, $\kappa = 1.67$; **Kent** with $N = 31972$, $N_{links} = 277044$, $\kappa = 2.65$; **East Anglia** with $N = 33623$, $N_{links} = 325967$, $\kappa = 5.50$; **Sussex** with $N = 54759$, $N_{links} = 804246$, $\kappa = 7.29$; **York** with $N = 59689$, $N_{links} = 414200$, $\kappa = 8.13$; **Bath** with $N = 73491$, $N_{links} = 541351$, $\kappa = 3.97$; **Glasgow** with $N = 90218$, $N_{links} = 544774$, $\kappa = 2.22$; **Manchester** with $N = 99930$, $N_{links} = 1254939$, $\kappa = 3.47$; **UCL (London)** with $N = 128450$, $N_{links} = 1397261$, $\kappa = 2.33$; **Oxford** with $N = 200823$, $N_{links} = 1831542$, $\kappa = 4.66$; **Cambridge (2006)** with $N = 212710$, $N_{links} = 2015265$, $\kappa = 1.71$.

We also developed a special code with which we performed crawling of university web networks in January - March 2011 with the data given below: **University of Cambridge (2011)** with $N = 898262$, $N_{links} = 15027630$, $\kappa = 30.0$; **École Normale Supérieure, Paris (ENS 2011)** with $N = 28144$, $N_{links} = 971856$, $\kappa = 1.67$ (crawling deepness level of 3 links), $N = 129910$, $N_{links} = 2111944$, $\kappa = 16.2$ (crawling deepness level of 5 links), $N = 1820015$, $N_{links} = 25706373$, $\kappa = 63.6$ (crawling deepness level of 7 links); **University of Bologna** with $N = 339872$, $N_{links} = 16345488$, $\kappa = 2.63$.

The data for hyperlink network of **Wikipedia English articles (2009)** are taken from [15] with $N = 3282257$, $N_{links} = 71012307$, $\kappa = 4.08$.

Transcription Gene networks are taken from [22]. We have for them: **Escherichia Coli** with $N = 423$, $N_{links} = 519$, $\kappa = -0.0645$; **Yeast** with $N = 690$, $N_{links} = 1079$, $\kappa = -0.0497$; for all links the weight is take to be the same.

Business Process Management network is taken from [17] with $N = 175$, $N_{links} = 240$, $\kappa = 0.164$.

Brain Model network is take from [16] with $N = 10000$, $N_{links} = 1960108$, $\kappa = -0.054$ (unweighted), $\kappa = -0.065$ (weighed).

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