Arbitrary accuracy iterative phase estimation algorithm as a two qubit benchmark

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We discuss the implementation of an iterative quantum phase estimation algorithm, with a single ancillary qubit. We suggest using this algorithm as a benchmark for multi-qubit implementations. Furthermore we describe in detail the smallest possible realization, using only two qubits, and exemplify with a superconducting circuit. We discuss the robustness of the algorithm in the presence of gate errors, and show that 7 bits of precision is obtainable, even with very limited gate accuracies.

Solid-state quantum computing is now entering the stage of exploration of multi-qubit circuits. Coherent two-qubit coupling has been experimentally realized for all major types of superconducting qubits[1,2,3,4,5,6,7,8,9,10,11], and two-qubit gates have been demonstrated for charge[8], phase[9,10] and flux qubits[11]. The question then arises, what kind of testbed application can be performed having at hand a very limited amount of qubits?

Here we propose to employ the Phase Estimation Algorithm (PEA), which can be implemented with just two qubits. Furthermore, we suggest how to use this algorithm to characterize (benchmark) qubit circuits. The PEA is an algorithm to determine the eigenvalue of a unitary operator $\hat{U}$; it is closely related to the Quantum Fourier Transform (QFT), which is a key element of many quantum algorithms, e.g., Shor's factoring algorithm[12] and in general Abelian Stabilizer type of problems[13]. The algorithm's relevance for quantum simulations was noticed by Abrams and Lloyd[14], and recently emphasized by Aspuru-Guzik *et al.*[15] simulating quantum computation of the lowest energy eigenvalue of several small molecules. It is clear that the PEA will be one of the important algorithms in future quantum information processing applications, and how accurately a phase can be determined will be an important figure of merit for any implementation.

The textbook[16] implementation of this algorithm requires n qubits representing the physical system in which $\hat{U}$ operates, and m ancillary qubits for the work register. The number m determines the algorithm's precision $1/2^m$, i.e the number of accurate binary digits extracted.

There is also an alternative algorithm proposed by Kitaev[13], where the Fourier transform is replaced with a Hadamard transform. In implementing this algorithm to obtain a precision of order $1/2^m$, it is possible to run either $\log m$ rounds (iterations) with m ancillary qubits or $m\log(m)$ rounds with only a single ancilla. The precision increases exponentially with the number of rounds, but each round requires exponentially many applications of $\hat{U}$, unless powers U^{2^k} are available by different means[17].

Also the QFT-based PEA can be implemented in a multiround fashion, using a single ancillary qubit, based on the semiclassical QFT[18]. In this paper, we refer to this single ancilla QFT based PEA as iterative PEA (IPEA). The iterative version of Kitaev's algorithm is referred to as Kitaev's PEA. The relevance of the iterative PEA as a viable alternative to the textbook version was noticed by Mosca & Ekert[19] in the context of the hidden subgroup problem, by Zalka[20] for factoring, and by Childs *et. al.*[21] and Knill *et. al.*[22] in more physical contexts.

As long as the number of qubits is a limiting factor, implementations of phase estimation with only a single ancillary qubit will be of foremost importance. Thus, it is instructive to compare the iterative PEA with Kitaev's PEA. In the IPEA scheme, the bits of the phase are measured directly, without any need for classical postprocessing. Moreover, each bit has to be measured only once, compared to $\log(m)$ times. When the phase ϕ has a binary expansion with no more than m bits, the IPEA deterministically extracts all bits, in contrast to Kitaev's PEA which is always probabilistic. The IPEA is also optimal in the sense that a full bit of information is gained in each measurement[23].

Theoretically the accuracy of the algorithm is limited only by the number of rounds, but in practice it will be limited by experimental imperfections. Thus, the experimentally maximally obtainable accuracy can serve as a benchmark for any multi-qubit implementation.

For benchmarking purposes a setup is needed where the phase to be measured can be set to an arbitrary value. We describe in detail such an implementation in a system of two superconducting qubits. Introducing gate noise, we also perform a robustness analysis, indicating which gates are most critical, and we calculate the number of repetitions needed as a function of noise levels.

Iterative PEA. We now describe the IPEA briefly, but still in some detail, in order to make the robustness analysis clear. The most straightforward approach for phase estimation is shown in Fig. 1. The upper line is the ancil-

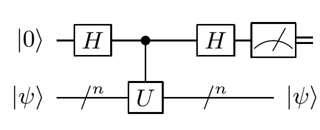

FIG. 1: Naive implementation of the phase estimation algorithm.

lary qubit which is measured, and the lower line describes the qubits representing the physical system in which $\hat{U}$ operates. Initially the ancillary qubit is set to $|0\rangle$ and the lower line register to an eigenstate $|\Psi\rangle$ of the operator $\hat{U}$ with eigenvalue $e^{i2\pi\phi}$. Right before the measurement the system state is $\frac{1}{2}\left[\left(1+e^{i2\pi\phi}\right)|0\rangle+\left(1-e^{i2\pi\phi}\right)|1\rangle\right]|\Psi\rangle$, giving the probability $P_0 = \cos^2(\pi\phi)$ to measure "0". By repeating this procedure N times, P_0 can be determined to an accuracy of $1/\sqrt{N}$. Thus, one needs to go through at least $N \sim 2^{2m}$ independent rounds to obtain m accurate binary digits of ϕ. The number of rounds corresponds to the number of measurements since each round is terminated with a measurement.

Kitaev's PEA allows the number of rounds and consequently the number of measurements to be drastically reduced, with the assumption that the controlled-$\hat{U}^{2^k}$ gates are available[13]. For each k, $1 \le k \le m$, the controlled-$\hat{U}^{2^{k-1}}$ gate is used to prepare an ancillary qubit in the state $\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i2\pi\left(2^{k-1}\phi\right)}|1\rangle\right)$. After a number of repetitions, the ratio of resulting zeros and ones is used as an estimate for the fractional part of $2^{k-1}\phi$. A classical algorithm with polynomial runtime is then used to assemble ϕ from the fractional parts. The whole algorithm performs estimation of ϕ with precision $1/2^{m+2}$ and error probability $\le \varepsilon$ after $O(m\log(m/\varepsilon))$ measurements. The gate $\hat{U}$ is applied $O(2^m\log(m/\varepsilon))$ times to create the powers $\hat{U}^{2^k}$, which is nearly a quadratic improvement compared to the naive version of the PEA.

The iterative PEA differs by the following modification of the above described procedure: first less significant digits are evaluated and then the obtained information improves the quantum part of the search for more significant digits. The information transfer is done with an extra single qubit Z-rotation that is inserted into the circuit, as shown in Fig. 2. Note that k is iterated backwards from m to 1.

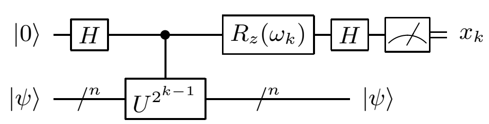

FIG. 2: The kth iteration of the iterative phase estimation algorithm. The feedback angle depends on the previously measured bits through $\omega_k = -2\pi(0.0x_{k+1}x_{k+2}\ldots x_m)$, and $\omega_m = 0$.

In order to derive the success probability for each bit being determined correctly, we first assume the phase ϕ to have a binary expansion with no more than m bits, $\phi = (0.\phi_1\phi_2\ldots\phi_m000\ldots)$. In the first iteration $(k=m)$ a controlled-$\hat{U}^{2^{m-1}}$ gate is applied, and the mth bit of the expansion is measured. The probability to measure "0" is $P_0 = \cos^2[\pi(0.\phi_m00\ldots)]$, which is unity for $\phi_m = 0$ and zero for $\phi_m = 1$. Thus, the first bit ϕ_m is extracted deterministically. In the second iteration $(k = m-1)$ the measurement is performed on the $(m-1)$th bit. The phase of the first qubit before the Z-rotation is $2\pi(0.\phi_{m-1}\phi_m00\ldots)$, and performing a Z-rotation with angle $\omega_{m-1} = -2\pi(0.0\phi_m)$, the measurement probability becomes $P_0 = \cos^2[\pi(0.\phi_{m-1}00\ldots)]$. Thus, using feedback the second bit is also measured deterministically, and generally using the feedback angle $\omega_k = -2\pi(0.0\phi_{k+1}\phi_{k+2}\ldots\phi_m)$ all m bits of ϕ are extracted deterministically.

Denoting the first m bits of the binary expansion of the phase ϕ as $\tilde{\phi} = 0.\phi_1\phi_2\ldots\phi_m$, there is in general a remainder $0 \le \delta < 1$, defined by $\phi = \tilde{\phi} + \delta 2^{-m}$. In this case, the probability to measure ϕ_m is $\cos^2(\pi\delta/2)$. If ϕ_m was measured correctly, the probability to measure ϕ_{m-1} in the second iteration is $\cos^2(\pi\delta/4)$, and so on. Thus, the conditional probability P_k for each bit to be measured correctly is $P_k = \cos^2(\pi\, 2^{k-m-1}\delta)$, and the overall probability for the algorithm to extract $\tilde{\phi}$ is

$$P(\delta) = \prod_{k=1}^{m} P_k = \frac{\sin^2(\pi\,\delta)}{2^{2m}\sin^2(\pi 2^{-m}\delta)}, \qquad (1)$$

which is the same outcome probability as the textbook phase estimation, based on the QFT[16]. For $\delta \le 1/2$ the best m-bit approximation to ϕ is indeed $\tilde{\phi}$, while for $\delta > 1/2$ rounding up to $\tilde{\phi} + 2^{-m}$ is better. The probability to extract $\tilde{\phi} + 2^{-m}$ is $P(1-\delta)$. The success probability $P(\delta)$ decreases monotonically for increasing m. In the limit $m \to \infty$, we find the lower bound for the probability to extract the best rounded approximation to ϕ as $P(1/2) = 4/\pi^2$. Best rounding implies an error smaller than $2^{-(m+1)}$, while an accuracy of 2^{-m} implies that we accept both answers $\tilde{\phi} + 2^{-m}$ and $\tilde{\phi}$. The success probability is then $P(\delta) + P(1-\delta)$, with a lower bound of $8/\pi^2$. In conclusion, iterative PEA determines the phase with accuracy $1/2^m$ and with an error probability $\varepsilon < 1 - 8/\pi^2$, which is *independent* of m.

Success probability amplification for the textbook PEA was discussed in Ref. 16. When the algorithm is executed with $m' = m + \log(2 + 1/2\varepsilon)$ ancillas, the estimate is accurate to m bits, with probability at least $1-\varepsilon$. A similar approach can also be used in iterative PEA, by determining m' bits and keeping only the m most significant bits. However, implementing the $\hat{U}^{2^k}$ gate for large k is the algorithm's bottleneck in a realistically noisy environment, as discussed below.

A different approach, avoiding this problem, is to repeat the algorithm a number of times, choosing the most frequent result. In natural ensemble systems, such as NMR, this is naturally exploited with advantage[24]. However, for single systems such as superconducting qubits, repetitions on all m bits are unnecessarily expensive. From Eq. (1) it is clear that the main contribution to the error probability comes from the least significant bits.

Thus, we may lower the error probability significantly by repeating the measurement of only the first few bits a limited number of times, and using simple majority voting. It is clear from the binomial distribution that the bitwise error probabilities decrease exponentially with the number of repetitions. Also, because of the feedback procedure the bare error probability $\sin^2(\pi\, 2^{k-m-1}\delta)$ already decreases exponentially with decreasing k. Thus, one needs only $O\left[\log^2(1/\varepsilon)\right]$ extra measurements to obtain an error probability smaller than ε, independently of m.

Benchmark circuit. The minimal system for implementing the iterative PEA is a two qubit system, where one qubit is a read-out ancilla, and the second qubit represents a physical system. From the work of Barenco *et al.*[25] we know an explicit construction of any controlled-$\hat{U}$ gate, where $\hat{U}$ is an arbitrary single qubit gate. This construct involves three single qubit gates and two controlled-NOT (CNOT) gates.

For benchmarking purposes we propose to use the very simple Z-rotation operator

$$\hat{U} = \begin{pmatrix} e^{-i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix}, \qquad (2)$$

where α is an arbitrary rotation angle.

The advantages of this operator are 1) it is diagonal in the qubit eigenbasis, thus the initial preparation of its eigenstate is straightforward, 2) the phase to be measured is controlled directly, and 3) controlled powers of this gate are generated by a single entangling gate, $ZZ(\alpha) = \mathrm{diag}\,(e^{-i\alpha}, e^{i\alpha}, e^{i\alpha}, e^{-i\alpha})$; this gate can be straightforwardly implemented by using most common superconducting qubit coupling schemes. As shown in Fig. 3, a step of the iterative PEA is implemented using one ZZ-gate, and in addition only three single qubit gates. The phase we are measuring in this case is set by the coupling strength λ, rather than by the free qubit energy that is the case using the general construction with two CNOT gates, $\alpha 2^{k-1} = \lambda T$, T is the pulse duration.

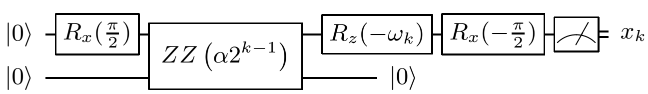

FIG. 3: A gate sequence implementing the k-th step of the iterative phase estimation algorithm, on a two qubit system using the entangling gate $ZZ(\alpha)$.

Let us consider implementation of the ZZ-gate with superconducting qubits in more detail. For superconducting charge and charge-phase qubits operated at the charge degeneracy point, and physically connected via a Josephson junction placed at the intersection of the qubit loop-shaped electrodes, inductive interaction of persistent currents circulating in the loops creates direct switchable zz-coupling[26,27,28]. Thus the implementation of the ZZ-gate is straightforward.

Furthermore, the ZZ-gate is a generic gate for the qubits coupled via tunable linear oscillator: this gate is generated by applying a composite dc-pulse sweeping through the qubit-cavity resonances as shown in[29].

For the permanent transverse coupling (xx-coupling in the qubit eigenbasis) frequently discussed in the context of charge[7], phase[9], and flux[30] qubits, the ZZ-gate can be realized with dynamic control schemes[30,31]. The parametric coupling method[30] suggests harmonic modulation of the coupling strength $\lambda(t)$ with the two resonant frequencies corresponding to the sum and the difference of the qubit energies. This induces the Rabi rotations, U_R^P, and U_R^S, in the parallel spin ($|00\rangle, |11\rangle$), and antiparallel spin ($|01\rangle, |10\rangle$) subspaces, respectively. The ZZ-gate is then obtained, up to a single qubit Z-rotation, by applying the Hadamard gates to both the qubits, $\mathrm{H} = \mathrm{H}_1 \otimes \mathrm{H}_2$, according to the scheme, $ZZ(\alpha) = \mathrm{H}\,[U_R^P(\alpha) \oplus U_R^A(\alpha)]\,\mathrm{H}$.

The method[31] is similar although more time consuming. In this case, the resonant rf-pulses are simultaneously applied to both the qubits, inducing Rabi rotations U_R. The pulse amplitudes are set equal to the half difference between the qubit energies. Such an operation produces the gate which is equivalent to the rotation in the $|00\rangle, |11\rangle$ subspace, $U_R(\alpha) = \mathrm{H}U_R^P(\alpha/4)\mathrm{H}$. Rotation in the $|01\rangle, |10\rangle$ subspace can be performed by first swapping the states of one of the qubits, and then applying the same pulse, $U_R^S(\alpha/4) = X_1 \mathrm{H} U_R(\alpha) \mathrm{H} X_1$. Thus the ZZ-gate is achieved applying the Rabi pulses twice, the full operation sequence taking the form, $ZZ(\alpha) = U_R(4\alpha)\, Z_1\, U_R(4\alpha)\, Z_1$.

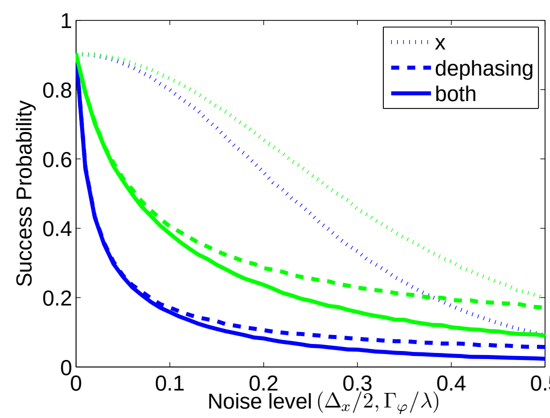

FIG. 4: The success probability of the IPEA to correctly determine the phase α, with precision better than 2^{-5} (green) and 2^{-7} (blue) as a function of the noise level. The three cases of pure X-gate errors (dotted), pure dephasing (dashed), and both types of noise acting simultaneously (full) are considered. The simulation was averaged over α evenly distributed in the range $-\pi \le \alpha < \pi$.

Robustness analysis. There are numerous imaginable sources of error, in all parts of the algorithm from initialization via gate manipulation to readout. With our setup initialization will probably be accurate, but the gates will certainly suffer from imperfections due to environmental noise. First we consider the effect of pure dephasing with rate Γ_φ, which eventually will limit the accuracy of the $ZZ(\alpha 2^k)$ gate. In addition, we consider imperfect x-rotations of the form $R_x(\pm\pi/2 + \delta_x)$, where δ_x is a normally distributed random angle with variance Δ_x. These errors modify the probability of measuring the correct value of the kth bit into

$$P_k' = \frac{1}{2}\left[1 + e^{-\Delta_x^2 - |\alpha| 2^k \Gamma_\varphi/\lambda} \cos\left(\pi 2^{k-m}\delta\right)\right]. \qquad (3)$$

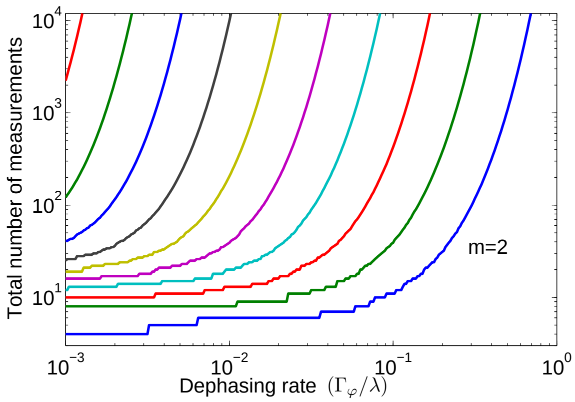

FIG. 5: The total number of measurements needed to obtain the phase α with precision better than 2^{-m} ($2 \leq m \leq 11$), with an error probability $\varepsilon < 0.05$.

In Fig. 4 the algorithm's success probability, as a function of the dephasing rate and variance Δ_x is shown for $m = 5$ and $m = 7$. The algorithm is rather robust against x-rotation errors, while being much more sensitive to dephasing, which is evident from the exponentially growing factor $2^k\alpha$ in the exponent of Eq. (3).

As discussed below Eq. (1), the success probability can be improved by repeated measurements of each bit. To achieve an overall success probability of $1 - \varepsilon$, we need to increase the per bit success probability to $(1 - \varepsilon/m)$, using N_k repeated measurements. For P'_k close to 1/2, many repetitions are needed, and the binomial distribution approaches the normal distribution giving,

$$N_k = \frac{1}{8}\left(\frac{erf^{-1}(1-\frac{2\varepsilon}{m})}{P'_k - \frac{1}{2}}\right)^2, \tag{4}$$

where erf^{-1} is the inverse of the error function. Considering the dominating effect of dephasing, we find that the number of repetitions grows quickly with the desired number of bits m, $N_k \propto e^{2|\alpha|2^m\Gamma_\varphi/\lambda}$. In Fig. 5 we plot the total number of measurements $N_{tot} = \sum_k N_k$, needed to obtain $2 \leq m \leq 11$ bits of the phase α, with an error probability $\varepsilon < 0.05$. For a realistic dephasing rate of 1 to 10 percent of the qubit-qubit coupling ($0.01 < \Gamma_\varphi/\lambda < 0.1$), between 5-8 binary digits of α can be extracted with less than 10^4 measurements.

In conclusion, we have described a benchmark implementation of the iterative PEA on two superconducting qubits, and analyzed its robustness towards dephasing and gate errors. The number of extractable binary digits is mainly limited by the dephasing, and for realistic parameters amounts to 5-8. We believe phase estimation will be an essential part of future applications of quantum computing, and propose that the number of accurate binary digits can be used as a benchmark for multi-qubit implementations.

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