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QUASIDEGENERACY AROUND A SINGLE NONLINEAR RESONANCE

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Abstract

The Shnirilman theorem on the global quasidegeneracy in the quasiclassical limit is discussed in detail using the single nonlinear resonance in the pendulum approximation as a typical example. Various tunneling asymptotics based upon the Mathieu and Hill equations are analysed. Particularly, a new intermediate asymptotics in the perturbation parameter of general Hill's equation has been found and was studied analytically. The main attention is paied to the model of no spatial symmetry with only time–reversal invariance left. A new, tunneling, time scale of quantum chaos is introduced, and its impact on the quantum dynamics is considered.

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Many years ago Shnirelman had announced [1] and recently proved [2] the theorem on asymptotic multiplicity of the quantum spectrum in a classically KAM–integrable system. Here is a few first lines from Ref.[1]:

"Let an arbitrary smooth Riemannien metrics, sufficiently close to the Euclidian one, be given on a 2D torus M, and let Δ be the Laplace operator of the former metrics, $\hat{\Lambda} = \sqrt{-\Delta}$, u_1, u_2... the eigenfunctions of $\hat{\Lambda}$ with eigenvalues $\lambda_1 \leq \lambda_2 \leq ...$

Theorem. $\forall N \, \exists C_N > 0, \quad \forall n > 1 \quad min\,(\lambda_{n+1} - \lambda_n,\, \lambda_n - \lambda_{n-1}) < C_N n^{-N}$

...Thus, the asymptotic multiplicity of the spectrum is a stable phenomenon which does not necessarily require the presence of any symmetry of the manifold. As a matter of fact the symmetry of the geodesic flow is of importance which is always present."

In Ref.[2] this theorem was formulated (and proved) in a more modest way:

AD.2.7. Theorem. If M is a 2D torus, and Condition AD.2.6 is valid, then the spectrum of $\hat{\Lambda}$ is asymptotically multiple, i.e. for each $N > 0$ there exists $C_N > 0$ such that

$$min\,(\lambda_{n+1} - \lambda_n,\, \lambda_n - \lambda_{n-1}) < C_N \lambda^{-N}$$

The principal difference is in omitting the statement $\forall n > 1$ (see above) even though in an informal explanation of the latter theorem Shnirelman still insists [2]: "...the whole sequence of eigenfunctions is asymptotically multiple. (We guess that in fact, in the generic case, this sequence is asymptotically *double*)."

Recently there were many discussions on a possible physical interpretation of this, less known, Shnirelman theorem. The correct interpretation is important for the attempts to extend its implications from a very specific Shnirelman's example onto a more broad class of quantum systems.

Two possible mechanisms were considered:

(i) the effect of the classical KAM structure of everywhere dense set of resonances, and

(ii) the quantum tunneling which transforms the exact degeneracy in the classical limit into a *quasidegeneracy* that is a splitting of the energy levels, relatively small compared to the mean level spacing.

The implications of the first mechanism are still vague but, most likely, that can provide the quasidegeneracy for a relatively small number of levels only [3].

Unlike this, the tunneling quasidegeneracy is well known from the beginning of quantum mechanics (a standard example is the spatially symmetric double–well potential). However, in the Shnirelman theorem a different symmetry is only required, one with respect to the time reversal. The corresponding quasidegeneracy, produced by the tunneling in *momentum* space was also studied recently [4–6] but for the *accidental* degeneracy in the frame of the theory of avoided energy level crossings.

The tunneling mechanism of the global quasidegeneracy, predicted by Shnirelman, was checked and received a preliminary confirmation [7] by computation of the level spacing distribution following a suggestion in Ref.[8]. Simple analytical estimates were also derived which further support such an interpretation [7]. The mechanism was conjectured to work in some completely integrable systems as well.

Here we analyze the latter case in some detail taking the approach similar to that in Ref.[5]. It is based on the reduction of the problem of tunneling through a nonlinear resonance to the Mathieu equation [7]. Two critical remarks on this approach in the first Ref.[4] seem to us irrelevant.

The first remark is: "It is unlikely that the approximations used by Uzer et al involving quantizing the Birkhoff – Gustavson effective Hamiltonian are valid in the limit $\hbar \to 0$." To the contrary, just in the quasiclassical region the classical canonical transformations, which form a basis of this approach, acquire unambiguous quantum counterpart, the corresponding unitary transformations [9,10].

The second remark: "Another difficulty with their approach is that it is the rational tori which are destroyed by perturbations in a generic system; thus their theory describes the quantization of tori which do not exist" is apparently a misunderstanding. Indeed, sufficiently weak perturbation does not destroy the resonance torus but only modifies it, particularly producing an exponentially narrow chaotic layer along the resonance separatrix. The accuracy of this approximation is the same as that for the adiabatic invariance (see, e.g., Refs.[11,12]).

1. Tunneling asymptotics of the Mathieu equation

In Refs.[5,6] the tunneling and quasidegeneracy were considered for *weakly nonlinear* resonances that is for a perturbed *linear* oscillator. The structure of such resonances is generally rather complicated and, moreover, essentially depends on a particular model. Here we consider a simpler case of the *strongly nonlinear* resonance when the unperturbed frequencies are energy–dependent. As is well known (see, e.g., Ref.[11]) such a resonance admits, for sufficiently weak perturbation $\epsilon \to 0$, a universal description by the pendulum resonance Hamiltonian (including many–dimensional oscillations). After appropriate canonical transformations this Hamiltonian can be written in the form:

$$H = \frac{n^2}{2} + \epsilon \cdot \cos(2\theta) \tag{1.1}$$

where $n,\ \theta$ are the action–angle variables, n being the angular momentum of free ($\epsilon = 0$) rotation in θ.

In quantum case $n,\ \theta$ become operators, particularly in quasiclassical region ($n \to \infty$) $\hat{n} = -i\partial/\partial\theta$ ($\hbar = 1$). The Schrödinger equation for eigenfunctions of system (1.1) is the Mathieu equation

$$\frac{d^2\psi}{d\theta^2} + (a - 2q \cdot \cos(2\theta))\,\psi = 0 \tag{1.2}$$

with $a = 2E_n(\epsilon)$ and $q = \epsilon$ where integer n enumerates energy levels identified in the unperturbed limit as $E_n^{(0)} = n^2/2$. The family $E_n(\epsilon)$ of periodic solutions to the Mathieu equation forms a well–known picture of the parametric resonance 'tongues' (see, e.g., Fig.8A in Ref.[13]).

Tunneling in momentum space, or the above–barrier backscattering. The resonance separatrix at energy $E_s = \epsilon$ corresponds to the parameter

$$g = \frac{2q}{a} = \frac{\epsilon}{E} = 1 \tag{1.3}$$

Consider the region $g < 1$ outside the resonance where the classical motion is a nonuniform rotation in θ. In model (1.1) there are two symmetric rotations in opposite senses which are exchanged under the exact *discrete* symmetry with respect to the time reversal. In quantum mechanics each rotation is represented by a wave propagating in one direction. Neither of these two waves can be eigenstate of Hamiltonian (1.1) because of the above–barrier backscattering for any $\epsilon > 0$ that is for any violation of the *continuous* symmetry of the free homogeneous rotation. In the latter case there is exact (double) *degeneracy* of each eigenstate. If $\epsilon > 0$ the eigenstates are formed by symmetric and antisymmetric superpositions of both rotations with different energies $E_+ - E_- = \Delta$. As $n \to \infty$ the energy *splitting* $\Delta \to 0$ which is called *quasidegeneracy.*

Equation (1.2) for the quantum eigenfunctions can be also viewed as the motion equation for a classical linear oscillator with unperturbed ($q = 0$) frequency $\omega_0 = \sqrt{a}$ under the parametric perturbation, phase θ playing the role of time. Quasiclassical region ($n \to \infty$) corresponds here to the adiabatic perturbation with respect to the unperturbed motion $\psi^{(0)}(\theta) \sim \exp(in\theta)$. Nevertheless, parametric resonances occur for arbitrarily large n satisfying

$$n \approx \sqrt{a} \tag{1.4}$$

The instability band has a finite width $\Delta a \approx 4n\mu_0$ proportional to the maximal instability rate μ_0 at the exact resonance [14,15,12]. On both edges of the band $\mu = 0$, and the motion is periodic. These solutions correspond to a splitted quantum eigenstate with

$$\Delta = \frac{\Delta a}{2} = 2\,n\,\mu_0 = 2\,|V_{-r,r}| \tag{1.5}$$

where $V_{-r,r}$ is the matrix element of the adiabatic perturbation between the two states of a nonuniform rotation. If the unperturbed energies of the two states differ by Δ_0 the relation (1.5) takes the form (see, e.g., Ref.[16])

$$\Delta^2 \,=\, \Delta_0^2 \,+\, 4\,|V_{-k,k}|^2 \tag{1.6}$$

and describes the so–called *avoided crossing* of the two energy levels.

The width of the instability band for the Mathieu equation with $g \ll 1$ was calculated in Ref.[14] (see also Refs.[18,19]):

$$\Delta_n \,\approx\, \frac{q^n}{\left(2^{n-1}\,(n\,-\,1)!\,\right)^2} \,\rightarrow\, \frac{2n}{\pi}\left(\frac{e^2\,q}{4\,n^2}\right)^n \approx\, \frac{2n}{\pi}\left(\frac{e^2}{8}\,g\right)^n \tag{1.7}$$

The latter, asymptotic, relation has a reasonable accuracy even for $n = 1$ (!). This result was confirmed in Ref.[15] by a different method using the asymptotic resonance theory.

The same result is obtained employing the standard quasiclassical relation for the tunneling energy splitting (see, e.g., Ref.[16], and Appendix A below):

$$\Delta_n \,\approx\, \frac{2\,\omega(g)}{\pi}\,\exp\left(-\,S_n\right) \tag{1.8}$$

Here

$$\omega(g) \,=\, \frac{\pi}{2}\,\frac{\sqrt{1\,+\,g}}{K(k)}\,\sqrt{a} \,\approx\, n \tag{1.9}$$

is the mean rotation frequency, $K(k)$ is the complete elliptic integral of the first kind with $k^2 = 2g/(1+g)$, and the tunneling action in n

$$S_n \,=\, \int_{-n}^{n} |\theta(n)|\,dn \tag{1.10}$$

is given by the integral over a classically forbidden θ path.

Since $g = \epsilon/E$ in Eq.(1.7) is a classical parameter the dependence of Δ/A on quantum parameter n (and, hence, on $\hbar$) is the simple exponential in agreement with Ref.[4]. However, the prefactor $A = 2n/\pi E \sim \hbar$ (in units of Ref.[4]) is different ($A \sim \hbar^{3/2}$). Apparently, the prefactor is not universal. Also, I wonder if it was really possible to discern the dependence $\sim \hbar^{3/2}$ from that $\sim \hbar$ numerically (see Fig.2 in second Ref.[4]).

Notice that dependence (1.7) holds true for a completely integrable system (1.1) in agreement with a conjecture in Ref.[7] (see also Ref.[6]) but contrary to the conclusion in Ref.[4]. We remind that a *single* resonance is integrable, including a many–dimensional case, in spite of broken continuous symmetry in θ. The perturbation parameter ϵ in this case does not introduce any chaos but only switches from continuous to a discrete symmetry. What is really necessary for quasidegeneracy is a violation of the continuous symmetry.

For a fixed n the dependence of the energy splitting on symmetry–breaking parameter ϵ is a power law with integer exponent (1.7) as was numerically found in Ref.[4]. However, this exponent is not always integer either (see below).

The quasidegeneracy can be distinguished from the level fluctuations if $\Delta \ll \overline{\Delta}$, the mean level spacing. The latter depends on the number of freedoms F, roughly as $\overline{\Delta} \sim n^{2-F}$. Hence, the condition for quasidegeneracy takes the form

$$\frac{\Delta}{\overline{\Delta}} \lesssim n^{F-1}\, g^n \tag{1.11}$$

and is always fulfilled, as $n \to \infty$, for any $g < 1$ that is for any rotational state but not only for $g \ll 1$ [6].

Tunneling in θ–space. Consider now the region $g > 1$ inside the resonance separatrix with bounded oscillations in θ. Here, there is also quasidegeneracy, for $g \gg 1$, which is explained by the tunneling through a potential barrier. Asymptotic relation $E_n(\epsilon)$ in this region is derived by the standard quasiclassical quantization (see, e.g., Ref.[16] and Appendix B). In the simplest approximation[13] it is given by

$$E_n \approx -\epsilon + 2\sqrt{\epsilon}\left(n + \frac{1}{2}\right) \tag{1.12}$$

which is reasonably good in the lower half of the potential well: $-\epsilon < E_n < 0$ (for exact relation and a better approximation in the whole range $|E_n| < \epsilon$ see Appendix B). Integer $n = 0, 1, 2...$ enumerates the energy levels from the bottom of the potential well and corresponds to that of symmetric (with respect to $\theta = 0$) eigenfunctions of $E_n(0)$.

The quasiclassical energy splitting is described by the standard relation (1.8) with oscillation frequency

$$\omega(g) = \frac{\pi\sqrt{\epsilon}}{K(k)} \approx 2\sqrt{\epsilon}\left(1 - \frac{1+f}{8}\right) \tag{1.13}$$

where $f = 1/g < 1;\ k^2 = (1+f)/2$, and with tunneling action ($f > 0$):

$$S_\theta \approx \frac{\pi}{2}\sqrt{\epsilon}(1 - f) \approx \pi\,(n_W - n) \le \pi n_W \tag{1.14}$$

Here $n_W \approx 4\sqrt{\epsilon}/\pi$ (B.3) is the total number of states within each of the two potential wells.

Multiplicity of quasidegeneracy. Energy splitting outside the resonance ($g < 1$) in 1D approximation (1.1) is double independent of m (in Eq.(A.1)) because there are only two classically separated symmetric domains on both sides of the resonance. Inside resonance ($g > 1$) the situation is more complicated. For the standard Mathieu equation (1.2) the splitting here is also double because the second–harmonic perturbation produces two classically separated and symmetric potential wells. In case of the first–harmonic resonance ($V(\theta) = \cos\theta$) the tunneling into a single barrier results in a slight shift of eigenvalues without any splitting. In the classical picture (1.1) the parametric resonance occurs here for any *half–integer* n. However, in quantum mechanics n must be *integer* if the physical perturbation is 2π–periodic. Hence, half of periodic solutions disappear together with the energy splitting for large q while the splitting for small q persists.

In case of the perturbation with arbitrary harmonic ($V(\theta) = \cos(m\theta)$, see Appendices A and B) the splitting is generally multiple because there are additional periodic solutions inside each stable region. The simplest example is the Mathieu equation itself which has both $2\pi-$ and $\pi-$periodic solutions, hence the multiplicity $M_{sp} = 2$. Generally M_{sp} is equal to the number of linearly independent solutions with a period T_θ such that both $mT_\theta/2\pi$ and $2\pi/T_\theta$ are integer. This is equivalent to a decomposition of harmonic number m into the product of two integers: $m = m_1 \cdot m_2$. If, for example, $m = p > 1$ is a prime number there are only two such decompositions: $m = 1 \cdot p = p \cdot 1$. This implies two solutions of periods $T_\theta = 2\pi/p$ and 2π, respectively, hence $M_{sp}(p) = 2$ (doublet). Since the total number of solutions is p, $p-1$ of them (with period 2π) are exactly degenerated in spite of perturbation. These are the solutions shifted by $2\pi/p$ in θ. Altogether, there are p such solutions but only $p-1$ of them are independent as the sum of the former is zero. Particular case of $m = 3$ was considered in detail in Refs.[17]. Another simple example is $m = p^k$ for which $M_{sp} = k+1$. Apparently, the upper limit $M_{sp}(m) = m$ is reached for $m = 1$ and 2 only.

In a many–dimensional system of F freedoms perturbed by a single resonance there are additional $F-1$ exact motion integrals I_r $(r = 2, 3, ..., F)$ beside the resonance one, the energy or the corresponding pendulum action I_1 [11]. Transitions $I_r \to -I_r$ increase the multiplicity of quasidegeneracy both outside and inside the resonance provided dependence of the perturbation on other phases θ_r. This is only possible in the presence of additional resonances when the motion is no longer completely integrable. The maximal multiplicity in this case is $M_{sp} = 2F$. This simple consideration gives some support on a rich quasidegeneracy structure in many freedoms [7]. In other words, the simple backscattering turns into a multidimensional scattering in the action space, the quasidegeneracy corresponding to some rational scattering angles on the lattice of quantum numbers.

Multidimensional tunneling was recently considered in Ref.[25].

2. The Hill equation: distorted resonance

The Mathieu equation posseses both temporal, or time–reversal, $(n \to -n)$ as well as spatial $(\theta \to \theta + 2\pi/m;\ \theta \to -\theta)$ discrete symmetries. In light of the Shnirelman theorem, discussed in the Introduction above, it is interesting to consider a model with only time–reversal symmetry left. To this end we need a more general periodic perturbation in resonance Hamiltonian (1.1):

$$V(\theta) \;=\; \sum_{m \neq 0}^{m_f} V_m \, \cos{(m\theta \,+\, \phi_m)} \tag{2.1}$$

represented by a finite or infinite $(m_f = \infty)$ Fourier series. The corresponding Schrödinger Eq.(1.2) is known as Hill's equation.

For a particular harmonic m the rotation energy splitting is given by (see Appendix A):

$$\Delta_n \;\approx\; \frac{m\,n}{\pi} \cdot (CgV_m)^{2n/m} \tag{2.2}$$

with factor $C \approx 1$.

Critical perturbation harmonic. Asymptotically ($n \to \infty$), the main contribution to Δ_n comes from a certain critical harmonic $m = m_c$ which, depending on the rate of V_m decay and other parameters, may be the lowest one ($m_c = 1$), the highest ($m_c = m_f$) or intermediate ($1 < m_c < m_f$, see Appendix C). In case of infinite Fourier series (2.1) the value of m in Eq.(2.2) is bounded from above by the first instability zone (the main parametric resonance): $m \le 2n$.

To the best of my knowledge the intermediate regime has not been considered as yet. On the contrary, there are various assertions in the literature that such regime does not exist at all. For example, in the second Ref.[18] there is a brief remark (without reference): "For a general Hill's equation (with arbitrary coefficient $V(\theta)$ [in our notations, see Eq.(2.1)]) the situation is completely different; the width of any zone decreases for typical $V(\theta)$ as the *first* power of ϵ". Apparently, the (implicit) formulation of the latter problem is different, namely, to study the *asymptotic* behavior of a resonance zone as $\epsilon \to 0$ for a *fixed* n. Then, m_c increases and eventually reaches the upper limit $m_c = 2n$ below which (in ϵ) Eq.(2.2) gives

$$\Delta_n \approx \frac{2}{\pi} C\, g\, n^2 V_{2n} \tag{2.3}$$

so that the splitting is indeed simply proportional to the perturbation strength $g = \epsilon/E_n$. Here we consider also the *intermediate* asymptotics: $1 \gg \epsilon \gg \epsilon_c$ where ϵ_c is determined by the condition: $m_c(\epsilon_c) = 2n$ (see Eq.(C.7)). This is only possible for a sufficiently fast decay of perturbation Fourier harmonics. Apparently, the critical decay is approximately the simple exponential: $V_m \sim \exp(-\sigma m)$.

The same is true for the finite Fourier series as well. In this case the critical ϵ_c is found from the equation $m_c(\epsilon_c) = m_f$. Again, there is an apparent contradiction with the rigorous results in Ref.[18] that $m_c = m_f$ always that is for arbitrary $V(\theta)$ but *only*, as far as I understand, in the limit $\epsilon \to 0$.

3. Statistics of quasidegeneracies

A global characteristic of quantum degeneracy is the statistics of energy splittings Δ recently studied in Ref.[7] in an attempt to clarify the physical meaning of the Shnirelman theorem. The model used was somewhat different from Eq.(1.1), namely, the kicked rotator on a torus specified by the Hamiltonian:

$$H = \frac{n^2}{2} + k \cdot V(\theta) \cdot \delta_T(t) \tag{3.1}$$

where δ_T is the δ–function of period T, and the following perturbation

$$V(\theta) = \cos\theta - \frac{1}{2}\sin(2\theta) \tag{3.2}$$

was chosen to completely destroy the spatial symmetry. For a sufficiently small classical perturbation parameter $K = kT \approx 0.2 \ll 1$ we may expect the global behavior, particularly, degeneracy to be close to that for integrable system (1.1) with perturbation (2.1) and $m_f = 2$ (see also below). There is, of course, a chaotic component of the motion but it is relatively small.

Circumference of the torus N (in n) is equal to the total number of quantum states. There are $r = NT/2\pi = 2$ identical resonances at $n = 0$ and $n = N/2$. Each of them is characterized by the potential well (3.2) with $|V(\theta)| \le 3\sqrt{3}/4 \equiv V_0$.

The Δ–statistics is described by integral probability

$$P(s) = \frac{1}{2} - \frac{2n}{N} \tag{3.3}$$

where $s = \Delta/\overline{\Delta} = 2\Delta$ is the quasienergy splitting normalized to the mean level spacing $\overline{\Delta} = 2\pi/TN = 1/r = 1/2$. The minimal $s \approx 0$ is reached at $n = N/4$ (because of the two resonances present), and the total number of splitted states cannot exceed $N/2$.

According to the above theory the splitting is determined, for sufficiently small s, by the second harmonic of perturbation (3.2) with $\epsilon = k/T$ and the classical parameter

$$g = \frac{k/T}{n^2/2} = \frac{K}{8\pi^2}\left(\frac{N}{n}\right)^2 \tag{3.4}$$

Using Eq.(2.2) we obtain

$$\ln s = \ln\left(\frac{4}{\pi}\right) + \ln n + 2n\,(L - \ln n);\;\; L = \ln\left(\frac{\sqrt{CK}\,N}{4\,\pi}\right) \tag{3.5}$$

The dependence of $I = NP$ on $\ln s$ was found in a certain range of s to be close to linear one (Fig.1). Indeed, Eq.(3.5) implies ($n \gg 1$):

$$\frac{d\ln s}{dI} \equiv \frac{1}{l} = 1 - \frac{1}{2n} - L + \ln n \approx 1 + l_{sp} > 1;\;\; l_{sp} = -\frac{1}{2}\ln\left(\frac{Cg}{2}\right) \tag{3.6}$$

where l is the empirical slope of function $I(\ln s)$, and l_{sp} is the slope for a *fixed* classical parameter n/N, or g (see Eqs.(2.2) and (3.4)). The former is always less than 1 contrary to empirical result $l \approx 1.8$ [7]. The difference is clearly seen in Fig.1.

To understand the origin of discrepancy the derivative $dI/dlns$ is shown in Fig.2 together with theoretical dependence (3.6). In case of a single harmonic (Mathieu's equation) the factor $C \approx 1$ with renormalized $\tilde{g} = gV_m = g/2$ in Eq.(3.6) (see Eq.(2.2) and Appendix A). For Hill's equation a plausible approximation would be renormalization to the full amplitude of the perturbation:

$$\tilde{g} = g\,V_0 \tag{3.7}$$

Then, $C \approx 2V_0 = 3\sqrt{3}/2 \approx 2.6$ in Eq.(3.6). The best fit, shown in Fig.1, gives a reasonably close value of $C = 3.1$ (fitting parameter $C_0 = C^{-1/2} = 0.57$). Beside a poor theoretical approximation for C used, apparently s is not small enough owing to numerical errors in computing eigenvalues for $s \lesssim 10^{-9}$. I think this is the main cause of the above discrepancy.

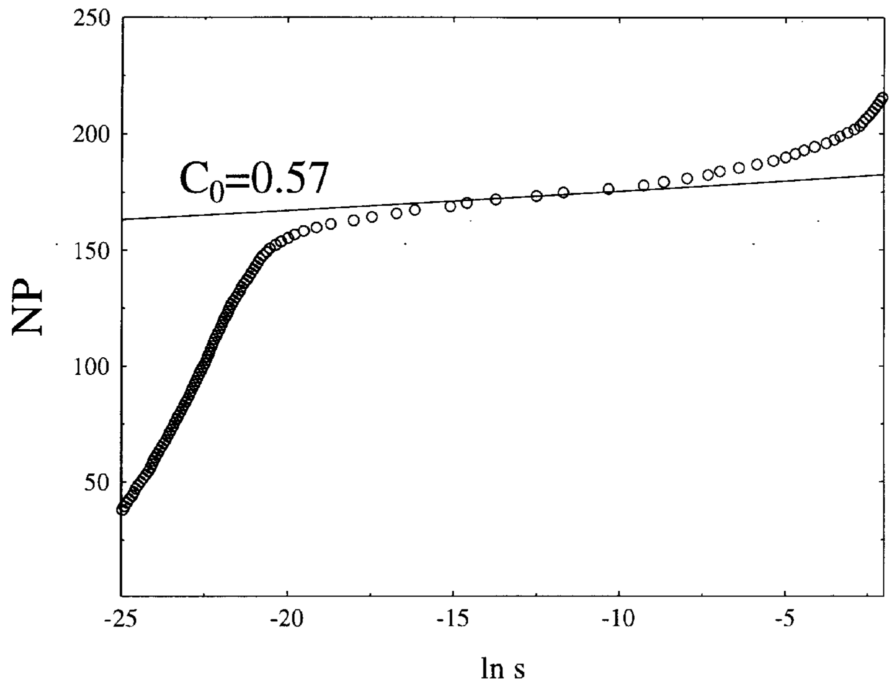

Figure 1: Statistics of quasidegeneracies in model (3.1): $k = 6 - 10$; $K \approx 0.15 - 0.25$; $N = 501$ (after Ref.[7]). Solid line is the best fit of numerical data[7] to theoretical dependence (3.5) with $C = 1/C_0^2$.

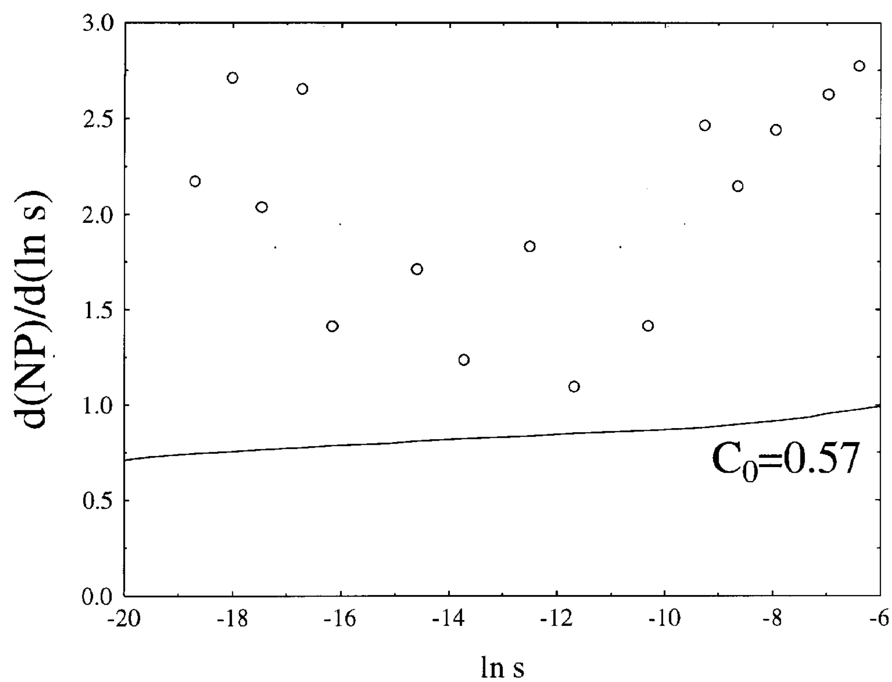

Figure 2: Derivative of the distribution function in Fig.1. Solid line shows dependence (3.6) with the same fitting parameter $C_0 = 0.57$.

Instead of the integrable approximation we may use Eq.(1.5) with matrix element for the direct one–kick transition $-n \to n$ of approximate kick operator [7]

$$\hat{V} = \exp\left(i\,\frac{k}{2} \cdot \sin(2\theta) \right)$$

Then,

$$|V_{-n,n}| = \frac{1}{\sqrt{2\pi n}} \cdot \left(\frac{e\pi}{2} \cdot \frac{g\,n}{N} \right)^n \approx \frac{\Delta}{2} \qquad (3.8)$$

which differs from Eq.(2.2) used above by a small classical factor n/N, thus giving negligible Δ for $g \ll 1$. Nevertheless, this approach provides a correct estimate for l_{sp} [7] but not for Δ. The main reason for underestimating Δ is apparently in that the uniform–rotation eigenfunctions are used in evaluating Eq.(3.8) instead the nonuniform ones (cf. Eq.(1.5)).

The region inside the resonance does not contribute to quasidegeneracy because of the broken spatial symmetry [7]. Apparently, the second (later) formulation of the Shnirelman theorem (see Introduction above) is more accurate or, perhaps, more general.

4. Two symmetric resonances

Hamiltonian (1.1) describes a nonlinear resonance to some approximation only. Next terms, e.g., $\sim n^3$ would generally destroy the symmetry of states $\pm n$ leaving behind only accidental degeneracy with some avoided level crossings. In case of exact time–reversal symmetry the only exclusion corresponds to the resonance exactly at $n_0 = 0$. We remind that in model (1.1) variable n is generally the difference $(n - n_0)$.

However, under time–reversal symmetry to each resonance shifted from zero ($n_0 \neq 0$) there is symmetric resonance at $-n_0$. This restores the global tunneling and quasidegeneracy.

Consider Hamiltonian

$$H = \frac{n^2}{2} + 2\,\epsilon \cdot \cos(m\theta) \cdot \cos(\Omega t)$$

$$= \frac{n^2}{2} + \epsilon \cdot \cos(m\theta - \Omega t) + \epsilon \cdot \cos(m\theta + \Omega t) \qquad (4.1)$$

which describes two resonances at $n_0 = \pm\Omega/m$. This system is no longer completely integrable but for a large adiabaticity parameter $\lambda = n_0/\sqrt{2\epsilon} = \sqrt{f_0}$ chaotic component is of exponentially small measure in λ [11] (the so–called KAM integrability). For the problem in question it is unimportant (cf. Section 3).

There are three regions different with respect to the tunneling in momentum space:

(i) outside resonances where the tunneling goes through both of them and the region inbetween,

(ii) between resonances with tunneling away from both, and

(iii) inside resonances where the energy splitting is caused by the tunneling in *momentum* space between two resonances but not inside them which is negligible for $m = 1$ (see Section 1).

In evaluating the energy splitting under condition $\lambda \gg 1$ the combined action of both resonances can be neglected. Then, by a change of variables the problem is reduced to a single resonance with

$$E = \frac{(n - n_0)^2}{2}, \qquad g = \frac{2\,\epsilon}{(n - n_0)^2} \tag{4.2}$$

In the simplest case $g \ll 1$ we can use expression (2.2) to obtain, assuming $C = 1$, the following rough estimates (see Appendix A). In regions (ii) and (iii), defined above, the tunneling action is given by Eq.(A.5), and we have:

$$\Delta_n \sim \frac{m}{\pi}\,|n - n_0|\,g_0^{\frac{2n_0}{m}}, \qquad 0 < n \lesssim n_0 \tag{4.3}$$

where $g_0(n) = g(0)$. In region (i) the tunneling is generally more complicated. If $n \gtrsim 2n_0$ the tunneling through each resonance is incomplete at one side: $0 < n < n_0$. Then, using Eq.(A.6) we obtain:

$$\Delta_n \sim \frac{m}{\pi}\,|n - n_0|\,g^{\frac{2n}{m}}, \qquad n \gtrsim n_0 \tag{4.4}$$

In the interval $n_0 \lesssim n \lesssim 2n_0$ there is a competition of two tunnelings, one between the resonances, Eq.(4.3), and another through both of them. If the latter is decisive that is providing less Δ_n then

$$\Delta_n \sim \frac{m}{\pi}\,|n - n_0|\,g^{\frac{4(n-n_0)}{m}} \tag{4.5}$$

otherwise estimate (4.3) holds. The transition between both tunnelings is roughly at $n \approx 3n_0/2$. In all regions the energy splitting $\Delta_n \to 0$ as quantum parameter $n \to \infty$.

For a time dependent Hamiltonian like (4.1) the mean quasienergy level spacing

$$\overline{\Delta} = \frac{\Omega}{N} \tag{4.6}$$

where N is the total number of states (cf. Section 3). As $N \to \infty$ mean spacing $\overline{\Delta} \to 0$. However, in a conservative system with *compact* energy surfaces $\overline{\Delta}$ is always finite and, for two freedoms, is independent of quantum parameter n.

5. Conclusion: a new time scale of quantum chaos

How simple and specific the model of a single (1.1) or even double (4.1) resonance may seem it actually represent, at least qualitatively, a rather general picture of the bulk quantum quasidegeneracy. Indeed, the principal condition for the latter is the existence of a *discrete and only discrete* symmetry between some well separated domains in phase space [7]. The present results provide additional confirmation to the physical interpretation of the Shnirelman theorem in Ref.[7].

The models discussed above do not include chaotic motion where the global quasidegeneracy may also occur [20,8,7] (see also Ref.[23] where a similar dual problem in symmetric random potential is considered). The main condition for the chaotic quasidegeneracy is a strong *quantum* localization which separates symmetric domains. For example, in model (3.1) this condition takes the form

$$l_s \ll N \tag{5.1}$$

where $l_s \approx D$ is the localization length (in the number of states) of the quantum steady state [21], D stands for the classical diffusion rate, and N is the total number of states. The quasienergy splitting is given by the estimate [22] (see also Ref.[8]):

$$\Delta \approx A \cdot \exp\left(-\frac{2\,n}{l_{sp}}\right) \tag{5.2}$$

where n is the distance of a localized state from the center of symmetry ($n = 0$, see Section 3), $l_{sp} \approx l_s$, and A is some constant.

In a sense, the tunneling counteracts quantum localization. It is characterized by the *tunneling time scale*

$$t_t \sim \frac{1}{\Delta} \sim \exp\left(\frac{2n}{l_{sp}}\right) \tag{5.3}$$

This is the third principal time scale in addition to random (t_r) and relaxation (or localization) (t_R) time scales which, for model (3.1), are given by the estimates [21]:

$$t_r \sim \ln k; \qquad t_R \sim \frac{1}{\overline{\Delta}} \sim l_s \sim k^2 \tag{5.4}$$

The quasidegeneracy can be observed only if (cf. Section 1)

$$\frac{t_t}{t_R} \sim \frac{\overline{\Delta}}{\Delta} \gtrsim 1 \tag{5.5}$$

Unlike two scales (5.4) which indefinitely *grow* with quantum parameter k the scale t_t *decreases* (5.3) until the quasidegeneracy gets lost (5.5) within level fluctuations.

The first, to my knowledge, direct observation of the tunneling time scale was reported in Ref.[20] (see Fig.2 there): a narrow wave packet was shown numerically to oscillate.[1] Due to dispersion of tunneling frequences (5.2) in the quantum steady state which we assume in the form

$$f_s(n) = |\psi_s(n)|^2 = \frac{4/\pi\, l_s}{e^{2m/l_s} + e^{-2m/l_s}} \tag{5.6}$$

where $m = n - n_0$, the oscillation decays, roughly as (see Appendix D)

$$\frac{< n >}{n_0} \sim \cos\left(\tau \cos\left(\frac{\pi\, q}{2}\right)\right) \cdot \exp\left(-\,\tau \sin\left(\frac{\pi\, q}{2}\right)\right) \tag{5.7}$$

with parameter $q = l_s/l_{sp}$, and $\tau = \Delta_0 t,\ \Delta_0 = \Delta(n_0)$ (5.2). From a single example of tunneling relaxation in Ref.[20] it is difficult to judge whether the variation of $< n >$ shows

[1] The tunneling through chaos between regular domains was studied much before in many papers (see [26] and references therein).

some residual oscillation or just a fluctuation. In the latter case $l_s = l_{sp}$ as expected [8], and the relaxation (5.7) would be a pure exponential. In any case the tunneling relaxation leads to a new, "double–hump", steady state with two symmetric "humps" at $n = \pm n_0$. The fluctuation of $< n >$ in this steady state can be roughly estimated in the same way as that for energy [24] (see also Ref.[8]): $\Delta < n > /n_0 \sim k^{-0.6} \approx 0.4$, which does not contradict with numerical data in Ref.[20].

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Appendix A: tunneling in momentum space

Consider a slightly different form of Hamiltonian (1.1), namely

$$H(I,\,\theta) \;=\; \frac{I^2}{2} \,+\, \epsilon\cdot\cos{(m\theta)} \tag{A.1}$$

Then, in classically forbidden domain of θ including resonance ($I^2 < 2\epsilon(f-1) = I_1^2$)

$$\cos{(m\theta)} \;=\; f \,-\, \frac{I^2}{2\epsilon} \;=\; \frac{e^{m|\theta|} \,+\, e^{-m|\theta|}}{2} \;\equiv\; F(I) \geq 1$$

whence

$$|\theta| \;=\; \frac{1}{m}\,\ln{(F \,+\, \sqrt{F^2 \,-\, 1})} \;\approx\; \frac{\ln{(2F)}}{m} \tag{A.2}$$

if $f \gg 1$. The action integral

$$S \;=\; 2\int_0^{I_1} |\theta(I)|\,dI \;\approx\; \frac{2}{m}\int_0^{I_1} \ln{(2F(I))}\,dI \;\approx\; -\frac{2I}{m}\,\ln{(Cg)} \tag{A.3}$$

where $F(I_1) = 1$ and $g = 1/f = 2\epsilon/I^2$. Asymptotic value of factor $C = e^2/8 = 0.92$ ($g \ll 1$) while $C \to 1$ as $g \to 1$. Using Eq.(1.8) with prefactor $m\omega(g)/\pi$ we arrive at Eq.(1.7) for $m = 2$ (standard Mathieu equation), and $I \approx n$. Multiplier m accounts for the coherent backscattering from m barriers of the potential in Eq.(A.1).

Notice also that approximations (1.4) and (1.9) are fairly good for small g:

$$a \;\approx\; n^2\left(1 \,+\, \frac{g^2}{8}\right), \qquad \omega(g) \;\approx\; n\left(1 \,-\, \frac{g^2}{16}\right) \tag{A.4}$$

There is also a classically forbidden domain in θ outside resonance ($I^2 > 2\epsilon(f+1) = I_2^2$) where Eq.(A.2) holds as well with $F = I^2/2\epsilon - f > 1$. Here the action integral

$$S \;=\; 2\int_{I_2}^{I_0} |\theta(I)|\,dI \;\approx\; \frac{2}{m}\int_{I_2}^{I_0} \ln{(2F(I))}\,dI \;\approx\; -\frac{2\,I_0}{m}\,\ln\left(\frac{e^2}{2}\,g_0\right) \tag{A.5}$$

if $I_0 \gg I = \sqrt{2\epsilon/g}$, and $g_0 = 2\epsilon/I_0^2 \ll 1$. In this approximation Eq.(A.5) remains unchanged upon substitution of any $I > 0$ instead of I_2 that is by starting integration inside the resonance. Relations (A.5) and (A.3) are similar, in both S is determined by the larger value of I on the integration path. However, asymptotic value of $C = e^2/2$ in (A.5) is larger than in (A.3).

Finally, consider the incomplete tunneling through half of a resonance (cf. Eq.(A.3)):

$$S = \int_0^{I_0} |\theta(I)|\, dI \approx -\frac{I_0}{m} \ln\left(\frac{g_0}{2}\right) \tag{A.6}$$

where $I_0 < I = \sqrt{2\epsilon/g}$. Again, the result is similar to Eqs.(A.3) and (A.5) with a different factor $C = 1/2$. For rough estimates to logarithmic accuracy we can use in all cases $C = 1$.

Appendix B: θ–tunneling within the resonance

The quasiclassical asymptotics of the spectrum for Hamiltonian (A.1) at $f = 1/g = E_n/\epsilon < 1$ is given by the action integral (see, e.g., Ref.[16]):

$$S_a = \oint I(\theta)\, d\theta = \frac{4\sqrt{2\epsilon}}{m} \int_0^{x_0} \sqrt{f + \cos x}\, dx =$$

$$\frac{16\sqrt{\epsilon}}{m} \left[E(k) - \frac{1-f}{2} K(k)\right] = 2\pi\,(n + 12) \approx 7.23\, \frac{\sqrt{\epsilon}}{m} \cdot \frac{1+f}{\left(\frac{\pi}{2} - f\right)^{0.15}}, \tag{B.1}$$

where $E(k)$ is the complete elliptic integral of the second kind, and $k^2 = (1+f)/2$. The last simple expression in (B.1) provides a fairly good approximation to S_a in the whole range $|f| \leq 1$. At the bottom of the potential well Eq.(B.1) gives the spectrum of a linear oscillator with frequency $m\sqrt{\epsilon}$:

$$E_n = -\epsilon + m\sqrt{\epsilon}\left(n + \frac{1}{2}\right) \tag{B.2}$$

which is also well known asymptotics of the Mathieu equation [13]. Near the separatrix we obtain the total number of states within each of m wells:

$$n_W \approx \frac{8\sqrt{\epsilon}}{\pi\, m} \tag{B.3}$$

which is very close to the total number of rotating states up to separatrix energy $E_s = \epsilon \approx \frac{n_R^2}{2}(1 + g^2/8) = 9n_R^2/16$ (see Eq.(A.4)). In this approximation we have ($m = 2$) $n_W/n_R \approx 3/\pi = 0.95$.

The tunneling action in θ is given by the integral (cf. Eq.(B.1)):

$$S_\theta(f) = \int |I(\theta)|\, d\theta = \frac{2\sqrt{2\epsilon}}{m} \int_0^{x_0} \sqrt{\cos x - f}\, dx = \frac{S_a(-f)}{2} \tag{B.4}$$

In the upper half of the potential barrier $f > 0$ (and in the lower half of the well) we can neglect a slight variation of denominator in approximate relation (B.1) to obtain still simpler expression

$$S_\theta \approx \pi\, \frac{\sqrt{\epsilon}}{m}(1 - f) \approx \pi\,(n_W - n). \tag{B.5}$$

It corresponds to the harmonic oscillator approximation for S_a.

Appendix C: intermediate critical harmonic in the Hill equation

Rewrite the contribution of m-th harmonic (2.2) in the form

$$\ln\Delta(m) \,=\, \ln\left(\frac{n}{\pi}\right) \,+\, \ln m \,-\, 2n\cdot G(m); \qquad G(m) \,=\, \frac{\ln f \,+\, v(m)}{m} \tag{C.1}$$

where $f = n^2/2\epsilon = 1/g > 1$, and $v(m) = -\ln V_m$. Asymptotically, as $n \to \infty$, the main contribution comes from the harmonic $m = m_c$ which minimizes $G(m)$, whence

$$\ln f \,+\, v(m_c) \,=\, m\, v'(m_c) \,\equiv\, \lambda_c\, v(m_c) \tag{C.2}$$

Then,

$$\Delta_n \,\approx\, \Delta(m_c) \,\sim\, \frac{n\, m_c}{\pi}\cdot g^{\frac{2n}{m_c}\cdot\frac{\lambda_c}{\lambda_c - 1}} \tag{C.3}$$

provided $m_c > 1$ and $m_c < min(m_f,\, 2n)$ (see Section 2).

For example, if $V_m \sim \exp(-\sigma m)$ Eq.(C.2) has no solution, and the critical harmonic $m_c = min(m_f,\, 2n)$ is as large as possible. Another example is a faster exponential decay of V_m with

$$v(m) \,=\, \sigma\, m^{\lambda} \tag{C.4}$$

Then, $\lambda_c = \lambda > 1$, and

$$m_c \,=\, \left(\frac{\ln f}{\sigma\,(\lambda \,-\, 1)}\right)^{1/\lambda} \tag{C.5}$$

The above inequality $m_c > 1$ implies

$$g \,<\, e^{-\sigma(\lambda-1)} \tag{C.6}$$

otherwise $m_c = 1$, and Eq.(2.2) should be used instead of Eq.(C.3). Another condition $m_c < m_l$ where $m_l = min(m_f,\, 2n)$ leads to

$$g \,>\, e^{-\sigma(\lambda-1)\, m_l^{\lambda}} \tag{C.7}$$

otherwise $m_c = m_l$ in Eq.(2.2).

A more interesting example is the perturbation

$$V(\theta) \,=\, \sin(\beta\,\sin\theta) \,=\, 2\sum_{m=1}^{\infty} J_m(\beta)\,\sin(m\theta) \tag{C.8}$$

where $J_m(\beta)$ are the Bessel functions, and all m are odd. For sufficiently large m

$$v(m) \,=\, (m \,+\, 1/2)\,\ln m \,-\, m\,\ln(\beta e/2) \,+\, \frac{1}{2}\,\ln(\pi/2)$$

and

$$m_c \,\approx\, \ln\tilde{f} \,+\, \frac{\ln\ln\tilde{f}}{2} \,\approx\, \ln\tilde{f}; \qquad \lambda_c \,\approx\, 1 \,+\, \frac{1}{\ln\left(\frac{2m_c}{\beta e}\right)} \tag{C.9}$$

where $\tilde{f} = f \cdot \sqrt{\frac{\pi}{2e}}$. For $\beta \ll 1$ the distortion of resonance (1.1) with $V(\theta) = \cos(\theta)$ by perturbation (C.8) is very small, yet its effect on the energy splitting may be quite big:

$$\Delta(m_c) \sim [\Delta(1)]^P ; \qquad P \approx \frac{\ln\left(\frac{2m_c}{\beta}\right)}{m} \ll 1 \tag{C.10}$$

if $g \ll \beta$, $(m_c \gg 1)$.

Appendix D: tunneling relaxation

For each degeneracy doublet the state initially localized in one of two symmetric domains, e.g., at $n = n_0$ (for model (3.1)) will oscillate so that

$$\nu(t) \equiv \frac{< n >}{n_0} = \cos(\Delta \cdot t) \tag{D.1}$$

where $< n >$ denotes the quantum averaging in an instantaneous state, and

$$\Delta = \Delta_0 \cdot \exp\left(-\frac{2m}{l_{sp}}\right) \tag{D.2}$$

is the energy splitting with $m = n - n_0$ and $\Delta_0 = \Delta(n_0)$ in Eq.(5.2). The relaxation $\nu \to 0$ as $t \to \infty$ is determined by the spectrum $w(\Delta)$ which, in turn, depends on the quantum steady state assumed in the form

$$f_s(m) = |\psi_s(m)|^2 = \frac{4/\pi \, l_s}{e^{2m/l_s} + e^{-2m/l_s}} \tag{D.3}$$

Combining Eqs.(D.2) and (D.3) we obtain for the spectrum

$$w(\omega) = f_s(m)\left|\frac{dm}{d\omega}\right| = \frac{2p}{\pi} \cdot \frac{\omega^{p-1}}{1 + \omega^{2p}} \tag{D.4}$$

where $\omega = \Delta/\Delta_0$ is dimensionless freguency, and $p = l_{sp}/l_s$ stands for the ratio of splitting and localization scales. Assuming decoherence of chaotic eigenstates, the relaxation is given by the integral

$$\nu(\tau) = \int_0^\infty \left(1 + \frac{m}{n_0}\right) \cdot \cos(\omega\tau)\, w(\omega)\, d\omega \approx \int_0^\infty \cos(\omega\tau)\, w(\omega)\, d\omega \tag{D.5}$$

neglecting a small term with $m/n_0 \sim l_s/n_0$ in the latter expression, $\tau = \Delta_0 t$. In a particular case $p = 1$ $(l_{sp} = l_s)$ the relaxation is a pure exponential

$$\nu = \exp(-\tau) \tag{D.6}$$

Otherwise the oscillation arises due to the singularity at $\omega = i^q$. Asymptotically as $\tau \to \infty$

$$\nu(\tau) \sim \cos\left[\tau \cdot \cos\left(\frac{\pi q}{2}\right)\right] \cdot \exp\left[-\tau \sin\left(\frac{\pi q}{2}\right)\right] \tag{D.7}$$

where $q = 1/p$.

References

[1] A.I.Shnirelman, Usp.Mat.Nauk **30**, #4, 265 (1975).

[2] A.I.Shnirelman, *On the Asymptotic Properties of Eigenfunctions in the Regions of Chaotic Motion,* addendum in: V.F.Lazutkin, *KAM Theory and Semiclassical Approximations to Eigenfunctions*, Springer, 1993.

[3] J.Bellissard, private communication, 1994.

[4] M.Wilkinson, Physica D **21**, 341 (1986); J.Phys. A **20**, 635 (1987); M.Wilkinson and J.Hannay, Physica D **27**, 201 (1987).

[5] T.Uzer, D.Noid and R.Marcus, J.Chem.Phys. **79**, 4412 (1983).

[6] A.Ozorio de Almeida, J. Phys. Chem. **88**, 6139 (1984).

[7] B.V.Chirikov and D.L.Shepelyansky, Phys. Rev. Lett. **74**, 518 (1995).

[8] G.Casati and B.V.Chirikov, Physica D **86**, 220 (1995).

[9] P.Dirac, The Principles of Quantum Mechanics, Clarendon Press, Oxford, 1958.

[10] W.Miller, Adv. Chem. Phys. **25**, 69 (1974); G.P.Berman and A.R.Kolovsky, Physica D **8**, 117 (1983).

[11] B.V.Chirikov, Phys. Reports **52**, 263 (1979).

[12] B.V.Chirikov, Particle Dynamics in Magnetic Traps, in: *Reviews of Plasma Physics*, Vol. 13, Ed. B.B.Kadomtsev, Consultants Bureau, 1987, p.1.

[13] N.McLachlan, Theory and Application of Mathieu Functions, Oxford, 1947.

[14] V.M.Frolov, Diff. Uravneniya **18**, 1363 (1982).

[15] B.V.Chirikov, Asymptotic Methods in Adiabatic Problems, preprint Budker INP 86-22, Novosibirsk, 1986.

[16] L.D.Landau and E.M.Lifshits, Quantum Mechanics, Pergamon,1958.

[17] V.I.Arnold, Funct. Anal. Appl. **6**, 94 (1972); M.Berry, J.Phys. A **10**, L193 (1977).

[18] V.I.Arnold, Izv.AN SSSR, mat. **25**, 21 (1963); Usp.Mat.Nauk **38**, #4, 189 (1983).

[19] D.Levy and J.Heller, Comm. Pure Appl. Math. **16**, 469 (1963); H.Hhochstadt, ibid. **17**, 251 (1964).

[20] G.Casati, R.Graham, I.Guarneri and F.M.Izrailev, Phys. Lett. A **190**, 159 (1994).

[21] B.V.Chirikov, F.M.Izrailev and D.L.Shepelyansky, Physica D **33**, 77 (1988).

[22] F.M.Izrailev, private communication, 1994.

[23] A.G.Moiseev and M.V.Entin, Semiconductors **28**, 727 (1994).

[24] G.Fusina, thesis, Milano University, 1992.

[25] S.Creagh, J.Phys. A **27**, 4969 (1994).

[26] O.Bohigas, S.Tomsovic and D.Ullmo, Phys. Reports **223**, 43 (1993).

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