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RESONANCE PROCESSES IN MAGNETIC TRAPS*

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Abstract—Consideration is given to resonances between the Larmor rotation of charged particles and their slow oscillations along the lines of force. Under certain conditions these resonances can result in a complete exchange of energy among the degrees of freedom of the particle, so that the particle escapes from the trap. The influence of resonances on adiabatic processes associated with a time variation of the magnetic field is also examined.

1. INTRODUCTION

ONE of the methods for thermally insulating a plasma in order to realize a controlled thermonuclear reaction is the use of so-called adiabatic traps, or traps with magnetic mirrors, proposed and calculated by BUDKER.(1) Similar systems have been proposed by YORK(2) and calculated by JUDD, MCDONALD and ROSENBLUTH.(3) Recently, considerable developments in this direction have occurred and therefore it is of interest to study further similar systems.

The action of an adiabatic trap is based(1) on the conservation of orbital magnetic moment of a charged particle in a magnetic field ($\mu = Mv_\perp{}^2/(2H)$ where $v_\perp$ is the component of the particle velocity in a direction perpendicular to the magnetic field H). It is a necessary, but of course not a sufficient condition for the usefulness of a trap that it can entrap a single charged particle. Generally speaking the lifetime of such a particle in the trap is not infinite because the magnetic moment is only an adiabatic invariant, i.e. it can change slowly and so allow a redistribution of energy among the longitudinal and transverse degrees of freedom of the particle and consequent escape from the trap.

The question of the time variation of an adiabatic invariant has been considered in a number of papers.(6–8) However, only KULSRUD(7) takes his calculations as far as concrete results for a harmonic oscillator, obtaining

$$\frac{\Delta I}{I} = \frac{2\,\Delta^{(q)}}{(2\omega_0)^{q+1}}\cdot\cos\left(2\theta_0 - \frac{\pi q}{2}\right) \qquad (1.1)$$

Here I is the adiabatic invariant, $\Delta^{(q)}$ is the discontinuity in the qth derivative of $\omega(t)$, θ_0 and ω_0 are the phase and the frequency of the oscillator at the time of the discontinuity in the derivative. The basically unsatisfactory feature of the above expression is its asymptotic nature. This means that it is correct only if $1/(\omega T) \to 0$ (T being the characteristic time for the variation $\omega(t)$. For finite values of the adiabaticity parameter $1/(\omega T)$ equation (1.1) is not always correct. (The conditions for its applicability are given in the Appendix.) In the particular case where $\omega(t)$ is an analytic function, equation (1.1) gives $\Delta I/I = 0$. This means that when $1/(\omega T) \to 0$ the quantity $\Delta I/I$ tends to zero faster than any power of the parameter $1/(\omega T)$ (for instance as $\exp(-\omega T)$), but it remains unknown how exactly it behaves. For this reason the normally used methods of asymptotic expansion in powers of a small parameter such as $(1/\omega T)$ are not applicable in this case.

In the present paper we consider a different approach to this problem. It is based on the simple physical model of resonances between the Larmor rotation of the charged particle and slow oscillations of the particle along the magnetic lines of force.† Such resonances are possible in spite of the differences in frequency if the slow oscillations of the particle are anharmonic and contain high harmonics of their basic frequency. The action of the resonances leads in particular to a change in the magnetic moment of an individual particle (ignoring collisions).

2. BASIC EQUATIONS

The present paper does not aim to produce formulae for computation. The main attention is directed to the physical processes taking place when a charged particle moves in a magnetic trap. We therefore confine ourselves to the study of the simple Hamiltonian used by FIRSOV(6) ($M = 1$)

$$\mathscr{H} = \frac{p_x{}^2 + p_y{}^2 + \omega^2(x)y^2}{2}\,;\quad p_x = \dot{x};\quad p_y = \dot{y}. \qquad (2.1)$$

Here x and y are the co-ordinates along and across the magnetic line of force respectively and ω is the Larmor frequency. The equations of motion have the form

$$\ddot{y} = -\omega^2 y;\quad \ddot{x} = -\omega\frac{d\omega}{dx}\,y^2. \qquad (2.2)$$

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† The importance of resonances for the change of adiabatic invariants has been pointed out by ANDRONOV, LEONTOVICH and MANDEL'SHTAM.(9)

Since the oscillations along the x-axis are slow ($\Omega \ll \omega$) the solution for y can be stated in the form

$$y = \rho \cos \theta; \quad \theta = \int \omega \, dt + \varphi. \tag{2.3}$$

Here ρ is the Larmor radius of the particle and

$$\ddot{x} = -\omega \frac{d\omega}{dx} \frac{\rho^2}{2} (1 + \cos 2\theta).$$

Since $\omega \rho^2/2 = I$, where I is the adiabatic invariant in which we are interested and which is connected to the magnetic moment by the relation $I = c\mu/e$, we obtain

$$\ddot{x} + I \frac{d\omega}{dx} = -I \frac{d\omega}{dx} \cos 2\theta. \tag{2.4}$$

Thus the motion along the x-axis is oscillatory with a potential energy $I\omega(x) = \mu H(x)$ and superimposed on it a fast periodic perturbation of frequency 2ω. This perturbation is usually neglected because $\omega \gg \Omega$. However, if the oscillations along the x-axis contain high harmonics of the basic frequency, a resonance is possible between the high frequency perturbation $I(d\omega/dx) \cos 2\theta$ and one of these harmonics.

The effect described can also be approached from a different direction. Let us consider the equation $\ddot{y} + \omega^2(t) y = 0$ in which the dependence of ω on time is related to the oscillations along the x-axis. The period of the function $\omega(t)$ is much greater than $1/\omega$, but if $\omega(t)$ contains high frequencies right up to ω, one of these may produce a parametric resonance.

Since $\mathscr{H}$, the total energy of the particle, is conserved, both the resonances mentioned lead to a redistribution of the particle energy among its degrees of freedom. To investigate these resonances, we shall use a method described elsewhere by the author.[10] First we introduce the Hamiltonian $\mathscr{H}_y$ which describes the motion along the y-axis

$$\mathscr{H}_y = \frac{p_y^2 + \omega^2(t) y^2}{2}. \tag{2.5}$$

Then $d\mathscr{H}_y/dt = \partial \mathscr{H}_y/\partial t = \omega \dot{\omega} y^2$. For the variation of the adiabatic invariant $I = \mathscr{H}_y/\omega$ we find

$$\frac{dI}{dt} = \dot{\mathscr{H}}_y/\omega - \mathscr{H}_y \dot{\omega}/\omega^2 = \dot{\omega}(y^2 - \overline{y^2}), \tag{2.6}$$

where the bar denotes an average over the phase which changes with the frequency ω. Considering ω as a parameter, we obtain a correction to the frequency as $\dot{\varphi}$ where φ is given by (2.3)[10]

$$\dot{\varphi} = -\frac{\omega \dot{\omega}}{\dot{y}} \left[\omega y^2 \left(\frac{\partial y}{\partial I} \right)_{\omega, \theta} + \left(\frac{\partial y}{\partial \omega} \right)_{I, \theta} \right] = -\frac{\dot{\omega}}{2\omega} \sin 2\theta. \tag{2.7}$$

We also introduce the Hamiltonian $\mathscr{H}_x$ which describes the motion along the x-axis as given by (2.4)

$$\mathscr{H}_x = \frac{p_x^2}{2} + I\omega(1 + \cos 2\theta). \tag{2.8}$$

Putting

$$x = x(I, \vartheta); \quad \vartheta = \int \Omega(I) \, dt + \psi \tag{2.9}$$

and taking into account that the quantity $W_x = p_x^2/2 + I\omega$, is equal to the total Hamiltonian $\mathscr{H}$ and is conserved, we get

$$\dot{W}_x = \frac{\partial W_x}{\partial t} + [W_x, \mathscr{H}_x] = \dot{I}\omega - \dot{x} \frac{d\omega}{dx} I \cos 2\theta = 0,$$

Here [,] denotes a Poisson bracket. Hence

$$\dot{I} = I \frac{\dot{\omega}}{\omega} \cos 2\theta, \tag{2.10}$$

in agreement with (2.6). In analogy with (2.7) we obtain for the $\dot{\psi}$ of (2.9)[10]

$$\dot{\psi} = \frac{\Omega}{\dot{x}} \left(\frac{\partial x}{\partial I} \right)_\vartheta \dot{I} = I \left(\frac{\partial x}{\partial I} \right)_\vartheta \frac{\Omega}{\omega} \frac{d\omega}{dx} \cos 2\theta. \tag{2.11}$$

We note that the equations (2.7), (2.10) and (2.11) are exact for the particular Hamiltonian (2.1) we have chosen.

3. FIRST ORDER RESONANCES

We integrate (2.10) by expanding the right-hand side into a Fourier series. The function $\cos 2\theta$ expresses the frequency modulated oscillation

$$\begin{gathered} \dot{\theta} = \omega + \dot{\varphi} = \bar{\omega} + \dot{\varphi} + \sum_n \omega_n \cos 2n\vartheta, \\ \theta = \varphi + \bar{\omega} t + \sum_n \frac{\omega_n}{2n\Omega} \sin 2n\vartheta. \end{gathered} \tag{3.1}$$

Here, and in the following, the bar denotes an average over phases which change with the frequency Ω. Performing some simple transformations we find

$$\cos 2\theta = \tfrac{1}{2} \sum_n (F_{1n} \pm F_{2n}) \cos 2(\bar{\omega} t + \varphi \pm n\vartheta). \tag{3.2}$$

Here we sum over the two sign combinations, with the upper and the lower signs taken together. The fact that we expand $\omega(t)$ only in cosines is related to the symmetry of the process with respect to the two points of reversal ($\dot{x} = 0$). The factor 2 in (3.1) characterizes the symmetry of $\omega(x)$ relative to the median plane of the magnetic field. The Fourier coefficients F_{1n} and F_{2n} are determined by the equations

$$\left. \begin{aligned} \cos \left(\sum_n \frac{\omega_n}{n\Omega} \sin 2n\vartheta \right) &= \sum_n F_{1n} \cos 2n\vartheta, \\ \sin \left(\sum_n \frac{\omega_n}{n\Omega} \sin 2n\vartheta \right) &= \sum_n F_{2n} \sin 2n\vartheta. \end{aligned} \right\} \tag{3.3}$$

Now let

$$\frac{\dot{\omega}}{\omega} = \sum_m \Omega_{1m} \sin 2m\vartheta. \quad (3.4)$$

Multiplying (3.2) and (3.4) we obtain the equation

$$\frac{dI}{dt} = I \sum_{\bar{\omega} \approx l\Omega} P_l \cos \psi_l. \quad (3.5)$$

The condition $\bar{\omega} \approx l\Omega$ shows that of the whole sum one should keep only the one term whose frequency is close to zero. This is just the resonance term which gives the greatest contribution to the variation of I.

$$\psi_l = 2(\bar{\omega}t + \varphi_1 - l\vartheta); \; \varphi_1 = \varphi - \pi/4, \quad (3.6)$$

$$4P_l = -\sum_{m-n=l} \Omega_{1m}(F_{1n} + F_{2n}) + \sum_{n-m=l} \Omega_{1m}(F_{1n} + F_{2n}) - \sum_{m+n=l} \Omega_{1m}(F_{1n} - F_{2n}). \quad (3.7)$$

Equation (3.5) must be supplemented by an equation for the $\dot{\psi}_l$ of (2.7) and (2.11)

$$\dot{\psi}_l = 2\left(\bar{\omega} - l\Omega - \frac{\dot{\omega}}{2\omega} \sin 2\theta - I \frac{\partial x}{\partial I} \frac{d\omega}{dx} \frac{\Omega}{\omega} \cos 2\theta\right).$$

Taking

$$2I \frac{\partial x}{\partial I} \frac{d\omega}{dx} \frac{\Omega}{\omega} = \sum_m \Omega_{2m} \cos 2m\vartheta, \quad (3.8)$$

we obtain

$$\frac{d\psi_l}{dt} = 2(\bar{\omega} - l\Omega) - \sum_{\bar{\omega} \approx p\Omega} Q_p \sin \psi_p, \quad (3.9)$$

where Q_p is determined by the expression

$$-4Q_p = \sum_{m-n=p} (F_{1n} + F_{2n})(\Omega_{2m} + \Omega_{1m}) + \sum_{n-m=p} (F_{1n} - F_{2n})(\Omega_{2m} - \Omega_{1m}) + \sum_{m+n=p} (F_{1n} - F_{2n})(\Omega_{2m} + \Omega_{1m}). \quad (3.10)$$

4. HIGHER-ORDER RESONANCES

The equations (3.5) and (3.9), which we will discuss further below, are approximate not only because we have discarded non-resonance terms in them (these only lead to small oscillations of I) but mainly because not all resonances have been taken into account. In deriving (3.5) and (3.9) we started from (3.1) and (3.2) assuming φ and ψ to be constant, while in fact these quantities contain small periodic components as shown by (2.7) and (2.11). It can be shown that this leads to additional resonances determined by the condition $k\bar{\omega} = l\Omega$ ($k \geqslant 2$). (We shall call k the order of the resonance.) There are also other effects which lead to resonances of higher orders, which we shall not take into account exactly. However, an examination of particular cases has shown that higher order resonances are described by equations similar to (3.5) and (3.9) with the replacements $\bar{\omega} \to k\bar{\omega}$; $P_l \to P_l^{(k)} \simeq P_l\alpha_k$ and $Q_l \to Q_l^{(k)} \simeq Q_l\alpha_k$ where α_k is taken to be equal to the greater of the quantities $(\rho/R)^{2(k-1)}$ and $(Q_l/\Omega)^{k-1}$ (R is the radius of curvature of the magnetic line of force). In the first case, in which ρ/R is involved, both integral and half-integral values of k are possible, while in the second case, involving Q_l/ω, only integral values are allowed. Of course, the rule just formulated is only valid for rough estimates and the whole question requires further study.

5. THE STATIONARY CASE

We shall call conditions stationary if in (2.1) ω is not explicitly dependent on time ($\partial\omega/\partial t = 0$), while in the non-stationary case we have $\omega = \omega(x,t)$.* The stationary case corresponds to a magnetic field which is constant in time and axially symmetric. Non-stationary conditions arise both as a result of azimuthal inhomogeneities† (owing to particle drift) and also when the magnetic field depends explicitly on time.

It is well known (see for instance Bogolyubov and Mitropol'skii[11]) that in the present case equations (3.5) and (3.9) determine regions of instability at $\bar{\omega} = l\Omega$ whose widths are $\Delta_r(\bar{\omega} - l\Omega) \sim Q_l$. As will be shown below there is no need to investigate these regions in detail. It is important merely to note that they do not overlap, i.e. that their width Q_l is less than the distance 2Ω between them. This follows directly from the estimate (1.2)‡ for Q_l which gives $Q/\Omega \lessdot \sqrt{(\Omega/\bar{\omega})} \ll 1$.

Higher-order resonances do not change this last inequality, since the total resonance width is

$$\sim \sum_k kQ_l^{(k)} \sim Q_l \sum_k k\alpha_k \sim Q_l; \; (\alpha_k \ll 1).$$

(See Section 4).

However in that case the regions of instability play no part at all because of the non-linearity of the oscillations, or more precisely because of the dependence of the frequencies $\bar{\omega}$ and Ω on I. Even for particles which fall into the unstable regions, I will not change monotonically, but will perform oscillations round its resonance value.[10] In the following

* The time dependence $\omega(t)$ which occurred above was not explicit in x.

† Strictly speaking the Hamiltonian (2.1) is only correct in the axially-symmetric field. However, if change in the magnetic field due to particle drift is small in a time $1/\Omega$ one can retain the form (2.1) with an explicitly time-dependent ω.

‡ (1.1)–(1.3) are the formulae of the Appendix.

we shall call such oscillations phase oscillations in analogy with the phase oscillations of charged particles in accelerators. They can be investigated with the aid of the equation which is obtained from (3.5) and (3.9) by elimination of I:

$$\frac{d^2\psi_l}{dt^2} = 2\frac{\partial}{\partial t}(\bar{\omega} - l\Omega) + 2IP_l\frac{\partial}{\partial I}(\bar{\omega} - l\Omega)\cos\psi_l. \quad (5.1)$$

Here and in the following, we neglect terms in Q_l which is permissible under the condition $Q_l^2 \ll |IP_l\,\partial(\bar{\omega} - l\Omega)/\partial I|$ which is always satisfied.[10] The amplitude of the frequency change in phase oscillations is

$$\Delta_{ph}(\bar{\omega} - l\Omega) \simeq 2\sqrt{\left|IP_l\frac{\partial}{\partial I}(\bar{\omega} - l\Omega)\right|}, \quad (5.2)$$

while the frequency of the phase oscillations is given by

$$\Omega_{ph}^2 \simeq \left|2IP_l\frac{\partial}{\partial I}(\bar{\omega} - l\Omega)\right|. \quad (5.3)$$

The case requiring further special discussion is when the amplitude (5.2) becomes comparable or exceeds the distance between the resonances 2Ω (see Section 7).

6. THE NON-STATIONARY CASE (FAST PASSAGE THROUGH RESONANCES)

If the frequencies $\bar{\omega}$ and Ω are explicitly time-dependent and if the difference $\bar{\omega} - l\Omega$ varies, a resonance is crossed and thus a change of I occurs. We first consider a fast passage through resonance, in which the rate of change of $\bar{\omega}$ and Ω due to the change in I can be neglected in comparison with the rate of change due to the explicit dependence on time.

$$A = \left|\frac{IP_l\dfrac{\partial}{\partial I}(\bar{\omega} - l\Omega)}{\dfrac{\partial}{\partial t}(\bar{\omega} - l\Omega)}\right| \ll 1. \quad (6.1)$$

In this case the equations (3.5) and (3.9) can be integrated immediately and in first approximation (in $\Delta I/I$) we have

$$\Delta I = \sqrt{\pi}I\sum_{\omega\approx l\Omega}\frac{P_l\cos(\psi_{0l} + \pi/4)}{\sqrt{\left|\dfrac{\partial}{\partial t}(\bar{\omega} - l\Omega)\right|}}. \quad (6.2)$$

Here ψ_{0l} is the value of the phase ψ_l at the instants of resonance, $(\dot{\psi}_e = 0)$. If the distribution of the phases ψ_{0l} in the different passages through resonance is random we have $\overline{\Delta I} = 0$, and

$$\overline{(\Delta I)^2} = \frac{\pi}{2}I^2\sum_l\frac{P_l^2}{\left|\dfrac{\partial}{\partial t}(\bar{\omega} - l\Omega)\right|}. \quad (6.3)$$

Here the summation extends over all resonances crossed and the bar denotes an average over the phases ψ_{0l}. In the next approximation we have

$$\overline{\Delta I} = \frac{1}{2}\frac{\partial}{\partial I}\overline{(\Delta I)^2} + \frac{\pi}{4}\sum_l\frac{IP_l}{\left|\dfrac{\partial}{\partial t}(\bar{\omega} - l\Omega)\right|}\left[Q_l - \frac{\partial}{\partial I}(P_l I)\right]. \quad (6.4)$$

The first moment (6.4) characterizes the systematic change of I while the second moment (6.3) characterizes its spread. Knowledge of the two moments is sufficient to establish Fokker-Planck type equations which can be solved to find the flux of particles into the forbidden cone.[1] However, as is shown by the expressions for the moments, this equation proves very complicated. We therefore consider a simpler method for estimating the change in I.

It follows from (6.4) that $\overline{\Delta I}/I \sim \overline{(\Delta I)^2}/I^2$. If $\Delta I/I \ll 1$ the influence of the first moment may be completely neglected ($[\overline{(\Delta I)^2}/I^2]^{1/2} \gg \overline{\Delta I}/I$). If $\Delta I \gtrsim I$ the influence of the two moments is of equal order of magnitude. For estimates it is therefore sufficient to investigate only the change of $\overline{(\Delta I)^2}$. The condition for escape from the trap then has the form $\overline{(\Delta I)^2} \simeq (I_0 - I_k)^2$, where I_0 is the initial value of I and I_k its value on the surface of the forbidden cone.

Since the number of resonances crossed in unit time is $(k/\Omega)\,|\partial(\bar{\omega} - l\Omega)/\partial t|$ we get, in view of (6.3)

$$\frac{d}{dt}\overline{(\Delta I)^2} = \frac{\pi}{2}\frac{kI^2(P^{(k)})^2}{\Omega}. \quad (6.5)$$

This ordinary differential equation is easy to solve either numerically or by approximate methods. Of course, for (6.5) to be valid it is necessary that the change of I in crossing a single resonance should be small compared to I itself.

The coefficient k in (6.5) denotes the order of the resonance crossed and is determined by the range of the variation of $\bar{\omega} - l\Omega$ due to the explicit time dependence.

$$k = \Omega/\Delta_t(\bar{\omega} - l\Omega). \quad (6.6)$$

In particular, if the magnetic field has some azimuthal inhomogeneity we may estimate k according to the formula $k \approx (\Omega/\bar{\omega})(\Delta H/H)$, which is obtained from $\Delta_t(\bar{\omega} - l\Omega)/\bar{\omega} \simeq \Delta H/H$.*

* It may appear that the azimuthal inhomogeneity does not lead to a change of the frequencies $\bar{\omega}$ and Ω because the particle drift is perpendicular to ∇H. However, if one takes into account the motion of the particles along the lines of force, it is not difficult to see that H will change along the particle orbit, in fact in such a way that one has to take for $\Delta H/H$ the maximum value of the inhomogeneity along the line of force.

As an example we give a comparison of the estimates obtained with the asymptotic formula (1.1).[7] We find the expression (2.3) for P and from (6.5) that $\Delta I/I \simeq \Delta^{(q)}(t\Omega)^{1/2}\pi^{-1/2}(2\omega)^{-(q+1)}$ or, in a time $\Delta t = \pi/(2\Omega)$, corresponding to the monotonic change of $\omega(t)$ in one direction that was assumed by KULSRUD,[7] $(\Delta I/I)_1 \simeq \Delta^{(q)}2^{-1/2}(2\omega)^{-(q+1)}$ which agrees with (1.1) in order of magnitude.

7. CRITERIA OF STOCHASTICITY

The equations (6.3)—(6.5) are based on the assumption of stochasticity, i.e. on the randomness of the phases ψ_{0l}. We now attempt to elucidate under what conditions this assumption may be justified.

For simplicity we discuss a periodic crossing of a single first-order resonance at equally spaced instants $T/2$. According to (6.2) we have $\Delta I \sim \sum_n \cos \psi_n$, where the ψ_n are the phases at the instants of resonance. If the frequencies $\bar{\omega}$ and Ω do not depend on I, we have

$$\psi_{n+1} - \psi_n = \psi_0 = 2.\int_{t_n}^{t_n+T/2} (\bar{\omega} - l\Omega)\, dt = \text{constant}$$

and therefore

$$\Delta I \sim \sum_n \cos n\psi_0 \leqslant 1/\sin \tfrac{1}{2}\psi_0. \tag{7.1}$$

In this case ΔI is bounded and there is no stochasticity. We now take into account the non-linear nature of the oscillations. After each passage through resonance the frequency $\bar{\omega} - l\Omega$ will be changed by the quantity $(\Delta\omega)_n = (\Delta I)_n \times \partial(\bar{\omega} - l\Omega)/\partial I$ and in the next passage through resonance this will lead to an additional change of phase by the amount $\psi_{1n} = T(\Delta\omega)_n$. If $\psi \ll 1$ we come back to the previous case. If however, $\psi_1 \gg 1$ then ψ_1 also determines the phase change ψ_n. Unlike (7.1) this phase change will not be uniform but will depend on the previous phase ($\psi_{1n} \sim (\Delta\omega)_n \sim (\Delta I_n) \sim \cos \psi_n$) and a slight change of the previous phase will give rise to a change of 2π in the following one. It is evident that in this case the sequence of phases ψ_n will be near to random. However, a rigorous proof of this statement does not at present exist and we accept it as a hypothesis. The criterion of stochasticity then has the form

$$\nu_1 = \frac{2}{\sqrt{\pi}} \left| \frac{IP_l T \dfrac{\partial}{\partial I}(\bar{\omega} - l\Omega)}{\sqrt{\left|\dfrac{\partial}{\partial t}(\bar{\omega} - l\Omega)\right|}} \right| \gg 1. \tag{7.2}$$

Similar criteria have been obtained by GOWARD[12] and HINE[13] by means of numerical calculations on the motion of a non-linear oscillator under the action of short periodic impulses. The authors derive ν_1 as the criterion for instability. Its connexion with stochasticity is not examined. According to these papers instability begins at $\nu_1 = \pi$.

We shall give this criterion a somewhat different physical interpretation. If $\omega(x, t)$ is periodic in t then each harmonic in (3.5) and (3.9) will be modulated with a certain frequency $\Omega_d = 2\pi/T$ and can be expanded into new harmonic components of frequencies $2(\bar{\bar{\omega}} - l\bar{\Omega}) \pm p\Omega_d$ ($\bar{\bar{\omega}}$ denotes an average over phases varying with the frequencies Ω and Ω_d, and $\bar{\Omega}$ is averaged over phases varying with the frequency Ω_d). The coefficients of this expansion are $P_{lp} \simeq P_l(\Omega_d/(2\Delta_t(\bar{\omega} - l\Omega))^{1/2}$ (see Appendix, paragraph 1).

Inserting the last expression into (7.2) and putting

$$\left|\frac{\partial}{\partial t}(\bar{\omega} - l\Omega)\right| \simeq \frac{\Delta_t(\bar{\omega} - l\Omega)}{2}\,\Omega_d,$$

we obtain

$$\nu_2 = 4\sqrt{\pi}\,\frac{\Omega_{ph}{}^2}{\Omega_d{}^2} \gg 1, \tag{7.3}$$

where $\Omega_{ph}{}^2 = \left| 2IP_{lp}\dfrac{\partial}{\partial I}\left(\bar{\bar{\omega}} - l\bar{\Omega} \pm \dfrac{p\Omega_d}{2}\right)\right|$ is the square of the frequency of the phase oscillations for the new system of resonances (5.3) and Ω_d is the distance between them.

We apply the criterion (7.3) to the stationary case (see Section 5). Using (5.3) and taking into account that the distance between resonances is 2Ω we obtain

$$\kappa = \frac{\left|2\sqrt{\pi}\, IP_l \dfrac{\partial}{\partial I}(\bar{\omega} - l\Omega)\right|}{\Omega^2} \gg 1. \tag{7.4}$$

If (7.4) is satisfied, the variation of I is stochastic.

Apparently this effect was observed by GARREN *et al.*[14] who used numerical methods to investigate the motion of a charged particle in an adiabatic trap. They discovered that near the forbidden cone there exists a region of 'unstable' orbits which escape from the trap stochastically. Let us apply criterion (7.4) to the data of this paper. We determine P_l from (3.5) in the same way as this was done in the example of Section 6.

$$(\Delta I/I)_{1/2} \simeq \pi P_l/\Omega,$$

where $(\Delta I/I)_{1/2}$ is the relative change of I in a half period (π/Ω) of the slow oscillation. This quantity was also evaluated by GARREN *et al.*[14] Putting $|\partial(\bar{\omega} - l\Omega)/\partial I| \simeq \bar{\omega}/I$, we find from (7.4) that instability begins at $(\Delta I/I)_{1/2} \geqslant 0{\cdot}1$, while from the data of GARREN *et al.* it follows that $(\Delta I/I)_{1/2} \gtrsim 0{\cdot}02$. The discrepancy should not be considered as very great, remembering that the estimates are very rough.

8. NON-STATIONARY CASE (SLOW PASSAGE THROUGH RESONANCE)

Evidently the condition for the slow passage through a resonance is $A \gg 1$ where A is defined in (6.1). As was shown by the author elsewhere[10] there are two basic regimes for the slow passage through a resonance: trapping and single passage. An important characteristic of the second regime is that ΔI is independent of the phase of the oscillations and, related to this, that the process is reversible even when the stochasticity condition (7.2) is satisfied. Therefore, if one and the same resonance is passed through periodically and slowly, in both directions, I will experience only small oscillations.

The trapping regime is characterized by the fact that when the frequencies $\bar{\omega}$ and Ω change I changes automatically owing to the explicit time dependence, in such a way that the condition of resonance $\bar{\omega} \approx l\Omega$ always remains satisfied. It is evident that if the frequencies $\bar{\omega}$ and Ω vary periodically the process will also be completely reversible.

Trapping may however exert an essential influence on the adiabatic processes when there is a considerable change of magnetic field with time, because the additional condition $\bar{\omega}/\Omega =$ constant must be satisfied. The main difficulty in utilizing this influence is connected with the fact that the region of trapping is small. Its width across each resonance is approximately equal to[1] $2\Delta_{ph}(\bar{\omega} - l\Omega) \sim \Omega_{ph}$ where Δ_{ph} is given by (5.2). If I has a uniform distribution among the particles and if therefore the frequencies are also uniformly distributed, the proportion of particles captured is Ω_{ph}/Ω. But this is just the ratio which must be small according to the stochastic criterion (7.4).

9. COMPARISON WITH EXPERIMENT

As far as we know, the only work in which quantitative data on the variation of the adiabatic invariant have been obtained is that of S. N. RODIONOV.[15] His comparison of experimental data with the theoretical results obtained above was performed for three magnetic field configurations (see Table 1). To find the Fourier coefficients Ω_{1m} the magnetic field on the axis of the system $(H(x) \sim \omega(x))$ was approximated by a function of the form $1/(x + b)^2$ where x is the distance from the turning point of the path $(\dot{x} = 0)$ and b was so chosen as to obtain the best agreement. The accuracy of the approximation was 20–50 per cent. The following numerical values were assumed: mean distance of the particles from the axis of symmetry in the median plane of the magnetic field: $r_1 = 2$ cm; radius of curvature of the magnetic line of force $R = 2/n$ cm; index of the fall-off of magnetic field $n = (\gamma - 1)/640$; here $\gamma = H_m/H_0$, where H_m is the maximum field on the axis of the system (the magnetic mirror) and H_0 the minimum field on the axis.*

The variation of I was found from equation (6.5). I_0 was so chosen that $v_\perp \sim v_\parallel$. Initially (for $I \sim I_0$) the change of I was at its slowest both because of the higher order of the resonances passed through (large Ω, see (6.6)) and because of the smallness of P_l (smoother function $\dot{\omega}/\omega$ smaller Ω_{1m}). It was assumed that it was sufficient to determine the number N of reflections on the magnetic mirrors which led to a change $(\Delta I_{1/2})$ of I such that the order of the resonances

* For a more accurate description of the experiment see RODIONOV.[15]

TABLE 1.—COMPARISON OF THEORETICAL AND EXPERIMENTAL DATA

No.	γ	ρ/R	$\Delta H/H(\%)$	k	A	ν	κ	N(calc.)	N(exp.)
1	13	6×10^{-3}	<4	2·5	—	$< 2 \times 10^{-6}$	2×10^{-8}	∞	$>6 \times 10^{6}$
			3	3·5	—	$<2 \times 10^{-6}$	2×10^{-8}	∞	$>6 \times 10^{6}$
2	40	5×10^{-2}	<4	4	—	3×10^{-3}	0·4	∞	10^{5}
			9	2	10^{-2}	2	0·4	8×10^{3}	2×10^{4}
			15	1·5	2×10^{-2}	2	0·4	6×10^{3}	2×10^{4}
3	5	3×10^{-3}	<0·5	13·5	—	$<8 \times 10^{-6}$	4×10^{-7}	∞	$>3 \times 10^{6}$
			4	2	—	$<8 \times 10^{-6}$	4×10^{-7}	∞	$>3 \times 10^{6}$

passed through decreased by a factor $\frac{1}{2}$,* because after this the rate of change of I increases. It was assumed that $\Delta I_{1/2}/I \simeq \frac{1}{2}k \ll 1$ where k is given by (6.6). Using the relation $dt = dN(\pi/\Omega)$ we obtain from (6.5)

$$N \approx \frac{\Omega^2}{2\pi^2 k^3 (P^{(k)})^2}. \tag{9.1}$$

The coefficient P was estimated according to the formula (1.1), the criteria A, ν and κ according to (6.1), (7.2) and (7.4) and it was assumed that

$$\left|\frac{\partial}{\partial I}(\bar{\omega} - l\Omega)\right| \simeq \bar{\omega}/I;$$

$$\left|\frac{\partial}{\partial t}(\bar{\omega} - l\Omega)\right| \simeq \frac{\Delta_t(\bar{\omega} - l\Omega)}{2}\Omega_d \simeq \frac{1}{2}\left(\frac{\Delta H}{H}\right)\Omega_d \bar{\omega}.$$

The drift frequency of the particles $\Omega_d = v_d/r$ was estimated according to the formula[11]

$$\frac{\Omega_d}{\Omega} \simeq \frac{n}{2}(\rho_1/r_1)^2 \frac{\omega_1}{\Omega}\frac{v^2}{v_\perp^2}\left(1 + \frac{\overline{v_\parallel^2}}{v^2}\right). \tag{9.2}$$

The results of the calculation are given in Table 1. The data of the last column are taken from Rodionov.[15] In fact a comparison can only be performed for his field configuration[2].† For $\Delta H/H = 9$ and 15 per cent we find satisfactory agreement with experiment. A sharp discrepancy for $\Delta H/H < 4$ per cent can apparently be explained by the large value of κ (for absence of stochastic changes of I it is necessary that $\kappa \ll 1$. For $\kappa \gg 1$ the number of oscillations would be only about 1000).

10. CONCLUSION

The resonance exchange of energy among the degrees of freedom of a charged particle that has been considered in this paper certainly takes place in any magnetic trap. However, it represents a danger, (allowing particles to escape,) only in systems which have 'forbidden' directions for the particle velocities. In addition to systems with magnetic mirrors some systems with compensation of the toroidal drift[4] are in this class. The most unfavourable conditions from the point of view of resonance exchange exist in traps with 'corrugated' magnetic fields.[16] In such fields the amplitudes of the resonance harmonics of the slow oscillations (Ω_m) are considerably greater than in a monotonically changing field.

We note finally that similar resonance phenomena can occur in devices to contain plasma by means of high-frequency fields.[17,18] In such cases one has not an exchange of energy, but a change of energy due to the action of the high-frequency oscillations.

* Naturally in the general case one can have both $\frac{1}{2}$ or 1 according to the ratio of the quantities ρ/R and Q/Ω (see Section 4).

† Special experiments for the verification of the present theory were not performed because at the beginning of the calculations (autumn 1958) the experiments of S. N. Rodionov were already completed and his apparatus dismantled.

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APPENDIX

Estimate of the coefficients P and Q

1. We assume that $\Omega_{1m} \sim \Omega_{2m}$ and that their signs are random. We assume also that the F_n are all of the same order for $\omega_1/\Omega < n < \omega_2/\Omega$ and that $F_n = 0$

outside this interval, while the signs of the F_n are random. ($\Delta\omega = \omega_2 - \omega_1$ is the range of variation of ω). Such a shape of spectrum is a good approximation in the case of harmonic frequency modulation and can also be utilized for estimates in other cases. Then, by Parseval's inequality $|F_n| \simeq (\Omega/\Delta\omega)^{1/2}$ and

$$|P_l| \simeq \frac{\sqrt{2}}{4}\sqrt{\frac{\Omega}{\Delta\omega}}\left(\sum_{m=\omega_1/\Omega}^{m=\omega_2/\Omega} \Omega_{1m}{}^2\right)^{1/2};$$

$$|Q_l| \simeq \sqrt{2}|P_l|. \qquad \text{(A1.1)}$$

Applying Parseval's equality to the sum under the the root, we find that it does not exceed Ω^2 (see (3.4), $\dot{\omega}/\omega \sim \Omega$). Assuming also that $\Delta\omega \sim \bar{\omega}$ we obtain from (A1.1)

$$Q/\Omega \leqslant \sqrt{\Omega/\bar{\omega}}. \qquad \text{(A1.2)}$$

2. If $\omega(t)$ has a discontinuity $\Delta^{(q)}$ in its qth derivative, so that its $(q+1)$th derivative contains a δ-function, direct calculation gives

$$\Omega_{1m} \to \frac{2}{\pi}\frac{\Omega\,\Delta^{(q)}}{\omega(2m\Omega)^q} \quad \text{for} \quad m \to \infty.$$

Inserting this into (A1.1) we get

$$P \simeq \frac{\sqrt{2}}{\pi}\frac{\Omega\,\Delta^{(q)}}{(2\omega)^{q+1}} \qquad \text{(A1.3)}$$

It is easy to see that for (A1.3) to be valid, it is not at all necessary that there should be a mathematical discontinuity in $\omega(t)$. It is only necessary that (a) at a certain instant the derivative of $\omega(t)$ should change by $\Delta^{(q)}$ during a time $\ll 1/\omega$, and (b) at all other times it should only change during times $\gg 1/\omega$. This is just the condition that (A1.1) be applicable.