STABILITY OF A PARTIALLY COMPENSATED ELECTRON BEAM

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The author discusses the conditions for stability of a partially compensated electron beam in relation to deflection ("snaking"). It is shown that, with a continuous spectrum of perturbation wave vectors, there is always a region of strong instability (with relatively large increments). With a discrete spectrum (e.g., with a beam of finite length in an accelerator), instability occurs only at beam currents greater than a certain critical value. Landau damping and radiation friction do not eliminate the instability. A weak dissipative instability is discovered, caused by radiation friction. In some cases Landau damping stabilizes this instability, but can also increase it.

The investigation is based on a model beam in the form of two pinches, electron and ion, with constant dimensions and uniform densities.

Studies of the stability of a particle beam in an accelerator are usually limited to the single-particle approximations, i.e., they discuss the motion of a single particle in the external fields. In this case the stability problem can practically be solved unambigously and reduces to a suitable choice of external fields.* To a first approximation, the interaction between particles can be regarded as the electrostatic repulsion, and hence we can estimate the limiting current. In actual fact, partly or wholly compensated beams in an accelerator form an unusual kind of plasma. It is well-known that in a plasma ther can be a number of instabilities due to the interactions of a large number of charged particles. The question arises: How far can these instabilities arise in accelerators? This problem was first dealt with by Budker [3] for a so-called stabilized electron beam. One of the most deleterious plasma instabilities was found to be beam deflection ("snaking"). In [3] is was shown that polarization of the beam, i.e., relative displacement of electrons and ions, eliminates this instability for sufficiently shortwave initial perturbations; it was suggested that long-wave perturbations might also be stabilized by external fields. This type of instability was further discussed in [4, 5]. The authors concluded that full stability can only be attained in a strong-focusing external magnetic field, and not by eddy currents or weak focusing. These results were obtained by treating separately stabilization by the external field and by polarization, the assumption being made that, to get stabilization, it is enough for these two stability regions to overlap. This treatment is in general incorrect, because new effects may arise from the simultaneous action of both forces. In this paper it will be shown that the simultaneous action of polarization and external forces always leads to instability for a certain range of wavelengths.

1. Dispersion Equation

Following [3-5], we shall begin by examining the stability of the simplest model: the electrons and ions are regarded as forming two cylindrical pinches of the same radius a, with constant densities n_e and n_i, for which we shall use the dimensionless values

$$\nu_e = \frac{\pi a^2 e^2 n_e}{mc^2}; \quad \nu_i = \frac{\pi a^2 e^2 n_i}{mc^2}, \tag{1}$$

* However, in systems with no damping (e.g., in proton storage rings), it is possible for delicate nonlinear effects to arise, of the stochastic-instability [1] or separatrix-splitting [2] type: these are difficult to calculate.

where m is the mass of an electron. The two pinches can move relative to each other, and polarization forces act between them: for one electron these are

$$f_p = 2\pi e^2 n_i (y_i - y_e) = 2m\nu_i \left(\frac{c}{a}\right)^2 (y_i - y_e), \tag{2}$$

where y is the transverse displacement of the corresponding pinch from its equilibrium position. Acting on the electrons there are also external forces proportional to the deviation from the equilibrium position:

$$f_{ext} = -\gamma m \lambda^2 y_e, \tag{3}$$

where $\gamma = (W/mc^2)$ is the relativity factor for electrons.

The fluctuations are assumed to be so small that the forces can be taken as pore transverse, so that the relativistic effects in the equations of motion reduce to the substitution $m \to \gamma m$. We shall ignore the magnetic forces (the magnetic "mass" of the current); this is permissible [3] because

$$\frac{\gamma}{\nu_e \ln R/a} \gg 1, \tag{4}$$

where R is the orbital radius of the beam. Finally, we shall assume that the ions execute only transverse vibrations (vibrations of the ion pinch), and that their longitudinal velocity is equal to zero. On these assumptions, the equations of motion take the form [3]

$$\begin{aligned} \ddot{y}_e + \lambda^2 y_e &= \Omega^2 (y_i - y_e); \\ \ddot{y}_i &= \xi \Omega^2 (y_e - y_i), \end{aligned} \tag{5}$$

where

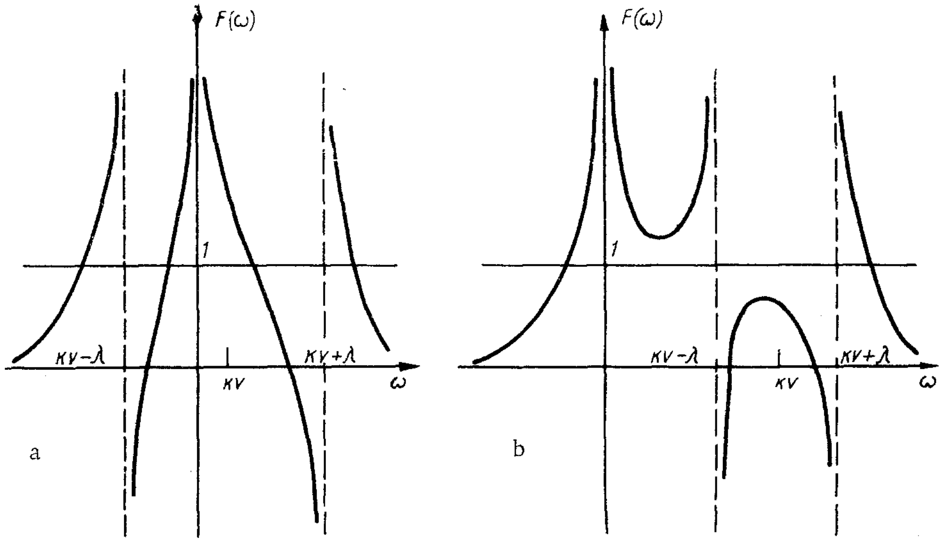
$$\Omega^2 = 2\left(\frac{c}{a}\right)^2 \frac{\nu_i}{\gamma}; \tag{6}$$

$\xi = (\gamma m/\alpha M)$ is the ratio of the electron to the ion mass of the beam ($\alpha = [\nu_i/\nu_e]$ being the compensation coefficient of the electron beam); dots indicate total derivatives with respect to time, $(d/dt) = [(\partial/\partial t) + (v\partial/\partial x)]$; $v = \beta c$ is the linear velocity of the electrons in the positive x-direction. We shall consider the stability problem in the linear approximation for a perturbation of the form $y \sim \exp i(kx - \omega t)$. Substitution of y in (5) leads to the dispersion equation derived in [3], which, however, is conveniently written in another form [6]:

$$F(\omega) = \frac{\xi\Omega^2}{\omega^2} + \frac{\Omega^2}{(\omega - kv)^2 - \lambda^2} = 1. \tag{7}$$

2. Instability

Certain properties of the function $F(\omega)$ are very helpful in the qualitative analysis of stability by means of the dispersion Eq. (7). There are clearly two possible shapes for the curves of $F(\omega)$; they are shown schematically in the diagram, for the case when $kv > 0$ (the case when $kv < 0$ reduces to changing the sign of ω). If $kv < \lambda$ all four roots are real (stability). On the other hand, if $kv > \lambda$, there are always values of k for which the line $F = 1$ falls exactly in the "gap" between the maximum and minimum of $F(\omega)$, corresponding to a pair of complex conjugate roots, and hence to instability. In fact, as kv goes from λ to ∞, the minimum of $F(\omega)$ goes from ∞ to 0, and for some value k_2 it is equal to unity. On the other hand, the maximum of $F(\omega)$ goes from some value $F(\omega_1)$ at $kv \to \lambda$ to $-\Omega^2/\lambda^2$ at $kv \to \infty$. If $F(\omega_1) < 1$ (small ξ), the wave vectors of unstable perturbations lie in the range $(\lambda/v) < k < k_2$; whereas if $F(\omega_1) > 1$ (large ξ), these wave vectors lie in the interval $k_1 < k < k_2$, where k_1 is the value of k for which the maximum of $F(\omega)$ is equal to unity. Clearly $k_2 > k_1$, since if this were not so (if there were not "gap"), (7) would have two extra roots. We thus infer that there is always a range of unstable perturbation wavelengths. This result differs qualitatively from the results of [4, 5], and has the following physical meaning: when both stabilizing factors act together, they interfere with each other so that there is always a residual range of instability. The interference effect is due to the fact that the external forces which reduce the vibration amplitude of the electrons also reduce the separation of the pinches, i.e., weaken the polarization forces.



Frequency dependence of $F(\omega)$. a) $kv < \lambda$ (stability); b) $kv > \lambda$ (instability).

3. Zones of Instability and Increments

Let us first consider the case $\xi \ll 1$, which is the case with relatively small compensation, $\alpha \gg \gamma m/M \ll 1$. As remarked above, the zone of instability is then $(\lambda/v) < k < k_2$, since the maximum of $F(\omega)$ is less than zero for all $k > (\lambda/v)$. Calculating k_2, we find the region of instability to be

$$\lambda^2 < (kv)^2 < \lambda^2 + \Omega^2\left[1 + 3\xi^{1/3}\left(1 + \frac{\lambda^2}{\Omega^2}\right)^{1/3}\right], \tag{8}$$

which becomes smaller but does not vanish when $\Omega < \lambda$, corresponding to overlapping of the regions of stabilization by polarization and by the external field.

The complex roots in the instability zone are

$$\omega \approx \xi \frac{kv\Omega^4}{[\Omega^2 + \lambda^2 - (kv)^2]^2} \pm i\sqrt{\xi\Omega^2 \frac{(kv)^2 - \lambda^2}{\Omega^2 + \lambda^2 - (kv)^2}}. \tag{9}$$

Hence it is seen that the increment is relatively small ($\sim\sqrt{\xi}$) and the instability is almost aperiodic ($\mathrm{Re}\,\omega \ll \mathrm{Im}\,\omega$). The most unfavorable part of the zone of instability is its right hand edge, $(kv)^2 \to \Omega^2 + \lambda^2$. In this case the approximate expression (9) is inapplicable and must be replaced by

$$\omega \approx \Omega\left(\frac{\xi\Omega}{4\sqrt{\Omega^2 + \lambda^2}}\right)^{1/3}(1 \pm i). \tag{10}$$

In practice, however, the maximum increment can be determined from the frequency scatter $\Delta\Omega$.* To make an exact allowance for these fluctuations, we must abandon our simple model. We can make a rough estimate of their effect if we assume that the minimum difference

$$\Omega^2 + \lambda^2 - (kv)^2 = 2\Omega^2\delta, \text{where } \delta = \frac{\Delta\Omega}{\Omega}.$$

From Eq. (9) we get

* It is important that there is a continuous frequency spectrum, i.e., a spectrum of random fluctuations of frequency Ω, which is just so for a beam which is usually located in a highly nonequilibrium state. On the contrary, a spatial and slowly changing inhomogeneity of the external fields leads only to displacement of the frequency λ and does not impose limits on ω. Exceptions to this are external forces caused by eddy currents, since these fluctuate proportionally to the beam current.

$$\omega \approx \xi \frac{\sqrt{\Omega^2+\lambda^2}}{4\delta^2} \pm i \sqrt{\frac{\xi\Omega^2}{2\delta}} . \qquad (11)$$

Let us now consider the opposite limiting case, $\xi >> 1$, which corresponds to very weak compensation of the beam [$\alpha << (\gamma m/M)$], and can be realized only by taking special precautions to free the beam from ions. Both boundaries of the instability region now correspond to the case when $F(\omega)$ touches $F = 1$ (see diagram). If the value of kv in the instability region is characterized by the parameter p ($-1 \leq p \leq 1$),

$$kv = \lambda + \omega_0 + p \sqrt{\omega_0^3/\lambda\xi}; \quad \omega_0^2 = \xi\Omega^2, \qquad (12)$$

the complex conjugate roots are

$$\omega \approx \omega_0 \left[1 + \frac{3p}{2}\sqrt{\frac{\omega_0}{\lambda\xi}} \pm \frac{i}{2}\sqrt{\frac{\omega_0(1-p^2)}{\lambda\xi}} \right] . \qquad (13)$$

4. Stable Case

In some of the cases discussed in Section 2 there is nevertheless no "universal" instability. This is due to the finite length $2\pi R$ of the beam in a ring-shaped accelerator, and the resultant discrete spectrum of perturbations, $k_1 = l R^{-1}$ (where l is an integer). Hence there are two possible ways of avoiding instability: 1) if the least value of $k = R^{-1*}$ is greater than k_2 (the right-hand edge of the instability region); 2) if the whole instability region lies between possible values of k_1.

For $\xi << 1$, both possibilites are, generally speaking, important. The first leads to the requirement that $\omega_H^2 > \lambda^2 + \Omega^2$ [here $\omega_H = (v/R)$]. If we consider the driving field of the accelerator as the external force, then $\lambda = \omega_H Q$, where Q is the number of betatron oscillations per revolution. Then we get $(\Omega/\omega_H)^2 < 1-Q^2$.

Note that this expression, like all its consequences, is also valid for strong focusing ($Q > 1$), where it corresponds to the so-called smoothed approximation. The higher harmonics, characterizing "deviations" of the trajectory, are always immobile, do not affect the local vibration frequency ω of the beam, and therefore do not alter the stability conditions.† From the above inequality it is seen that the first stabilization mechanism is realized only with weak focusing and leads to the following limitation on the beam current:

$$\nu_e < \frac{\gamma\beta^2}{2\alpha}\left(\frac{a}{R}\right)^2 \cdot Q^2. \qquad (14)$$

The second possibility is realized if $\sqrt{Q^2+\left(\frac{\Omega}{\omega_H}\right)^2} - Q < 1 - \{Q\}$, where the symbol { } implies the fractional part of the argument. The difference from the case just now considered clearly arises for strong focusing. The limiting current

$$\nu_e < \frac{\gamma\beta^2}{2\alpha}\left(\frac{a}{R}\right)^2 [(1+Q-\{Q\})^2 - Q^2], \qquad (15)$$

which is approximately Q times greater than for weak focusing, and is less than the estimate in [5] by the same factor.

For $\xi >> 1$ it is sufficient to consider only the second stabilization mechanism, in view of the relatively narrow region of instability. For this case we get

$$2\sqrt{\frac{\omega_0^3}{\lambda\xi\omega_H^2}} < 1 - \left(Q + \frac{\omega_0}{\omega_H} - \sqrt{\frac{\omega_0^3}{\lambda\xi\omega_H^2}}\right) . \qquad (16)$$

The most favorable conditions correspond to a choice of working point for which the term in brackets is equal to zero. Then the limiting current

* We shall not consider the value $l = 0$, which always satisfies the stability condition $kv < \lambda$ (cf. Section 2).

† Hence, in particular, the first stabilization mechanism suggested in [5] does not work.

$$\nu_e < \frac{\gamma\beta^2}{2\alpha}\left(\frac{a}{R}\right)^2\left(\frac{Q}{16\xi}\right)^{1/3}. \tag{17}$$

This value, though less than Eq. (16) [($\xi >> 1$)], is still very large, owing to the smallness of $\alpha << (\gamma m/M)$.

5. Frictional Forces

Let us consider the effect of friction. By this we mean any force directed against the electron velocity. Remembering that $y_e \sim \exp i\,(kx - \omega t)$, we can write ($\lambda_1 > 0$)

$$f_T = -\gamma m \lambda_1 \dot{y}_e = -i\,(kv - \omega)\,\lambda_1 y_e \gamma m. \tag{18}$$

The frictional force is equivalent to an imaginary term added to the external force:

$$\lambda^2 \to \lambda^2 + i\lambda_1\,(kv - \omega). \tag{19}$$

Assuming that this added term is sufficiently small ($\lambda_1 \to 0$), we can find a correction $\Delta\omega$ in the formula

$$\frac{\partial F(\omega,\lambda^2)}{\partial\lambda^2}\Delta\lambda^2 + \frac{\partial F(\omega,\lambda^2)}{\partial\omega}\Delta\omega + \frac{\partial^2 F(\omega,\lambda^2)}{\partial\omega^2}\cdot\frac{(\Delta\omega)^2}{2} = 0, \tag{20}$$

where $\Delta\lambda^2 = i\lambda_1\,(kv - \omega)$, and for ω we are substituting the roots of the dispersion Eq. (7).

Let us consider the expression for the correction to the frequency in the linear approximation (20):

$$\Delta\omega = -i\lambda_1\,(kv - \omega)\,\frac{\partial F/\partial\lambda^2}{\partial F/\partial\omega}. \tag{21}$$

Since $\dfrac{\partial F}{\partial\lambda^2} = \dfrac{\Omega^2}{[(kv-\omega)^2 - \lambda^2]^2} > 0$, the sign of $\mathrm{Im}(\Delta\omega)$ is determined by the signs of $kv - \omega$ and $\partial F/\partial\omega$ and can be either negative (damping) or positive (instability). Since $\mathrm{Im}(\Delta\omega) \sim (\partial F/\partial\omega)^{-1}$, it is clear that the strongest effect of friction corresponds exactly to the maximum and minimum of $F(\omega)$. In this case, by Eq. (20),

$$\Delta\omega = \pm\sqrt{-i\lambda_1\,(kv - \omega)\,\frac{\partial F/d\lambda^2}{\partial^2 F/d\omega^2}}. \tag{22}$$

Radiation friction, which is most important for electrons, is unfortunately too weak to suppress the type of instability under consideration. However, appreciable instability may arise under the action of frictional forces.

The physical significance of this dissipative instability is that the velocity of the electrons ($\dot{y} \sim kv - \omega$) may be directed in a sense contrary to the local wave velocity [$(\partial y/\partial t) \sim -\omega$]. Then the frictional force coincides in direction with the wave velocity and may lead to oscillation. The mechanism of the oscillation is associated with scattering of electrons in the field of the ion pinch, which vibrates with a certain phase difference from the electron pinch. Hence, it is clear that dissipative instability based on frictional forces is possible only in the presence of ions.

6. Landau Damping

Let us now consider the scatter of the longitudinal velocities of electrons and ions,* which is known to cause damping of the vibrations [7]. We shall confine ourselves to the discussion of a simplified dispersion equation [8]. This equation can be derived from the expression for the polarization force Eq. (2), in which y_e and y_i must be replaced by the electron and ion displacements averaged over the distribution function. This calculation yields

$$\xi\Omega^2\int\frac{f_i\,du}{(\omega - ku)^2} + \Omega^2\int\frac{f_e\,dv}{(\omega - kv)^2 - \lambda^2} = 1. \tag{23}$$

The exact theory [7] shows that the integration in Eq. (23) must be carried out in the complex plane of the variables v, u, bypassing the zero denominators (v_0, u_0) by a circuit from below. The ionic and electronic Landau damping are

* We regard the ions as nonmagnetic.

proportional to $f_i(u_0)$ and $f_e(v_0)$, respectively. If they are negligible, we return to the original dispersion Eq. (7).

Since we are interested in the boundary of the instability, we must take ω in Eq. (23) to be real. If $\xi << 1$, this quantity is small in the region of instability, and therefore $f_e(v_0) = f_e[(\omega \pm \lambda)/k]$ can be neglected.* Thus only the ion temperature is important. To make ω real, we shall choose it so that the imaginary part of the integral $(i\pi / k^2) \cdot [\partial f_i(\omega/k)/\partial u]$ vanishes. Assume that the maximum of $f_i(u)$ corresponds to $u = 0$, so that $\omega = 0$. It is convenient to express the real part of the integral in terms of the distribution function $\varphi(x)$ of the dimensionless velocity $x = u/\Delta u$, where Δu has the scatter

$$\frac{f_i\,du}{(\omega-ku)^2}=\frac{1}{(k\,\Delta u)^2}\int\frac{\varphi'(x)\,dx}{x}=-\frac{I_i}{(k\,\Delta u)^2}\,. \qquad I_i>0\,(\sim 1). \tag{24}$$

Substituting this value in Eq. (23), we find the stability limit in the form

$$\frac{I_i\xi\Omega^2}{(k\,\Delta u)^2}=\frac{\Omega^2+\lambda^2-(kv)^2}{(kv)^2-\lambda^2}\,. \tag{25}$$

This expression is meaningful only in the zone of instability Eq. (8). In the stable region of k, Eq. (8) cannot be satisfied by real ω, which indicates the presence of Landau damping, of which, in the region near Eq. (8), the decrement is approximately equal to $(k\Delta u)/\sqrt{I_i}$ [cf. Eqs. (25) and (9)]. From Eq. (25) it is seen that the region of unstable values of (kv) is somewhat restricted,

$$\lambda^2+\frac{\Omega^2}{1+\dfrac{\xi\Omega^2 I_i}{(k\,\Delta u)^2}}<(kv)^2<\lambda^2+\Omega^2, \tag{26}$$

but does not vanish. The shortening of the region is considerable for small values of

$$\frac{\xi\Omega^2 I_i}{(k\,\Delta u)^2}=\left(\frac{\Omega}{kv}\right)^2\cdot\frac{I_i}{\alpha}\cdot\frac{\gamma m v^2}{M\,(\Delta u)^2}\,. \tag{27}$$

Since the stable case corresponds to $\Omega \sim kv$ (cf. Section 4), it follows from Eq. (7) that the contraction of the region is small.

When $\xi >> 1$, $\omega \approx kv - \lambda \approx \omega_0$ is large, so that $f_i(u_0) = f_i(v - \lambda/k)$ can be neglected. The dispersion equation becomes

$$\frac{\xi\Omega^2}{\omega^2}-\frac{\Omega^2/2\lambda}{\omega-kv-\lambda}+\frac{\Omega^2}{2\lambda}\int\frac{f_e\,dv}{\omega-kv+\lambda}=1. \tag{28}$$

In contrast to the preceding case, the imaginary part of the integral does not vanish for any value of ω:

$$\operatorname{Im}\frac{\Omega^2}{2\lambda}\int\frac{f_e\,dv}{\omega-kv+\lambda}=-\frac{\pi\Omega^2}{2k\lambda}f_e\left(\frac{\omega-\lambda}{k}\right). \tag{29}$$

This means that there is no boundary of the stable region, i.e., scatter of the electron velocities leads to increase of the increment. In fact the correction to the frequency can be found from Eq. (20), by replacing its first term by Eq. (29) multiplied by i. For the extra increment due to Landau damping, we get

$$\operatorname{Im}(\Delta\omega)=\frac{\pi}{20}\cdot\frac{\omega_0^3}{\xi k\lambda}\cdot f_e\left(\frac{\omega-\lambda}{k}\right)>0. \tag{30}$$

The extra damping may equal the main damping Eq. (13). Since the quantity Eq. (29) is always negative, Landau damping leads to instability for $(\partial F/\partial\omega) < 0$ Eq. (20). In practice this instability is important only when

* Assuming that $kv \neq \lambda$.

$\omega \approx kv \pm \lambda$, as the velocity distribution of the electrons is usually fairly narrow. The physical meaning of the instability is the same as that discussed in Section 5, as in the ultimate analysis Landau damping is due to particle collisions, i.e., it is a special kind of friction. The importance of the collisions follows from the assumption [7] that the distribution function is constant. The part played by collisions was demonstrated clearly in [9, 10]. Instability due to Landau damping is evidently similar in its mechanism to the so-called universal instability in a plasma [11].

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All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. ***Some or all of this periodical literature may well be available in English translation.*** **A complete list of the cover-to-cover English translations appears at the back of this issue.**