The Nature and Properties of Dynamic Chaos

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1. INTRODUCTION

Until recently statistical properties of a physical system have been derived with the help of some special postulates, or Ansätze, such as Gibbs' microcanonical distribution in equilibrium statistical mechanics, Boltzmann's molecular chaos, Bogolyubov's correlation decay. Prigogine's causality condition, van Hove's random phase approximation, and the like in nonequilibrium mechanics. Those assumptions are especially plausible and natural in the so-called thermodynamic limit, that is, in the limit of infinite number of degrees of freedom (DF) for a given density. In a finite dimension, and the more so in just a few DF, these conjectures are generally false. Yet, such a classical approach to the statistical mechanics can be preserved by introducing an extrinsic "noise" with given statistical properties (correlations)—a method which can be traced back to Langevin.

Nowadays, tremendous progress in modern ergodic theory enables us in principle both to find the conditions for and derive the properties of the statistical or chaotic (stochastic) behavior on a purely dynamic basis, i.e., by the analysis of dynamic (deterministic) evolution equations free of any random element or assumption. In other words, in this new approach chaotic behavior becomes a particular regime of dynamic evolution. To reveal the peculiarities of this regime one needs of course to make use of some special statistical description and notions, such as the distribution function, coarse graining, symbolic trajectories, etc. The latter, however, should not be confused with statistical properties or chaotic behavior, which do not depend on the description.

As far as it concerns classical mechanics that dynamic chaos happens to be observationally undistinguishable from a "true"

random process, the latter being understood as unpredictable or unreproducible (see Section 5). Quantum dynamics appears to be less chaotic, and is certainly much less known thus far.[7,8]

I shall briefly discuss some of these more recent developments, referring to a couple of simple models of the Yang–Mills field as examples.

A mathematical review of the topic in question can be found in Refs. 1–5, that by a physicist in Refs. 6–9, and a recent popular presentation in Refs. 10–12.

In what follows we shall restrict ourselves to classical Hamiltonian dynamics, that is, to nondissipative but not necessarily conservative (in energy) dynamic systems. A typical problem to be studied—the Poincaré fundamental dynamic problem—is the influence of a weak perturbation on a completely integrable unperturbed system, which for the time being we assume to be nonlinear or nondegenerate. It means that determinant

$$\left|\frac{\partial \omega_i}{\partial I_k}\right| = \left|\frac{\partial^2 H_0}{\partial I_i \cdot \partial I_k}\right| \neq 0 \tag{1}$$

with ω_i, I_k as the frequencies and action variables, respectively, and H_0 the unperturbed Hamiltonian. A case of the linear (isochronous) unperturbed system more difficult for analysis will be considered in Section 4. Equation (1) implies a one-to-one correspondence between the I- and ω-subspaces of the dynamic (phase) space.

2. KAM integrability

The concept of integrability has little in common with the actual evaluation of a motion trajectory; the latter can always be done numerically if necessary. Instead, it is a basic characteristic of the motion (phase space) structure, namely, of its decomposition into elementary (irreducible) dynamic components. In group-theoretical language the problem of integrability is also the problem of the maximal symmetry group of a dynamic system. For a closed system the minimal one is the Poincaré group implying the ten well known integrals of motion. For the sake of brevity we discuss energy only, assuming the others to have been eliminated beforehand. Thus, integrability refers to additional specific isolating integrals related to

some "hidden" symmetry of the system. If there is nothing to hide, the motion is said to be transitive, or ergodic, on the energy surface. In this case the whole phase space decomposes into the one-parameter family of ergodic components, each comprising the entire energy surface. The opposite case corresponds to the full set of N (the number of DF) integrals in involution which reduces the elementary dynamic component to a quasiperiodical motion on a torus. For almost all tori the motion is ergodic (on the torus) or nonresonant. This is the classical view of the integrability problem going back to Poincaré.

A lucid presentation of our current understanding of this problem can be given with resonance analysis. This latter is especially convenient to carry out in the ω-subspace where each resonance

$$\sum_{i=1}^{N} m_i\omega_i + \sum_{k=1}^{M} n_k\Omega_k \equiv (m,\omega)+(n,\Omega)=0 \tag{2}$$

is clearly a plane. Here m_i, n_k are integers, and the external perturbation is assumed to be quasiperiodic with M basic frequencies Ω_k. Note that generally m_i, n_k are *any* integers, and hence the resonance set is everywhere dense in the phase space. Yet, its measure (volume) is zero, and this constitutes a singularity whose dynamic implications were not properly recognized until the Kolmogorov–Arnold–Moser theory was created[1–3, 13] (also see Ref. 18). In particular, the famous Poincaré theorem[14] (Secs. 81–83) stating that a generic Hamiltonian system is nonintegrable being formally true, has been actually misleading for a long time, at least for physicists.[19] The trick is that the theorem proves the absence of *analytic* integrals of motion. This mathematical subtlety is not merely a technical requirement. On the contrary, it is essential since the actual destruction of the motion integrals does occur, as Poincaré was aware of, on the resonance set due to the well known small denominators in the perturbation series. Obviously, the function which does not exist on an everywhere dense set cannot be analytic. But what of that? Do we really need analytic integrals of motion? Or, to put the question in another way, do we have to follow Poincaré in his somewhat implicit but fairly obvious assumption that the motion integrals comprise some solid region including resonance surfaces? The KAM theory teaches us that we do not. Yet, one can say also that there are (at least!) two different

notions of integrability:

i) *the Poincaré, or global, integrability* which refers to the overall integrals throughout the phase space; this notion somewhat exaggerates the letter of Poincare's idea retaining, however, its spirit, and is close to the modern notion of complete integrability;

ii) *the KAM integrability* restricted to a nondense, nonresonant set in the phase space.

Let us dwell on this latter notion somewhat longer. First, what are the conditions for KAM integrability? There are essentially three:

i) the unperturbed globally, or Poincaré, integrable system is nonlinear or nondegenerate (1);

ii) the perturbation is sufficiently smooth, i.e., it belongs to the class C^l with some $l > l_{cr}$;

iii) the perturbation is sufficiently weak, i.e., the perturbation parameter $\varepsilon \lesssim \varepsilon_{cr}$.

The critical perturbation smoothness l_{cr} (the number of continuous mixed partial derivatives both with respect to N angles θ_i and to M time-variables $\varphi_k = \Omega_k t + \varphi_k^0$) can be estimated by the resonance overlap criterion[6] (Section 4.5) as

$$l_{cr} \approx 2(N+M) - 2, \tag{3}$$

which is a generalization of the result in Ref. 6 to an explicit quasi-periodic time dependence of the Hamiltonian. This may be compared to the rigorous upper estimate for $M=0$ due to Moser[15]:

$$l_{cr} \leqq 2N + 2. \tag{4}$$

For a mapping the quantity l_{cr} (3) increases by 2. In particular, $l_{cr} = 2$ if $M=0$, and $N=1$.[6] The best rigorous upper bound in the latter case, again due to Moser,[3] is $l_{cr} \leqq 4$. For $l \leqq l_{cr}$ the critical perturbation does not exist, i.e., the system is not even KAM integrable at any nonzero $\varepsilon \to 0$. Takens[16] has proved this for a particular mapping with $l=2$ ($M=0$; $N=1$). Note that our l characterizes the Hamiltonian (or generating function) and is bigger by one than that in Refs. 3 and 16.

Evaluation of the critical perturbation ε_{cr} (for $l > l_{cr}$) is a more

tricky problem. We mention here only the analytical case (for detail see Ref. 6, Section 4.6). Let the perturbation be analytic in both θ_i and φ_k within the strip of half-width σ. A particular question is: How does ε_{cr} scale with σ? An interesting point is that the powerful techniques developed by Moser[3] to deal with a smooth perturbation can be applied to analytic perturbation to get a fairly efficient estimate[6]:

$$\varepsilon_{cr} \propto \sigma^{(\frac{3}{4}l_{cr}+C)}, \tag{5}$$

where the constant $C=\frac{3}{2}$ from the overlap criterion, and $C \leqq \frac{5}{2}$ from the rigorous upper bound (4).

3. SEPARATRIX STOCHASTIC LAYERS AND ARNOLD DIFFUSION

What is the nonintegrable set under the conditions of KAM integrability? The first estimate for its relative size and total measure was of the order of $\sqrt{\varepsilon}$, that is of the order of the full width of a nonlinear resonance $(\Delta\omega)_r$. However, further studies[6,9,17] revealed that the actual measure of the chaotic component, which forms very narrow stochastic layers along resonance separatrices, is a great deal smaller:

$$\xi = \frac{(\Delta\omega)_s}{(\Delta\omega)_r} \sim \exp\left[-\frac{A(\omega_i)}{(\Delta\omega)_r}\right] \tag{6}$$

where $(\Delta\omega)_s$ is the width of stochastic layer, and the factor $A(\omega_i)$ depends on dynamic variables. In simple cases the evaluation of estimate (6), based upon the overlap criterion, is fairly accurate (see Ref. 6, Section 6.2, and Ref. 20). It is worth noting that estimate (6) and the like have been obtained not by means of the powerful KAM techniques for the construction of convergent perturbation series, but using instead a routine asymptotic method of averaging.[21] However, the crucial new feature of our approach is introducing a new (resonant) perturbation parameter ξ (6) (instead of the original ε) which has to be explicltly calculated.[6,20]

What are the dynamic implications of the nonintegrable component of motion? They depend on the topology of the phase space. In

the case of $N=2$ (for a closed system) stochastic layers are separated from each other by integrable components, and their influence is negligible.[1,2] The motion remains stable (in the action variables) for all initial conditions, that is, the averaged globably integrable system approaches true motion with only an exponentially small ($\sim\xi$) ineradicable error.

However, in many dimensions ($N>2$) the motion changes drastically since all those layers merge and form an overall dense "web" along which the trajectory can and generally does approach arbitrarily closely any of the points on the energy surface.[17] Yet this does not mean ergodicity of motion, since the total measure of the web is still exponentially small. For this reason the motion on the web, which is called Arnold diffusion, may appear to be practically unimportant. In many cases this is true, especially when the diffusion rate is also exponentially small.[6,9,22] Nevertheless, Arnold diffusion certainly signifies a real instability of motion contrary to the assertion in Ref. 23. The confusion is caused by an artificial notion of allowable solutions introduced in Ref. 23, which excludes the unstable initial conditions on the web.

On the other hand, the whole problem is improper since the stochastic web is everywhere dense. There are several methods of so-called regularization of the problem, i.e., of its unambiguous formulation which would not depend on infinitesimal changes in initial conditions. One way is to bound the motion time interval from above by an arbitrary but finite value. It converts the everywhere dense resonance web into a finite mesh grid of "working" (sufficiently strong) resonances, and the problem acquires physical meaning.

Another regularization method is introducing, taking account of some always present arbitrarily weak but finite external "noise" which results in additional diffusion.[6] It also leaves the finite resonance set where the rate of Arnold diffusion exceeds the effect of the noise. Besides, this brings all the trajectories to one of the working stochastic layers and thus provides Arnold diffusion for all initial conditions.

Thus, Poincaré or local nonintegrability does not generally imply any significant change in the motion structure. Instead, KAM integrability generally takes place with, at worst, an exponentially slow Arnold diffusion.

Note that separatrix splitting, and hence formation of a stochastic layer, which is sometimes used as a criterion of nonintegrability (see,

e.g., Refs. 24,25), relates only to local nonintegrability and does not contradict KAM integrability.

Large-scale chaos sets in as a result of breaking down the KAM integrability as perturbation exceeds the critical level ($\varepsilon \gtrsim \varepsilon_{cr}$).

4. WEAK NONLINEARITY

If unperturbed oscillations are linear (isochronous) and hence $|\partial\omega_i/\partial I_k \equiv 0$ (1), the nonliearity emerges from perturbation terms only. This situation is rather typical in applications. Consider, for example, one of Matinyan's models for a classical spatially homogeneous Yang–Mills (YM) field in a Higgs vacuum[26]:

$$H(I_k, \theta_k) = H_0(I_k) + V(I_k, \theta_k)$$

$$H_0 = \tfrac{1}{2}(E_1^2 + E_2^2 + \omega_1^2 A_1^2 + \omega_2^2 A_2^2) = \omega_1 I_1 + \omega_2 I_2$$

$$V = \frac{A_1^2 A_2^2}{2} = \bar{V} + V_r + \tilde{V} \tag{7}$$

$$\bar{V} = \frac{I_1 I_2}{2\omega_1\omega_2}; \quad V_r = \frac{I_1 I_2}{4\omega_1\omega_2}\cos(2\theta_1 - 2\theta_2)$$

$$\tilde{V} = \frac{I_1 I_2}{2\omega_1\omega_2}[\cos 2\theta_1 + \cos 2\theta_2 + \tfrac{1}{2}\cos(2\theta_1 + 2\theta_2)]$$

Here $E_k = \dot{A}_k; I_k, \theta_k$ are the action-angle variables; $\bar{V}$, V_r and $\tilde{V}$ are mean, low frequency (resonant), and high frequency (nonresonant) parts of perturbation, respectively, while the small perturbation parameter is the Hamiltonian itself ($\omega_i \sim 1$): $\varepsilon = H \approx H_0 \sim I$; $V \sim \varepsilon H$. This model describes the simplest ($N = 2DF$) nontrivial case of the internal dynamics of a YM field.

In spite of degeneracy the KAM integrability still holds for this model under the additional condition $\omega_1/\omega_2 \neq p/q$ for integers p, q satisfying $|p| + |q| \leqq 4$ (see Ref. 2). An interesting case for the theory of YM fields is $\omega_1 = \omega_2$, where an exact resonance occurs at $H \to 0$. The crucial point is that there exists only one single resonance related to the low frequency perturbation term $V_r(7)$. In this case the averaged

system ($\omega_1 = \omega_2 = 1$),

$$\langle H \rangle = H_0 + \bar{V} + V_r = I_1 + I_2 + \frac{I_1 I_2}{2}[1 + \tfrac{1}{2}\cos(2\theta_1 - 2\theta_2)], \tag{8}$$

is always globally integrable due to a specific resonant symmetry (only one of N linearly independent phase combinations is present in the Hamiltonian) which implies $(N-1)$ resonant integrals (see Ref. 6). One of these integrals is always the unperturbed Hamiltonian $H_0 = I_1 + I_2$ for system (8). Since $\langle H \rangle$ is also an integral the averaged perturbation $\langle V \rangle = \bar{V} + V_r = \text{const.}$ is also conserved. The two independent integrals determine phase curves of the averaged system (8)[27]:

$$J^2 = 1 - \frac{v}{1 + \frac{1}{2}\cos\varphi} \tag{9}$$

where $J = (I_1 - I_2)/H_0$; $\varphi = 2\theta_1 - 2\theta_2$, and $v = 8\langle V \rangle / H_0^2$. Thus, the motion structure is independent of H_0 in the limit $H_0 \to 0$, the H_0 value determining only the time scale of the motion. For example, the frequency of small oscillation around the stable periodic orbit $J = \varphi = 0 (v = \frac{3}{2})$ is $\omega_0 = \sqrt{\frac{3}{8}} H_0$. There exists a separatrix corresponding to the unstable periodic orbit $J = 0$; $\varphi = \pi$ $(v = \frac{1}{2})$ which is split by the high frequency perturbation $\tilde{V}$, the resonant small parameter being

$$\xi \sim \exp\left(-\frac{C}{\varepsilon}\right), \tag{10}$$

where $\varepsilon \approx H_0 \sim \omega_0$ and $C \sim 1$, a numerical factor. Due to weak nonlinearity this latter parameter and hence the stochastic layer width is much smaller than Eq. (6), while the resonance width is much bigger, and is actually independent of ε ($J_{\max} = \sqrt{\frac{2}{3}}$) (9).

Thus, the model (7) is KAM integrable for $H \ll H_{cr} \sim C \sim 1$ (10). In Ref. 26 $H_{cr} \approx 6.7$ was accepted from numerical simulation of this model. At $H \gg H_{cr}$ the motion is globally chaotic,[26, 27] although with a small regular component incorporated.[27]

At larger N the KAM integrability is generally destroyed even in the limit $H_0 \to 0$. Consider[27] the model (7) with $N = 3$:

$$\begin{aligned} H_0 &= \omega_1 I_1 + \omega_2 I_2 + \omega_3 I_3 \\ 2V &= (A_1 A_2)^2 + (A_1 A_3)^2 + (A_2 A_3)^2. \end{aligned} \tag{11}$$

If $\omega_1=\omega_2=\omega_3$ there are three resonances, and even the averaged system $\langle H\rangle$ is no longer integrable, as numerical simulation in Ref. 27 confirms. The latter is always desirable to rule out any hidden symmetry in the system. The motion structure again depends only on the ratio $\langle V\rangle/H_0^2$, not on H_0 as $H_0\to 0$ [cf. Eq. (9)]. According to numerical data in Ref. 27 this structure includes both chaotic as well as regular (quasiperiodic) components of comparable measure.

5. THE NATURE OF DYNAMIC CHAOS

A typical (and the most important) feature of chaotic motion is a fast (exponential[11]) local instability, that is, divergence of a beam of close trajectories. This local dynamic behavior is described by the equations of motion linearized about one of chaotic trajectories. For a closed, time-reversible system the instability is characterized by the n Lyapunov exponents $\Lambda_i>0$, the dimensionality of the chaotic component being $(2n+1)$ (see Ref. 28). If the latter comprises an energy surface or a part of it, $n=N-1$. Lyapunov exponents determine the metric entropy of a dynamic system[28]:

$$h=\sum_{i=1}^{n}\Lambda_i\geqq\Lambda_m \tag{12}$$

which is also sometimes called the KS entropy (Krylov–Kolmogorov–Sinai entropy). The maximal Lyapunov exponent Λ_m is determined by

$$\Lambda_m=\lim_{t\to\infty}\frac{1}{t}\ln\frac{|\rho(t)|}{|\rho(0)|} \tag{13}$$

where $\rho(t)$ is vector of the linearized solution. For almost any initial vector $\rho(0)$ the solution $\rho(t)$ rapidly approaches the eigenvector related to Λ_m. This greatly simplifies the numerical procedure for calculating Λ_m. On the other hand, it is also sufficient to calculate only Λ_m since all we need to know is whether $h>0$. If so, almost all trajectories of the chaotic component are random according to the algorithmic theory of dynamic systems.[5] In this theory random means unpredictable or uncomputable. It may be elucidated as follows. Consider a coarse-grained, or symbolic, trajectory which is a

sequence of integers q_t $(0 \leqq q_t < M)$ where t is the integer time (of step T), and M is the number of finite elements of the phase space partition. Each q_t describes an instant position of the system to a finite accuracy. The latter is a crucial point for two reasons. First, it takes account of an unavoidable uncertainty of any observation or measurement in physics. Second, and most importantly, it enables us to rigorously discern "simple" (regular) and "complicated" (chaotic) trajectories. The important concept of complexity, introduced by Kolmogorov for sequences, can be in this way extended to dynamic trajectories.[5] Loosely speaking, the complexity of a finite sequence is the amount of information necessary to reproduce this sequence. If the complexity is maximal, that is, proportional to the sequence length, it is natural to define that sequence as random. To reproduce such a sequence one needs just to know it as given either explicitly or implicitly, i.e., encoded somehow, in particular in the initial conditions of motion. In the former case the reproduction is merely copying, and in the latter it is deciphering, by using equations of motion. The crucial point of the algorithmic philosophy is the impossibility of separating the two above cases in view of a possible continuous transition between them. For example, an intermediate step could be the change from binary to decimal numbers or vice versa. Thus, a given random trajectory is just given and cannot be reproduced in any simpler way.

The principal result, due to Brudno, in the algorithmic theory is[5, 29]:

$$K = h \tag{14}$$

where K stands for the mean Kolmogorov complexity of trajectories, per unit time. Thus, if $K > 0$ the origin of the maximal complexity of the trajectory and hence of its randomness is in the initial conditions and, ultimately, in the continuity of the phase space of classical mechanics.

The role of the equations of motion themselves in producing random trajectories turns out to be secondary. It is to provide the local instability of motion, and hence to grant dynamic significance to arbitrarily diminutive details of trajectory initial conditions. As such, the dynamic algorithm can be very simple which has appeared so puzzling until recently. We now understand that simplicity of the dynamic system does eclipse the true origin of dynamic chaos.

The above concept of algorithmic randomness appears to be in conformity with our intuitive idea of what randomness is like. Moreover, the developing of this concept has been actually guided by this conformity. At any event, this randomness does not mean complete randomness, in particular, it does not exclude the correlations in motion. It only implies the correlation decay, and moreover, in both directions in time ($t+ \pm \infty$) in accordance with the dynamic reversibility. For a given motion there is a continuous transition from deterministic to chaotic behavior. Generally, this transition can be characterized by the following randomness parameter:

$$r=\frac{T \cdot h}{\ln M} \tag{15}$$

the limit $r \to 0$ corresponding to an approximate and temporary deterministic evolution while the chaos is developing as $r \to \infty$.

The random, dynamically unstable motion possesses an important property of statistical stability, i.e., stability of any averaged quantity, which in turn is a corollary of the structural stability of dynamic chaos.

On the other hand, the randomness of motion does not fix its statistical properties. For example, the exponential local instability of motion generally does not imply any exponential relaxation, nor does it even appear to be a typical case.[20,30] The reason lies in different averaging for the entropy and the correlation. In particular, the chaos border in the phase space, i.e., a coexistence of chaotic and regular components of motion, inevitably leads to a power-type relaxation ($\sim t^{-p}$; $p \sim 1$).[20] Such a relaxation apparently has been observed both numerically[31,33] and experimentally[32] (see Ref. 20). Note that a power correlation decay generally implies a fairly complicated statistical description of dynamic chaos.

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