Chaotic dynamics of comet Halley

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Summary. A simple model of the dynamics of Halley’s comet is developed, and its motion is shown to be chaotic due to the perturbations by Jupiter. Estimates for the error growth in the extrapolation of the comet’s trajectory are obtained which particularly explain a sharp divergence of different extrapolations of comet Halley’s orbit previously obtained. Various mechanisms limiting the full sojourn time of the comet in the solar system are considered. These include the orbit diffusion under the perturbations by Jupiter and by Saturn, the orbit drift due to weak nongravitational forces as well as the prompt ejection of the comet from the solar system upon its very close encounter with Jupiter. The estimated sojourn time of comet Halley in the solar system (t ~ 10 Myr) is compared to the period of hypothetical comet showers from the Oort cloud which is about 30 Myr.

Key words: comets – celestial mechanics – chaotic dynamics

1. Introduction

Celestial mechanics, i.e. the dynamics of the solar system, has been always a perfect example of the regular, deterministic, motion which allows a long-term prediction to a fairly high accuracy. Yet, as in almost any other many-dimensional nonlinear oscillator system, the motion of a qualitatively different nature is possible here. We mean the so called dynamical chaos when a trajectory becomes random, i.e. highly irregular and unpredictable, irrespective of any noise (see, e.g., Lichtenberg and Lieberman, 1983; Zaslavsky, 1985). Moreover, according to Arnold’s conjecture (Arnold, 1964), which has been well confirmed in numerical experiments (Lichtenberg and Lieberman, 1983; Chirikov, 1979), the chaotic components of motion for the special initial conditions of a positive measure is a generic phenomenon in nonlinear oscillations. The origin of chaos lies in a neighborhood of any separatrix, the trajectory with zero frequency of the unperturbed motion.

In celestial mechanics the simplest example of separatrix is the parabolic trajectory in the two-body problem which separates the bounded and unbounded motions. As is well known by now (Lichtenberg and Lieberman, 1983; Zaslavsky, 1985; Chirikov, 1979), in this case any perturbation, particularly, a regular one, by a uniformly rotating third body, for instance, produces a finite chaotic layer at the side of unperturbed elliptic trajectories. This has been explicitly shown in a recent paper of Petsosky (1986) for a particular case of the plane circular restricted three-body problem.

The orbits close to parabolic, i.e. ones of a large eccentricity ε ~ 1/10 < ε < 1), are typical for comets, those “test particles” in celestial mechanics. The most detailed observational data exist for comet Halley due to the various historical records dating back to the year — 239 (240 B.C.). The analysis of these data allowed us to conclude that the motion of Halley’s comet is chaotic. We present some of its statistical characteristics, particularly, the diffusion rate in energy, the estimates for the comet’s life time in the solar system, and the increment of its motion local instability which sets the limit for the extrapolation of comet’s trajectory in both directions of time.

Our analysis is based upon the construction of a simple 2-dim. model (a map) for the comet’s dynamics, and on the subsequent study of this model by means of the modern theory of dynamical systems.

The motion of comet Halley is a new example of dynamical chaos in celestial mechanics. For earlier studies of chaos in celestial mechanics see, e.g. Everhart, 1979; Wisdom, 1980; Froeschle and Scholl, 1981; Wisdom et al., 1983. A general discussion of the dynamics of comets, chaos included, is presented in a recent paper by Sagdeev and Zaslavsky (1987). Interesting data on unusual motion of the third Soviet spacecraft are given by Gontkovskaja and Chebotarev (1964); they look very irregular and were, apparently, also chaotic.

Extensive numerical simulations of the dynamics of Halley’s comet (Yeomans and Kiang, 1981; Kalyuha et al., 1985; Landgraf, 1986) are a striking illustration of the difficulties and limitations in predicting chaotic motion (see Sect. 4 below).

2. The model

The strong instability of a chaotic trajectory restricts its extrapolation to a relatively short time interval irrespective of the modelling accuracy. On the other hand, for studying statistical properties of the motion, one can use a relatively simple model which includes the essential part of dynamics of the real system. In the problem under consideration, we assume it to be the dynamics of the phase of perturbations of the comet by Jupiter. As a conjugate variable it is convenient to choose some quantity proportional to the comet’s energy which determines the period of its motion and, hence, a change in the perturbation phase.

In constructing the model we have used, as the original input data, 46 values of τ, the comet’s perihelion passage time, as presented by Yeomans and Kiang (1981) and repeated in Table 1 (τ1 value after Kalyuha et al., 1985). The values for τ cover a fairly

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Table 1. Comet Halley’s dynamics: perihelion passage times (after Yeomans and Kiang, 1981)

<table>
<thead>
<tr>
<th>n</th>
<th>Year</th>
<th>Perihelion passage, $t_n$ (JD)</th>
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<th>Saturn’s phase $Y_n$</th>
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* After Kalyuha et al., 1985.
Effective periods for Jupiter 4332.653; for Saturn 10759.362 (days).
large interval in time from 1986 back to $-1403$ yr. Notice that only 27 values ($n=2-28$) are reconciled with the observations while the remaining 18 ones ($n=29-46$) have been predicted from the numerical orbit simulations of the comet (Yeomans and Kiang, 1981).

Define the global perturbation phase via Jupiter's position, with respect to the comet's orbit, at a perihelion passage time:

$$X_s = \frac{t_n}{P_s}$$  \hspace{1cm} (1)

and set $X_s = 0$ (Table 1). Jupiter is assumed to move uniformly in a circular orbit with an effective period $P_s = 4332.653$ days. As a matter of fact, $P_s$ includes various perturbations, particularly, Jupiter's and the comet's orbit precession. The above $P_s$ value has been empirically adjusted from the best intrinsic agreement of the original $t_n$ data (see below). Measured in years, $P_s$ is close to the ratio of the Earth's and Jupiter's mean motions.

The comet's period is $P_c = t_{n-1} - t_n$. Define a quantity

$$w_n = \left(\frac{P_n}{P_c}\right)^{-2/3} = (X_s - X_{s-1})^{-2/3} \approx -2E_n$$  \hspace{1cm} (2)

where $E_n$ is the comet's total energy, far from Jupiter, within the interval $(t_{n-1}, t_n)$. We set Jupiter's velocity and radius of the orbit to be unity while its mass $\mu_j = 9.54 \times 10^{-4}$ in solar units is the small perturbation parameter. The time unit is then $P_j(2\pi) = 437.563$ days = 1.88 years.

The change in $w$ (per comet's period) depends on the perturbation phase $x = X \,(\text{mod} \, 1)$. Together with Eq. (2) it leads to a canonical map of the plane $(w, x)$ (cf. Petrosky, 1986)

$$w_{n+1} = w_n + F(x_n) \quad x_{n+1} = x_n + w_n^{-3/2} \tag{3}$$

Apparently, it is the simplest (very restricted though) model of the dynamics of the comet (backwards in time).

The unknown perturbation function $F(x)$ can be found directly from the original data $t_n$ (Table 1) using the same Eq. (3). The result is depicted in Fig. 1. The scattering of points turned out to be caused by the perturbation due to Saturn.

The two perturbations can be separated as follows: let us approximate the dependence in Fig. 1 by a Fourier series $F(x)$ and plot the difference $F(x_n) - F_j(x_n)$ vs. Saturn's phase $y = Y \,(\text{mod} \, 1)$ where $Y = r_s X$ (Table 1), and $r_s = 0.4026868$ is Saturn's revolution frequency. The latter has been also empirically adjusted and it turned out to be equal to the ratio of Saturn's and Jupiter's mean motions. The difference $F - F_j$ as a function of $y$ was again approximated by another Fourier series $F_S(y)$, and the whole procedure repeated for the function $F(x_n) - F_S(y)$ instead of the initial $F(x_n)$. After about 10 such successive approximations the following decomposition of the total perturbation into that by Jupiter, and by Saturn has been obtained (Fig. 2):

$$F(x) = F_j(x) + F_S(y) + F_k(x).$$  \hspace{1cm} (4)

The final Fourier spectrum of the perturbation is shown in Table 2 where

$$F_j(x) = \sum_m [a_m \cos(2\pi mx) + b_m \sin(2\pi mx)]$$

for Jupiter, and similarly for Saturn. The mean values $\langle F \rangle \equiv a_0 \equiv 0$ were put equal to zero for both planets because the

---

**Table 2. Perturbation Fourier spectrum in model (3)**

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<tr>
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<th>$a_m \times 10^2$</th>
<th>$b_m \times 10^2$</th>
<th>$\sqrt{a_m^2 + b_m^2} \times 10^2$</th>
<th>$a_m \times 10^3$</th>
<th>$b_m \times 10^3$</th>
<th>$\sqrt{a_m^2 + b_m^2} \times 10^3$</th>
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</thead>
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Hamiltonian of the system is periodic in all phases. Notice that in the presence of Saturn's perturbation one should either use the global phase $X(A_x=r_x X_s)$ in map (3), as we did, or add the equation $y_{n+1} = y_n + r_y w_{n+1}^{1/2}$. In Fig. 2c the residual perturbation $F_R(z)$ is also plotted vs. phase of the Earth $z = Z(\text{mod} \; 1)$ where $Z = r_E X$, and $r_E = P_J$ (years) = 11.86241 is the Earth's frequency in the units adopted. We failed to find any simple dynamical interpretation for $F_R$ which, therefore, characterizes the accuracy of our model (3):

$$\left(\frac{\langle F_R^2 \rangle^{1/2}}{\langle F_F^2 \rangle^{1/2}}\right) \approx 0.030; \quad \frac{\langle F_R^2 \rangle^{1/2}}{w} \approx 3.5 \times 10^{-4}; \quad \langle (\Delta t)^2 \rangle^{1/2} \approx 14 \text{ days}$$

Here $\Delta t$ is the error in "prediction" of the next (or preceding) perihelion passage time.

In the process of successive approximations of $F_J$, $F_S$ the parameters $P_J$ and $r_S$ as well as the number of Fourier harmonics have been also optimized by minimizing residual $\langle F_R^2 \rangle$. Interestingly, the optimizing proved to be very sensitive to the value of $P_J$ so that the empirical uncertainty in this model parameter is only 0.01 days $\approx 15 \text{ min}$! Similarly, the relative uncertainty of effective frequency of Saturn $r_S$ is $\sim 10^{-6}$.

The developed model (3), together with the empirical perturbation (Table 2), is of course nothing more than a physically meaningless interpolation of the original data $t_a$.

Besides the Fourier approximation (FA) of the perturbations we made also use of a simpler "saw-tooth" approximation (STA) where each of the functions $F_J(x)$, $F_S(y)$ was represented by the two straight lines as shown in Fig. 2. The amplitudes and vertex positions have been assumed as follows

$A_J = 6.35 \times 10^{-5}; \quad x_+ = 0.552; \quad x_- = 0.640;$$\quad A_S = 1.05 \times 10^{-5}; \quad y_+ = 0.305; \quad y_- = 0.385;$$2d_J = x_+ - x_- = 0.088; \quad 2d_S = y_+ - y_- = 0.080$ (6)

Naturally, the accuracy of the latter approximation is much worse [cf. Eq. (5)]:

$$\left(\frac{\langle F_R^2 \rangle^{1/2}}{\langle F_F^2 \rangle^{1/2}}\right) \approx 0.10; \quad \frac{\langle F_R^2 \rangle^{1/2}}{w} \approx 1.2 \times 10^{-2}; \quad \langle (\Delta t)^2 \rangle^{1/2} \approx 50 \text{ days}$$

We mention that the dynamics of 2-dim maps with a "saw-tooth" perturbation, similar to map (3), was studied before by Chirikov et al. (1971), and Chirikov and Izrailev (1976); see also Lichtenberg and Lieberman (1983) and Chirikov (1979).

A surprisingly sharp phase dependence of the perturbation (Fig. 2) is explained by relatively close encounters of the comet and planets due to a small inclination angle $i$ of the orbit of the comet $\sin i \approx 0.3$. Indeed, two encounters per turn are possible, both corresponding to approximately the same phases $x$ and $y$. Recall that we define the perturbation phase at the perihelion passage time while the perturbation actually takes place at a different instant. The closest encounter corresponds to some "encounter phase" $x_c \approx 0.60$. Due to approximate symmetry of the encounter the value $F_J(x_c) \approx 0$. Saturn's encounter phase is $y_c \approx 0.35$. 

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Assuming the straight and uniform motion of both Jupiter and the comet at right angle to each other for a very close encounter \( \sin i \approx 1, \cos i \approx 1 \), the following simple analytical relation for the perturbation

\[
-F_j(x) = \frac{2A_j(x-x_j)\,d_j}{(x-x_j)^2 + d_j^2};
\]

\[F_j(x_j) = 0;\]

\[A_j = \frac{4\sqrt{2}}{3\,\sin i} \approx 0.0060;\]

\[d_j = \sqrt{\frac{3}{2\pi}} \sin i \approx 0.059\]  \(\text{(7)}\)

can be shown to hold within some interval around \( x_j \) including both \( |F_j| \) maxima. Numerical values are given for comet Halley. They agree quite well with Eqs. (6), and do so still better with a more accurate FA presented in Fig. 2a which gives \( A_j \approx 0.0059 \) and \( d_j \approx 0.062 \). Notice, however, that the empirical function \( F_j(x) \) is slightly asymmetric with respect to \( x_j \). For Saturn’s perturbation only the values of phase \( y \), and of amplitude \( A_S \) in Eq. (7) change, namely

\[A_S = \frac{\mu_S}{\mu_J} A_j \approx 0.163\]  \(\text{(8)}\)

where \( \mu_S, \mu_J \) are the mass and radius of the orbit of Saturn. The data in Fig. 2 give \( A_S/A_j \approx 0.175 \).

Other planets as well as the nongravitational forces (see Yeomans and Kiang, 1981; Landgraf, 1986) yield a perturbation comparable with the residual one (5): \( \left( <F_Z^2> / <J_Z^2> \right)^{1/2} \approx 0.025 \) (Chirikov and Vecheslavov, 1986). The latter includes, of course, the effect of some other model approximations, particularly, the assumed circular orbits of Jupiter and Saturn.

At small \( w \) the perturbation \( F(x) \) is nearly independent of \( w \) [see Eq. (7)] as the energy exchange with Jupiter is determined by the “local” (osculating) speed of the comet \( u^3 \approx 2 \gg w \). This was the reason to choose the quantity \( w \) as a dynamical variable of our model.

### 3. Local instability of motion

A strong local instability of motion – the exponential divergence of close trajectories – is commonly accepted by now as the simplest and most reliable criterion for dynamical chaos, at least, in numerical experiments (Lichtenberg and Lieberman, 1983; Zaslavsky, 1985; Chirikov, 1979). We studied this instability via the linearized equations of model (3) (for Jupiter’s perturbation only):

\[
\delta w_{n+1} = \delta w_n + F'(x^0_n)\delta x_n, \\
\delta x_{n+1} = \delta x_n - \frac{1}{2} \left( w^0_{n+1} \right)^{-5/2} \delta w_{n+1},
\]

where \( (x^0_n, w^0_n) \) is a reference trajectory, and \( (\delta x_n, \delta w_n) \) the tangent vector \( I \). The nature of the dynamics of vector \( I \) is determined by the Lyapunov exponent

\[\Lambda \approx -\frac{1}{n} \ln \frac{1}{I}, \quad n \to \infty\]  \(\text{(10)}\)

For a 2-dim. map \( \Lambda = h \), the Kolmogorov-Sinai entropy. Dynamical chaos occurs under the conditions \( h > 0 \), or \( \Lambda > 0 \).

The eigenvalues of matrix (9) satisfy the condition \( \lambda_1 \lambda_2 = 1 \). Denote the largest eigenvalue modulus by \( \lambda^* \); it depends on the iteration serial number \( n \). Then:

\[
\ln \lambda^* = \ln |1 - k_n + \sqrt{k_n^2 - 2k_n}|;
\]

\[
k_n = \frac{2}{w_n^{5/2}} F'(x_n),
\]

where we drop the superscript zero for the reference trajectory. In the STA (6)

\[
\begin{align*}
&\begin{cases}
0.108 & \text{if } 0.552 < x_n < 0.640 \\
0.0104 & \text{if } w_n^{5/2} \sqrt{2/\pi} < x_n < 0.552
\end{cases} \\
&\text{otherwise}
\end{align*}
\]

(12)

At the present value \( w_n = w_1 \approx 0.3 \), the instability occurs only within the phase interval given in the first line of Eq. (12) around encounter phase \( X, \approx \frac{1}{2} 0.60 \). The phases unstable for other \( x \) values \( \lambda^* = 1 \). Using Eqs. (11) and (12) we conclude that for \( w_n < w_{cr} \approx 0.12 \), all the phases become unstable as \( |k_n| > 2 \). In this domain there is a single solid chaotic component of motion. By contrast, at \( w > w_{cr} \), large regions of stable motion arise around the fixed points of map (3):

\[
w = w_m = m^{-2/3}, \quad x = x_f, \quad F_j(x_f) = 0, \quad F_j'(x_f) > 0,
\]

with \( m \) any integer (Fig. 3a). The oscillation period about a fixed point \( P_0 \approx 2 \pi m^{1/3} [(1 - 2d_j)/3A_j]^{1/2} \approx 700 \text{ yr} \) (in STA). Remnants of this periodicity persist in the chaotic component.Apparently, they were noticed and discussed in Yeomans and Kiang (1981),

![Fig. 3a and b. Phase trajectory of map (3) in the STA (6). Initial conditions (crosses): \( w_1 = 0.29164 \); \( x_1 = 0 \) (in 1986, see Table 1): a Jupiter’s perturbation only, \( N = 1.5 \times 10^4 \) iterations; b perturbation by both Jupiter and Saturn, \( N = 4000 \).](image-url)
Kiang (1979). There are fixed points at phase $x_0$, (7) as well, yet all the latter are unstable since $F'_x(x_0)<0$.

In region $w>w_c$, the motion instability grows only within the narrow interval of unstable phases (12). Let $p(w)$ be the probability for a trajectory to enter this unstable phase interval. Numerical simulation at $w=0.3$ gives $h \approx 0.26$ and $p \approx 0.19$. Notice that the probability $p$ considerably exceeds the interval width $2d_x \approx 0.088$ (6). This is explained by a decrease in the area of the chaotic component outside of the unstable phase interval due to large stable regions there (Fig. 3a).

The motion of the comet changes substantially if Saturn's perturbation is "switched on" (Fig. 3b). Notice that upon including Saturn's perturbation the phase plane point $(x, w)$ does no longer completely determine the trajectory which now also depends on Saturn's phase $y$. In other words, the plane in Fig. 3b is a 2-dim. projection of the 3-dim. phase space of map (3) (see Sect. 2). With Saturn's perturbation included, the stability domains decrease noticeably but persist. This leads to a reduction of the probability to $p \approx 0.13$ and, consequently, of the entropy to $h \approx 0.16$ while the unstable eigenvalue $\lambda \approx 6.2$ remains nearly constant. The latter is explained by a weak influence of Saturn upon parameter $\lambda$ in Eq. (11).

By contrast, the perturbation caused by the Earth, being relatively weak, completely dominates nevertheless, upon a close encounter with the comet as the perturbation is concentrated within a very narrow interval of Earth's phase $z$. Destabilizing effects of close encounters with the Earth are well known from numerical simulation of cometary trajectories (Yeomans and Kiang, 1981; Brady and Carpenter, 1971; Landgraf, 1986). However, the Earth's contribution to the entropy is insignificant (Chirikov and Vegheslav, 1986).

Within stability domains the motion is quasiperiodic, i.e. of a discrete spectrum, and the w variation is strictly bounded and small, while entropy $h=0$. Interestingly, the present value of comet Halley's energy is only 3% above the nearest stability region. However, the residual perturbation $F_R$ makes the existence of such regions questionable.

4. The error factor in numerical simulations of a chaotic trajectory

The local instability of comet Halley's motion is the cause of its chaotic behaviour, particularly, of the diffusion in energy (Sects. 5, 6). Moreover, the instability sharply restricts any extrapolation of the comet's trajectory both forward and backward in time. The most error-sensitive quantity is the perihelion passage time $t_\phi$ or the perturbation phase $x_\phi$. Just those $t_\phi$ errors are given usually in the papers on numerical simulations of the comet's dynamics (Yeomans and Kiang, 1981; Kalyuka et al., 1985; Brady and Carpenter, 1971; Landgraf, 1986). On the other hand, $x_\phi$ errors significantly change the motion of the comet as the trajectory may pass from stable to unstable phases and vice versa. Define the error factor

$$f_m = \frac{\delta x_m}{\delta x_0}$$

which describes the growth of phase errors over $m$ revolutions of the comet.

For $m \gg 1$ the mean error factor relates to the entropy (Sect. 3):

$$\bar{f}_m \approx e^{h m}$$

Assume the largest tolerable error $|\delta x_m|$ to be of the order of $d_\phi \sim 0.05, |\delta x_m| \sim 200$ days [a half-width of the unstable phase interval, see Eq. (6)]. Then, the extrapolation is restricted to

$$N_{\text{ext}} \approx \frac{1}{h} \ln \frac{d_\phi}{|\delta x_0|} \approx -6.3 \ln |\delta x_0| - 19$$

It grows only logarithmically with the modelling accuracy $|\delta x_0|$. In Eq. (16) we use the value $h=0.16$. Assuming an effective initial error $|\delta x_0| \approx 5 \times 10^{-4}, |\delta t_\phi| \approx 2$ days (see below) we obtain $N_{\text{ext}} \approx 39$ revolutions.

This estimate is rather crude, due to big fluctuations during relatively short times, particularly, because of a narrow interval of unstable phases. A more accurate evaluation of the error factor can be performed as follows.

Consider the linearized map (9) on some interval $(t_n, t_m)$ using the "true" orbit (Table 1) as reference trajectory. Then, the error factor can be estimated, within this interval, as the biggest eigenvalue module of the corresponding transfer matrix $F_{m,n}$, $\lambda_{m,n}$.

The quantity $(\ln \lambda_{m,n})/(m-n)=h_{m,n}$ describes a "current" entropy on this interval $(t_n, t_m)$. For instance, $h_{146 \rightarrow 48}=0.24$, which noticeably exceeds the mean value $h=0.16$. The latter is reached in a longer time interval. The former value may be compared to $h=0.30$ as measured by means of maps (3) and (9) (Sect. 3) for 50 revolutions of the comet.

As the errors of the linearized map also grow exponentially, the transfer matrix is better evaluated by multiplying the matrices of each iteration rather than numerically iterating map (9).

As a particular example we estimate the extrapolation accuracy for comet Halley's trajectory in Yeomans and Kiang (1981). The main parameters of this trajectory were determined from apparitions of the comet in 1759, 1682 and 1607. However, the most error-sensitive quantity $t_\phi$ was set to the observational value in 837 ($n=16$). So we assumed just the latter date as the beginning of the extrapolation in evaluating $\lambda_{16, \infty}$. These values are depicted

![Fig. 4. Error factor $f$ in extrapolation of comet Halley's chaotic trajectory (in 837 through 1403). Dependence of $\ln(|\delta x_m|/|\delta x_0|)=ln f$ vs. $m=17\rightarrow 46$ is shown after Yeomans and Kiang, 1981 (squares); Landgraf, 1986 (diamonds); our model (3) (circles), and $\ln(\lambda_{16, \infty}$ (crosses), see text](image)
in Fig. 4 for \( m = 17 \pm 46 \) (crosses) which correspond to the extrapolation back to \(-1403\) in Yeomans and Kiang (1981). A peculiar feature of this dependence is a rather long interval of stable motion (\( \lambda = 1, m = 17 - 29 \)) except \( \lambda_{16.33} \approx 1.89 \) followed by a fairly steep instability. It is important to note that the earliest reliable observations of comet Halley fall just on the stable interval (in \(-86, m = 28\), Yeomans and Kiang (1981)). The previous apparition in \(-163 (m = 29)\) dated to some accuracy in Stephenson et al. (1985), is located near the beginning of unstable region (see Fig. 4). In other words, the proper extrapolation, in absence of any observational data, gets actually into the unstable part of the trajectory. At the end of the extrapolation the error factor amounts to \( f_{16.44} \approx \lambda_{16.44} \approx 700 \).

As the initial error \( \delta_{in} \) it is natural to assume the rms deviation of the computed \( \tau_{m} \) from their observational values within the stable interval \( m = 17 - 28 \). Using the data of Table 5 in Yeomans and Kiang (1981) we find \( \delta_{in} = 2.7 \) days. At the end of extrapolation interval the error \( \delta_{err} \approx \delta_{in} f_{16.44} \approx 5 \) yr becomes prohibitively large. The extrapolation holds, therefore, only up to about \( m = 35 \) (in \(-615\)) when \( \delta_{err} \approx 200 \) days.

As a check of these estimates we made a similar extrapolation with our model (3) in the FA with the perturbation by Saturn included. We have slightly corrected the value \( w_{16} \) in order to decrease \( \delta_{err} \) (in 141) as well as it had been done in Yeomans and Kiang (1981). Actually, the eccentricity was corrected there but, curiously, both relative changes proved to be very close: \( \delta_{err}/(1-e) \approx \delta_{err}/w \approx 2.2 \times 10^{-4} \). The initial error for our model, calculated in the same way as above, \( \delta_{err} \approx 19 \) days, is in a reasonable agreement with the rms accuracy (5), and is only 7 times the error in Yeomans and Kiang (1981). The model values of \( \ln(\delta_{err}/\delta_{in}) \) are represented in Fig. 4 by circles, and they clearly demonstrate a “sudden” burst of instability. Our analysis explains a surprising strong divergence of different extrapolations of the orbit of comet Halley prior to \(-86\) presented recently by Stephenson et al. (1985) (see their Fig. 2).

Finally, one may compare trajectory of the comet in Yeomans and Kiang (1981) with the recent computation in Landgraf (1986), Table 8, sample 1, for instance. The result is shown in Fig. 4 by diamonds. The rms difference between the two trajectories within the stable section is \( \delta_{err} = 0.9 \) day. At the end of the extrapolation of the trajectory in Landgraf (1986) the error factor reaches the value \( f_{16.33} \approx \lambda_{16.33} \approx 30 \), and the separation of the trajectories becomes \( \delta_{err} \approx 27 \) days. Why in the final version of trajectory of the comet as presented in Table 9 in Landgraf (1986) the same separation lies within about one day remains a mystery for us.

In any event, all the above data definitely point out a rapid growth of the extrapolation errors in the time interval under consideration. This may explain some difficulties in reconciling the observational data in \(-465\) and \(-617\) with extrapolated trajectories of comet Halley as mentioned in Landgraf (1986).

We emphasize that the error growth in our model relates to the perturbation by Jupiter (and Saturn) only, without any contribution from the Earth, which would increase the errors still more. We mention that the instability of the motion and the growth of the error in a simple three-body model were noticed in Kiang (1979) and certainly follow from the results of Petrovsky (1986).

For comparison we note that the computational accuracy for the stable (quasiperiodic) motion of planets in the solar system on the same time interval is equivalent to \( \delta t \sim 1 \) day, and weakly depends on time (see, e.g. Table 2 in Yeomans and Kiang, 1981).

Contrary to the extrapolation, the interpolation of a chaotic trajectory gives much more accurate results. Particularly, it is demonstrated by surprisingly small errors of our fairly simple model (3). How strange it may seem at first glance, the interpolation is the simpler (requires the less changes in initial conditions and/or system parameters) the stronger the local instability of motion is. It is the property of structural stability (robustness) of chaotic dynamics which also provides stability of the statistical description.

Notice that big absolute errors of \( \tau_{m} \) for the computed trajectory of the comet in Yeomans and Kiang (1981) do not prevent us from using this trajectory for the reconstruction of the perturbation \( F(x) \) in Sect. 2. The point is that we need three successive values \( \tau_{m} \) only, and the reconstruction accuracy depends on trajectory errors within two revolutions of the comet.

5. The local diffusion rate of a chaotic trajectory

Within a chaotic component of the motion of a comet, the perturbation \( F(x) \) causes a diffusion in \( w \). If perturbation phases \( \chi_{m} \) would not only be random (which they are due to the local instability) but also statistically independent (which they generally are not in spite of randomness) then the diffusion rate would be determined simply by the mean square of the perturbation (see, e.g., Lichtenberg and Lieberman, 1983; Zaslavsky, 1985):

\[
D = \frac{\langle (\Delta w)^2 \rangle}{m} = D_{0} = \frac{\langle F^{2}(x) \rangle}{3} \approx \frac{A_{2}^{3}}{A_{2}} \approx 1.3 \times 10^{-5}
\]

That limiting case holds at \( w < w_{e} \) when all the phases are unstable (Sect. 3). Here Saturn’s contribution is negligibly small \( \langle A_{2}^{3}/A_{2} \rangle \approx 0.03 \).

As \( w \) grows, the entropy drops (11) which results in a time correlation, and in diffusion deceleration. This becomes especially significant for \( w > w_{e} \) due to the formation of domains with a regular motion (Fig. 3). This is just the case for the present value \( w_{e} \approx 0.3 \).

We numerically measured the local (a small change in \( w \)) diffusion rate by averaging over 1024 trajectories of 46 iterations each with slightly different initial conditions. In the FA the rate \( D_{w} \approx 5.6 \times 10^{-6} \), while the STA gives \( 6.0 \times 10^{-6} \). It is about two times less than \( D_{0} \) (17). “Switching-off” Saturn’s perturbation somewhat decreases the diffusion \( 4.4 \times 10^{-6} \) due to a stronger correlation. A residual perturbation \( F_{R} \) (Sec. 2) in the form of a random noise with the same rms magnitude does not change the rate: \( 5.5 \times 10^{-6} \) (FA).

Finally, we directly used the data of Table 1 in Yeomans and Kiang (1981), which is equivalent to one trajectory in the previous method. It gives a close value of \( 7.4 \times 10^{-6} \).

We also mention that for larger \( w \) the diffusion rate drops, e.g., \( D(0.7) \approx 2.7 \times 10^{-6} \).

6. Global dynamics of comet Halley

The simple model (3) does not take any other orbit parameters into account besides \( w \), and therefore its straightforward application is restricted by a relatively short time interval.

The most significant effect seems to be a periodic crossing of Jupiter’s and the comet’s orbits due to the perihelion precession with a period \( N_{p} \approx 440 \) revolutions of the comet (see Yeomans and Kiang, 1981). This leads to a considerable decrease of the minimal Jupiter-comet distance \( s \) as compared to the present value...

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$s_i = \sin i \approx 0.3$. As a result, the prompt ejection of the comet out of the solar system may happen in a single very close encounter with Jupiter. A rough estimate for the ejection mean life time of the comet

$$N_{ej} \approx \frac{\pi}{2d_j} \left( \frac{w}{A_j} \right)^2 \approx 10^5$$

(18)

turns out to be surprisingly long, well in excess of the diffusion life time (see below). We mention that the probability for the comet to drop on Jupiter is still about 100 times lower. On the other hand, the diffusion rate remains of the same order in spite of the crossing of the orbits (Chirikov and Vecheslavov, 1986).

Another omitted effect, which is important for the global dynamics, is apparently the diffusion in inclination $i$, as perturbation $F(x) \propto (\sin i)^{-1}$ (7). A rough estimate, obtained from the data in Yeomans and Kiang (1981), Table 4 (see Chirikov and Vecheslavov, 1986) shows that even though the diffusion in $i$ can hardly be neglected completely it does not seem to change the order of magnitude for life time of the comet in the solar system, especially in view of big fluctuations of the latter (see below).

In the STA a connected chaotic component is unbounded in $w$ because of a slow decay of the Fourier harmonics of the perturbation (Chirikov, 1979; Chirikov et al., 1971; Chirikov and Izrailev, 1976). For the true smooth perturbation, the chaotic component is limited from above: $w < w_0$. Yet, the limit is much higher than $w_1 \approx 0.3$, and therefore is unimportant here. Numerical simulation shows that, at any rate, $w_0 > 0.6$, $P_r < 25.5$ yr. What is important is that the chaotic component extends down to $w = 0$, i.e. the comet leaves eventually the solar system along a hyperbolic orbit.

In the independent-phase approximation (17) the diffusion life time of the comet would be

$$N_D \sim \frac{w^2}{D_0} \approx \frac{3w^2}{A_j}$$

(19)

which is much shorter than the ejection life time: $N_{ej} / N_D \approx 2d_j, \approx 0.1$. Notice, that even though Eq. (18) for $N_{ej}$ has a “diffusion appearance” it is not necessarily related to any chaotic motion, and it holds for regular trajectories as well provided the phase $x$ varies over the whole interval, i.e. is rotating.

The diffusion proceeds down to $w_{min} \approx A_1 \approx 0.06$ which corresponds to the comet’s period $P_{max} \approx 2.6 \times 10^4$ yr, and to its apheleon $2a_1 \approx 1700$ AU.

We carried out numerical simulations of the global dynamics using 40 trajectories of map (3) with initial conditions from Table 1. Because of the local instability all these trajectories rapidly diverge and show quite different values of the life time: $1374 \leq N_p \leq 5 \times 10^3$, $5.3 \times 10^4 \leq \tau_r, \tau_f \leq 2 \times 10^7$.

The scattering is due to big diffusion fluctuations, especially, at $w > w_{cr}$. An example of the full phase trajectory, projected onto plane $(x, w)$, is depicted in Fig. 5a.

The mean diffusion life time of the comet is equal to $N_D \approx 1.8 \times 10^4$ years, the average period being $P_D = \tau_D / N_D \approx 220$ yr, and the mean rate of global diffusion $D_G \approx w_1^2 / N_D \approx 5 \times 10^{-6}$. The latter is close to the local diffusion rate (Sect. 5).

The global diffusion primarily proceeds downwards in $w$ because of the increase of $D(w)$ in this direction.

The comet’s life time, unlike the local diffusion rate, crucially depends on a relatively weak perturbation by Saturn. If it is switched-off”, the life time jumps up to $N_D \approx 6 \times 10^4$ (2) $\tau_f \approx 6 \times 10^7$ yr, i.e. by a factor of 20, due to a long-time “sticking” of the trajectory in some narrow $w$ layers. Under these circumstances even a weak additional perturbation may greatly change the life time. Notice that the comet’s mean period $P_D \approx 100$ yr remains close to the initial $P_1 \approx 76$ yr.

As the diffusion proceeds symmetrically in both directions of time, the full sojourn time of the comet in the solar system is twice as big, $2N_D$. Certainly, the comet’s actual life time may be determined by totally different physical processes, for example, by its evaporation. Recent data in Boyarchuk et al. (1986) indicate that the evaporation time may be as short as $N_{e} \approx 4000$ revolutions. However, there is no such limitation backwards in time.

Another important effect is a systematic variation of $w$ (a drift). The physical cause of the drift is the so-called transverse nongravitational acceleration (force) related to evaporation of cometary material near the Sun (Marsden et al., 1973). Using the latest data on the parameters of non-gravitational forces (Kalyukha et al., 1985; Landgraf, 1986) we find

$$\frac{dw}{dn} \approx F(x) \approx 3 \times 10^{-5}$$

backward in time. Forward in time $F < 0$ which would result in the comet leaving the solar system after about $N_D \approx w_1 / F \approx 10^4$ revolutions, that is somewhat faster than $N_D \approx 1.8 \times 10^4$. The combination of both diffusion and drift would decrease the life time still further; numerically, $N_{tot} \approx 6600$.

We mention that the change of phase volume (dissipativity), which is inevitably related to the drift, is, nevertheless, much smaller, so that the corresponding time $N_{diss}$ can be shown to have the order $N_{diss} / N_D \approx (q \tau_f)^{-1} \sim 30$ where $q \approx 0.6$ AU $\approx 0.12$ is the comet’s perihelion distance. Hence, map (3) can be treated as a canonical one even in the presence of drift.
The effect of the drift is much stronger backward in time ($\tilde{F} > 0$). In this case the variation of $\mathbf{w}$ is eventually determined by the drift only, that is $\mathbf{w}$ continuously grows after some time because of a sharp decrease in the diffusion rate with $\mathbf{w}$ (Sect. 5).

Now, we take into account a possible time-variation of the nongravitational forces. For example, evaporation would cause the comet's mass to decrease thereby increasing the nongravitational acceleration. A natural time scale for that process would be the evaporation time of $N_e \approx 4000$ revolutions of the comet (Boyaruch et al., 1986). Because of this we modified the drift equation as follows (backward motion, $\tilde{F} > 0$):

$$\frac{dw}{dn} = \tilde{F} = \frac{\tilde{F}(n)}{1 + \frac{n}{N_{ev}}}$$ \hspace{1cm} (20)

The decrease of the drift speed $\tilde{F}$ with $n$ leads eventually to a purely diffusive motion. Yet, the life time of the comet increases considerably as compared to that without drift. This is because the drift, decreasing though, still leaves enough time for the comet to move to a bigger $\mathbf{w}$ where the diffusion rate sharply drops (Sect. 5).

According to numerical simulations, the life time for Eq. (20) is $N_{ev} \gtrsim 5 \times 10^6$. Even upon reducing $N_{ev}$ to $10^5$, the mean life time $N_{ev} \approx 1.4 \times 10^5$ is still much longer than $N_e \approx 1.8 \times 10^4$. An example of the latter trajectory with $N_{ev} = 3.1 \times 10^5$ is given in Fig. 5b. Under these circumstances the variation of the inclination $i$ as well as the single ejection of the comet by Jupiter (18) may play an important role. Even though the models of nongravitational forces used above, especially Eq. (20), remain highly hypothetical, it is completely clear from our numerical simulations that their impact on the global dynamics and life time of comet Halley is crucial.

### 7. Conclusion

A fairly simple model for the comet Halley's dynamics, developed in the present paper, allows to study essential features of its short- and long-term evolution in both directions of time. Numerical simulations as well as the analytical calculations reveal that the motion of the comet is chaotic, and allow to evaluate the error growth in the extrapolation of its trajectory. We would like to emphasize that the mean error growth is determined primarily by Jupiter's perturbation, and not by the comet's encounters with the Earth which seems to be a common belief (see, e.g. Yeomans and Kiang, 1981; Landgraf, 1986). Also, for any chaotic trajectory the mean error growth in time is exponential but not a power law one as is sometimes assumed. Increasing the computational accuracy helps, therefore, only on a short time interval as is easily verified by a slight change in initial conditions or by the time reversal.

The chaotic (non-periodic) nature of the motion is one of its important characteristics, and we propose to mark it by a special letter C (e.g. C/Halley instead of P/Halley) as a warning against underestimation of the errors in extrapolation of the chaotic trajectory.

Since chaotic motion has a continuous temporal Fourier spectrum, the so called “cyclic method”, i.e. the search of commensurabilities in motions of the comet, Jupiter and Saturn (Kamcinsky, 1962) is totally inapplicable here. This highlights the qualitative distinction of chaotic motion from a regular (quasiperiodic) one, as the motion of the planets, for example, where this method is successfuly used. We remind the reader that the perturbation in our model is a regular (quasiperiodic) function of time.

Dynamical chaos results in the diffusion of the orbit of comet in both directions of time, so that the comet is found eventually outside of the solar system. Numerical simulations show that the sojourn time of comet Halley within the solar system crucially depends on weak nongravitational forces acting upon the comet near the Sun. Interestingly, repeated crossings between the orbits of the planets and the comet only insignificantly affect the life time of the comet. The estimated sojourn time of comet Halley in the solar system ($N \sim 10^5$; $t \sim 10^2$ years) turns out to be much smaller than cosmological time scale which poses a serious problem related to the origin of comets. In this connection we would like to attract attention to the fact that the above time is of the same order of magnitude as the period of hypothetical comet showers from the Oort cloud conjectured recently in Hut et al. (1987).

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