CHAOS in SIBERIA
A story for history

Short (40') account of a long (40 yrs) work
Boris Chirikov & many coworkers in
Budker Institute of Nuclear Physics
& elsewhere

My (accidental) start
($\approx$ 1958, Kurchatov Institute, Moscow)

a simple–looking but turned–out rich
Budker's problem: single particle confinement in
Budker's (adiabatic) magnetic trap
(for the great END – controlled nuclear fusion !)

Toulouse
16 July 1998

BACKGROUND (retrospectively)

Revival of intensive studies
into nonlinear dynamics,
surprising rediscovery of chaos (stochasticity) after
Boltzmann ... Poincare ...

- new applications:
  strong focusing accelerators,
  controlled nuclear fusion

- computers, NUMERICAL EXPERIMENTS !!

– Lehmer, 1951
pseudorandom number generators after
Galton Board (= Lorentz gas in external field !)

– Goward and Hine (CERN), 1953, accelerators

– Fermi, Pasta and Ulam, 1955
foundations of statistical mechanics

– Symon and Sessler, 1956, accelerators

– Kolmogorov, 1954, KAM–integrability
(in spite of Poincare theorem !)

Budker's problem:

2–freedom nonlinear oscillations with a weak adiabatic coupling

$$(e = m = c = 1)$$

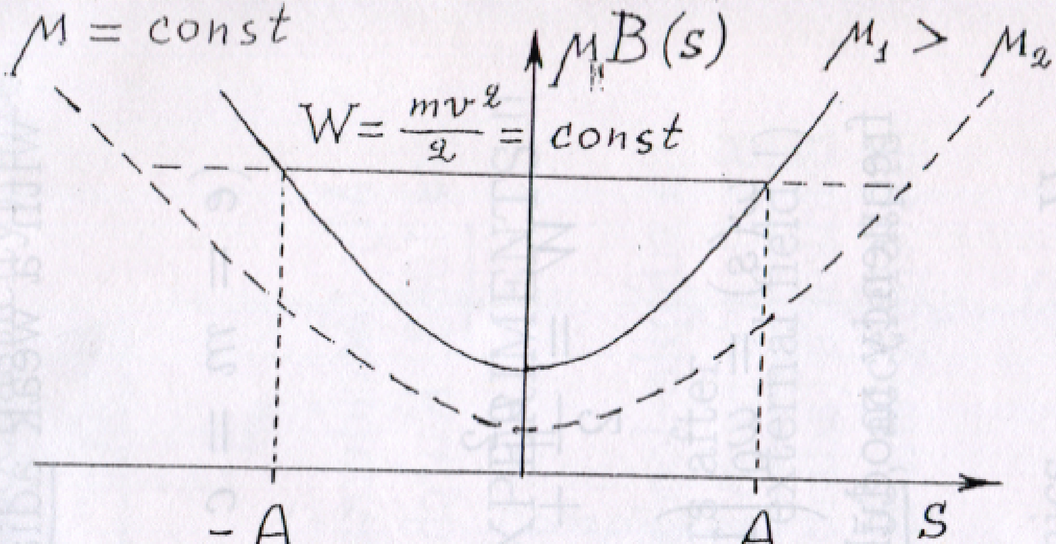
$$W = \frac{v_\parallel^2}{2} + \frac{v_\perp^2}{2} = \frac{p^2}{2} + \mu \cdot \omega(s) = const$$

$$\omega(s) = \omega_0 \left(1 + \frac{s^2}{2L^2} + ...\right) = < \omega > + \tilde{\omega}(t)$$

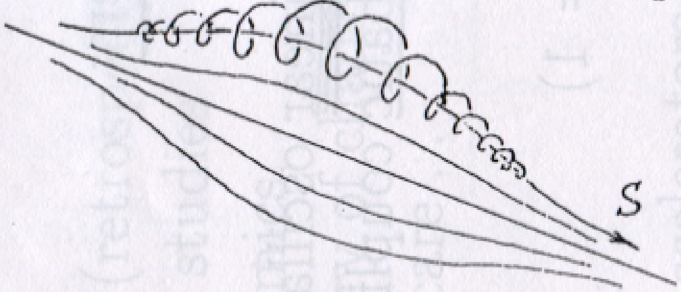
frequency modulated Larmor spiralling

$$H = \underbrace{\mu\omega_0 + J\Omega(\mu)}_{\text{adiabatic}} + \underbrace{V_a}_{\text{coupling}}$$

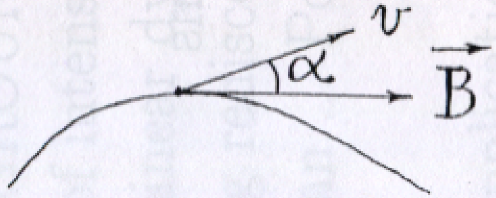
$$\mu,\ J = const \qquad H = const$$



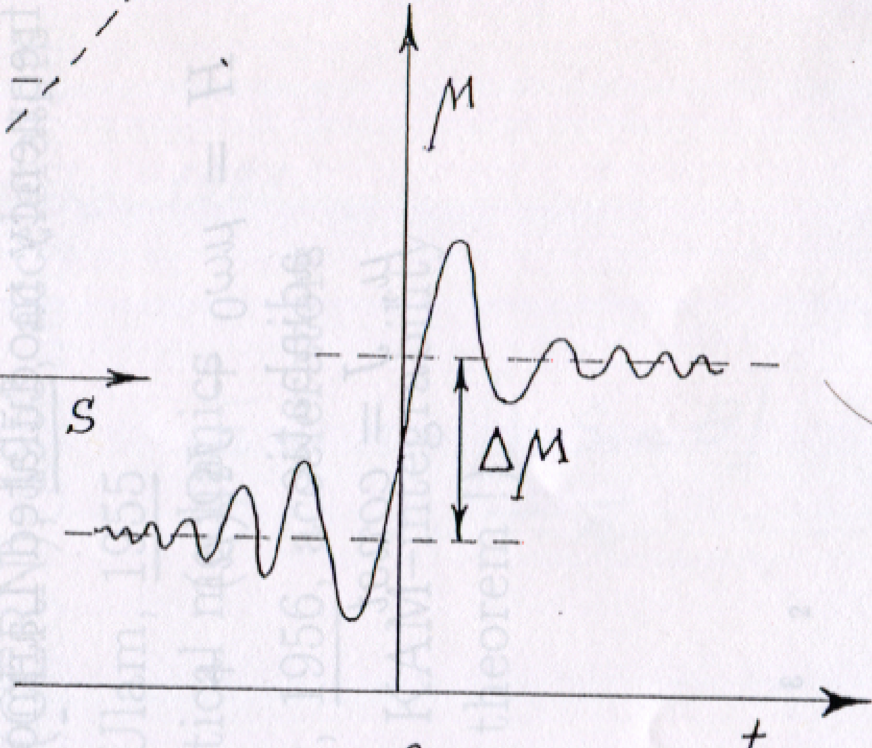
б



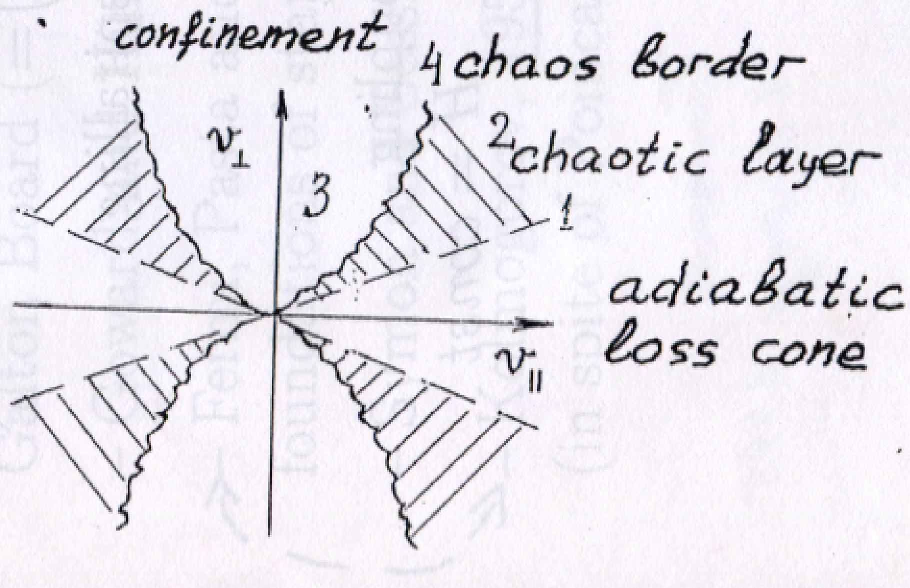
a



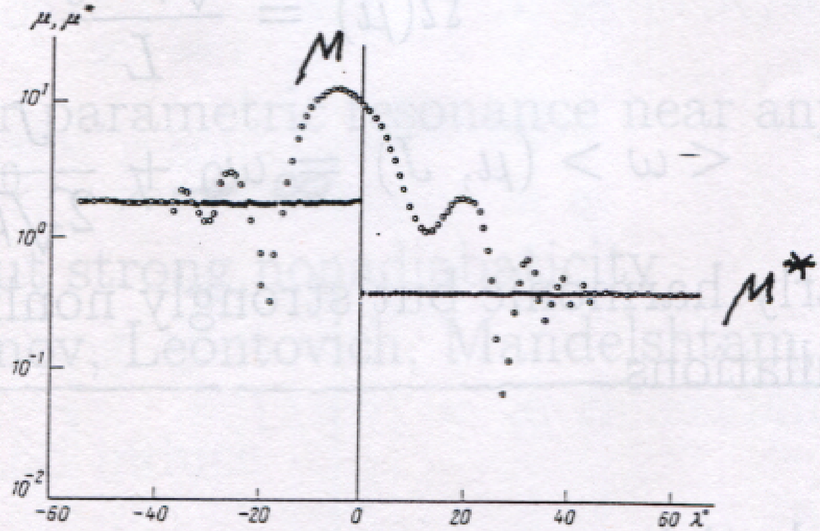
$$\mu \doteq \frac{mv^2}{2B(s)} \cdot \sin^2\alpha(s)$$



в



Ilyin, 1992



Зависимость $\mu \sim \sin^2\alpha/B$ (белые кружки) и $\mu^* \sim \sin^2\alpha^*/B$ (черные кружки) от широты λ при движении частицы с энергией $W = 200$ МэВ между точками отражения в геомагнитной дипольной ловушке. На экваторе $(\lambda = 0)$: R_{ei} (в радиусах Земли) $= 2,9$ $(\chi_i = 0,272)$: $\alpha = 20^0$: $\phi = 108^0$

$$\alpha_n^*, \phi_n \to \alpha_{n+1}^*, \phi_{n+1}, \tag{5}$$

где $\alpha_{n+1}^* = \alpha_n^* + \Delta\alpha_n^* \cos\phi_n$, $\phi_{n+1} = \phi_n + \Delta\phi(\alpha_{n+1}^*)$. Изменение фазы (набег фазы) между двумя последовательными пересечениями экваториальной плоскости приближенно описывается выражением [3]

$$\Delta\phi = \frac{\pi\bar{\omega}}{\Omega} = \frac{6F(\alpha)}{\chi}, \quad F(\alpha) \approx \sin^{-1,348}\alpha - 0,255, \quad 1,25^0 \le \alpha \le 90^0, \tag{6}$$

где $\bar{\omega}$ – ларморовская частота, усредненная по продольному колебанию, Ω – частота продольных осцилляций. Окончательно с учетом (3)–(6) приходим к отображению

$$\cos\alpha_{n+1}^* = \cos\alpha_n^* \cos\nu + \sin\alpha_n^* \sin\nu \cos(\phi_n - \bar{\phi}_0), \tag{7}$$

$$\phi_{n+1} = \phi_n + \frac{6F(\alpha_{n+1})}{\chi_i},$$

где $\bar{\phi}_0 = \arcsin(\sin\bar{\alpha}_0 \sin 2\delta / \sin\nu)$. По аналогии с [1] рекуррентному соотношению (5) можно придать вид стандартного отображения Чирикова

$$I_{n+1}^* = I_n^* + K^* \sin\theta_n, \quad \theta_{n+1} = \theta_n + I_{n+1}^*, \tag{8}$$

где $I^* = \frac{6}{\chi}\left(\frac{\partial F}{\partial\alpha^*}\right)_{\alpha_r^*} \cdot (\alpha^* - \alpha_r^*)$, $K^* = \frac{6\nu}{\chi}\left(\frac{\partial F}{\partial\alpha^*}\right)_{\alpha_r^*}$, $\theta = \phi - \bar{\phi}_0 - \pi/2$, α_r^* находится из условия резонанса $\bar{\omega} = 2r\Omega$ (r – любое целое число). В итоге получается, что рассматриваемая модель движения включает в себя "адиабатическое" движение ($\mu^* = \mathrm{const}$) между экватором и точкой отражения и дискретную модель (7) (или 8) для многократных продольных колебаний частицы. Данная

- nonlinearity

$$\Omega(\mu) = \frac{\sqrt{\mu\omega_0}}{L}$$

$$<\omega>(\mu,\, J) = \omega_0 + \frac{J}{2\sqrt{\mu}} \cdot \frac{\sqrt{\omega_0}}{L}$$

nearly harmonic but strongly nonlinear oscillations

- adiabaticity

(big) adiabaticity parameter

$$\lambda \sim \frac{<\omega>}{\Omega} \sim \frac{L}{\rho} \gg 1$$

a single swing:

$$\Delta\mu \sim \mathrm{e}^{-\lambda} \to \exp\left(-\frac{2\sqrt{2}}{3} \cdot \frac{L\omega_0}{v}\right)$$

(nonperturbative theory)

- RESONANCES

a linear parametric resonance near any integer $r = \frac{2\omega_0}{\Omega} \sim \lambda \to \infty$

slow but strong nonadiabaticity

Andronov, Leontovich, Mandelshtam, 1928

- NONLINEAR RESONANCES ???

Poincare map (standard) $(p, x) \to (\overline{p}, \overline{x})$:

$$\overline{p} = p + k \cdot \sin x$$
$$\overline{x} = x + T \cdot \overline{p}$$

with single parameter $K = kT$

$$H(x,p,t) = T\frac{p^2}{2} + k \cdot \sum_{r=-\infty}^{\infty} \cos(x - 2\pi r t)$$

adiabatic $\qquad$ perturbation

$p = const$

resonances at $p_r = 2\pi r/T$ $\qquad$ r integer

a single nonlinear resonance ('pendulum')

width $\Delta p = 4\sqrt{k/T}$, $\Delta\omega = 4\sqrt{K}$, $(\omega \approx \dot{x})$

(a new) frequency $\Omega_r = \sqrt{K} \ll 1$

completely integrable

suppresses resonant perturbation

(a big hope failed !)

INTERACTION of RESONANCES

2 (or more) resonances

with spacing $\delta p = 2\pi/T, \quad \Delta\omega = 2\pi$

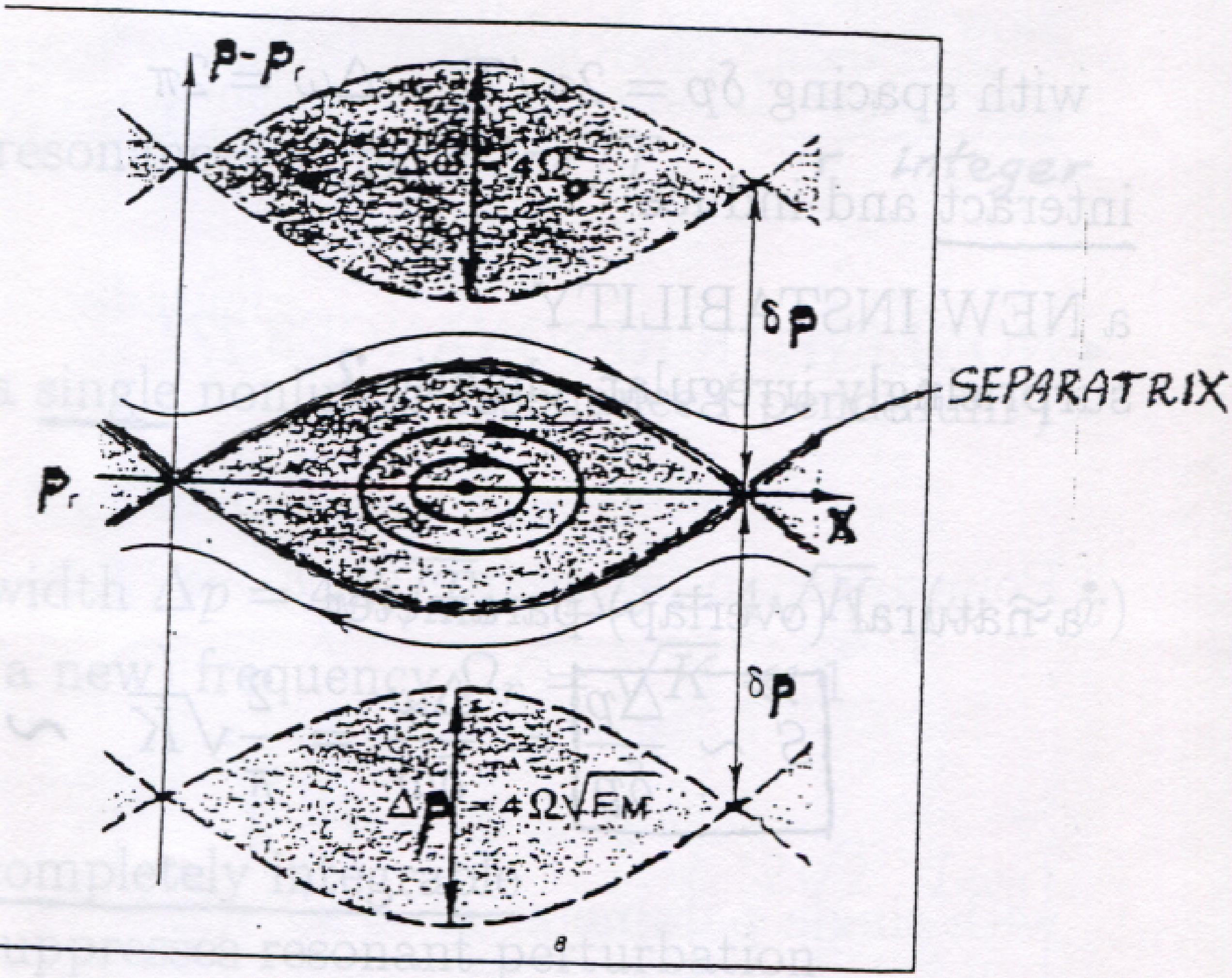
interact and induce

a NEW INSTABILITY
surprisingly irregular, chaotic ?

a natural (overlap) parameter

$$\boxed{S \sim \frac{\Delta p}{\delta p}} = \frac{\Delta\omega}{\delta\omega} = \frac{2}{\pi}\sqrt{K} \quad \sim 1$$

Б. В. Чириков



· Значение величин $\dot\theta$ и θ в начальный момент времени, для которых имеют место

$S \gg 1$ apparently global instability,
surprisingly irregular, chaotic ?

KS–entropy $\qquad h = \Lambda > 0$

Kolmogorov, 1958 $\qquad$ Sinai, 1962

diffusion

$$D = \frac{<(\Delta p)^2>_t}{t} = \frac{k^2}{2} C(K) \longrightarrow \frac{k^2}{2}$$

$K \to \infty$

Rechester, Rosenbluth, White, 1981

no confinement (**?**)

$S \ll 1$ nearly (but not completely **!**) integrable,
apparently stable,
confinement (**?**)

2F (axisymmetric trap):
eternal particle confiniment for some $K \to 0$
Arnold, 1962

3F (asymmetric) trap **?**
the universal instability, Arnold, 1964
Arnold diffusion

SEPARATRIX: most unstable place,
the universal (for any $K \to 0$) origin of chaos,
chaotic layer *etc*

CROSS–ROAD

→ other sophisticated applications to the great ENDs

→ simple models for fundamental laws (of the random)

STRATEGY (spontaneous)

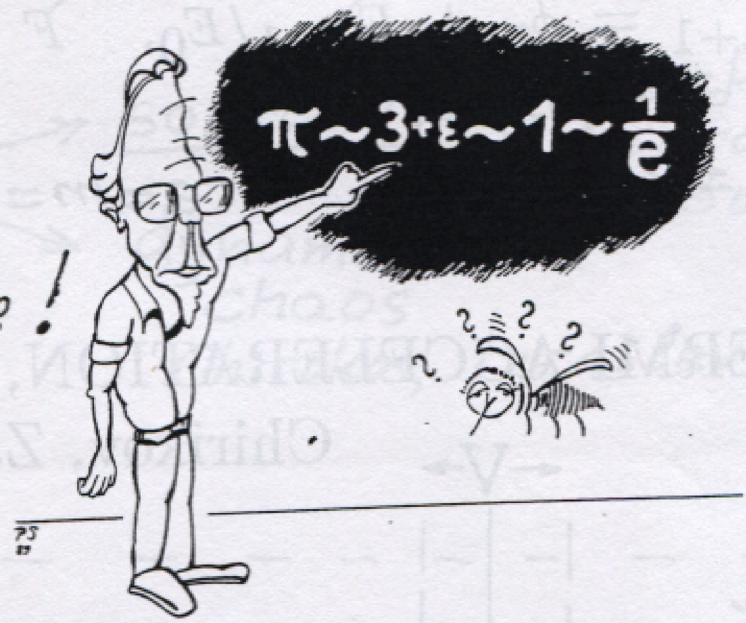
a physical RESONANCE theory

$$\Updownarrow$$

NUMERICAL EXPERIMENTS !!

A physical theory...

Chercher
la resonance!



instructive examples

SIMPLE CHAOS:

most complex dynamics

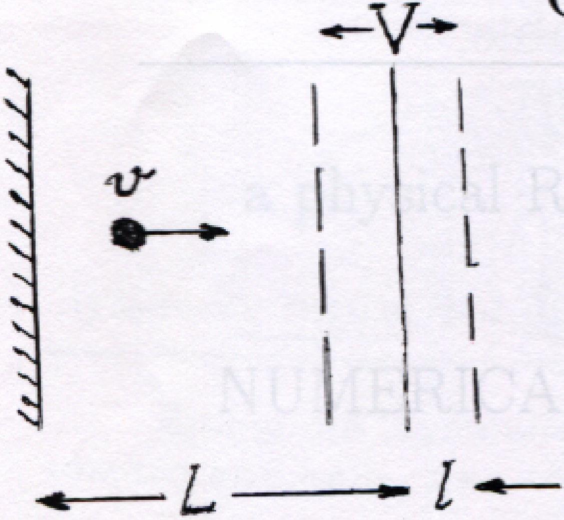
BUT simple statistics

- MICROTRON, Veksler, 1944

$$E_{n+1} = E_n + V_0 \sin \phi_n$$
$$\phi_{n+1} = \phi_n + E_{n+1}/E_0, \quad F = 2 \textit{ freedoms}$$
$$K = V_0/E_0 \gg 1$$
$$E_0 = \omega_0/2\pi\Omega, \qquad e = m = c = 1$$

- FERMI ACCELERATION, Ulam, 1961

Chirikov, Zaslavsky, 1964



$$\frac{v^2}{V^2} < \frac{4L}{l} \ !$$

(adiabatics: $\frac{v^2}{V^2} \lesssim \left(\frac{L}{l}\right)^2$)

- COMET HALLEY

diffusion backwards in time

$t_L \sim 10\, Myrs$?

Chirikov, Vecheslavov, 1989

Fermi's problem (1923)

ergodicity of nonlinear oscillation:

$$H=\sum_{i=1}^{N} \frac{\dot{x}_i^2}{2} + \frac{(x_i-x_{i-1})^2}{2} + (x_i-x_{i-1})^p, \quad p=3;4$$

$$N \gg 1$$

Fermi, Pasta, Ulam, 1955

nonergodic (generally) → solitons — Kruskal, Miura, Zabusky, 196

nonergodic (generally) → dynamical chaos — Chirikov, Izrailev, 1966

De Luca, Lichtenberg, Lieberman, 1995

Shepelyansky, 1997

Fermi, Pasta, Ulam, 1955

NO RELAXATION ???

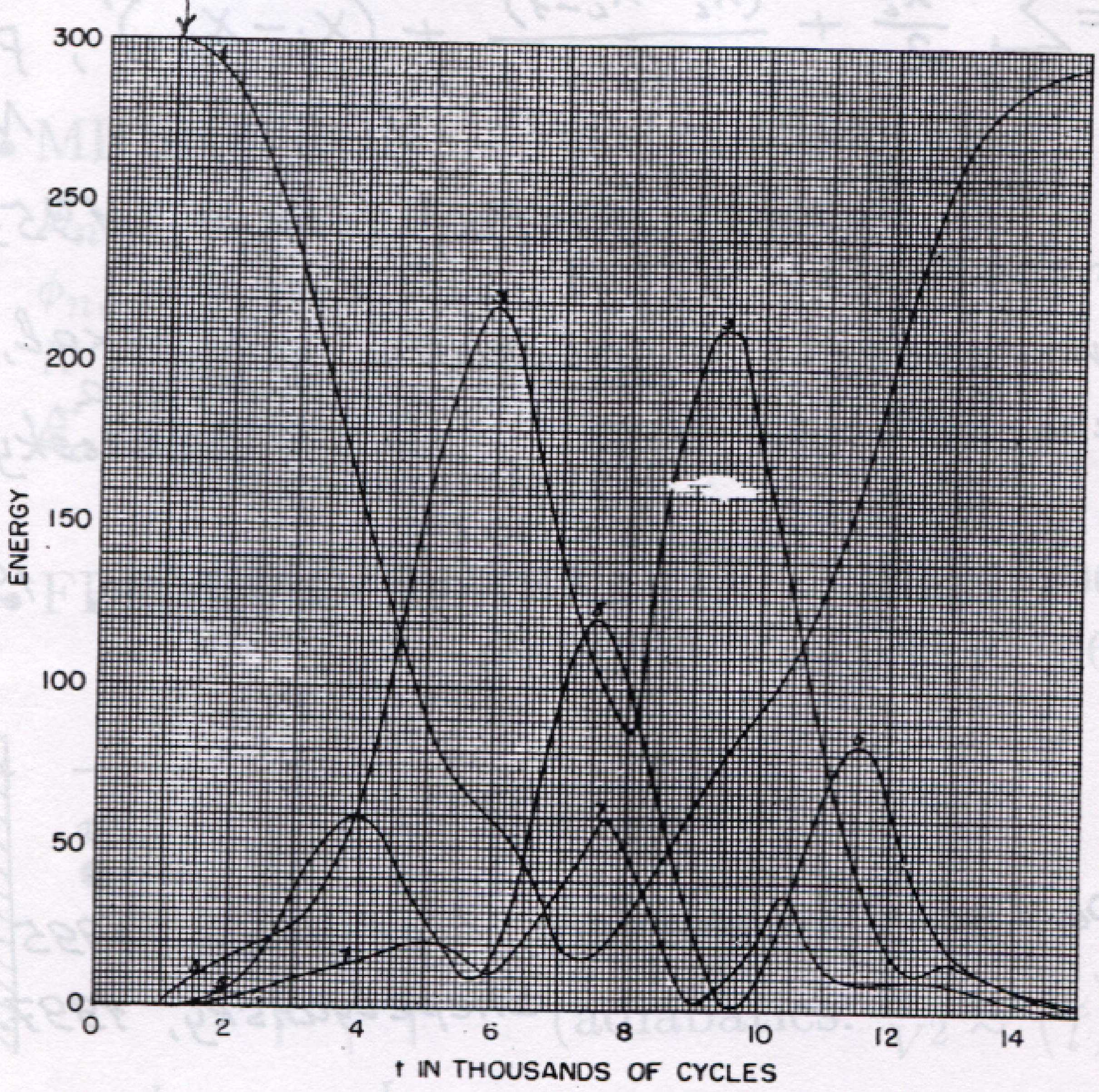


Fig. 4. The initial configuration assumed was a single sine wave; the force had a cubic term with $\beta = 8$ and $\delta t^2 = 1/8$. Since a cubic force acts symmetrically (in contrast to a quadratic force), the string will forever keep its symmetry and the effective number of particles for the computation $N = 16$. The even modes will have energy 0.

Fermi, Pasta, Ulam, 1955

RELAXATION !!!

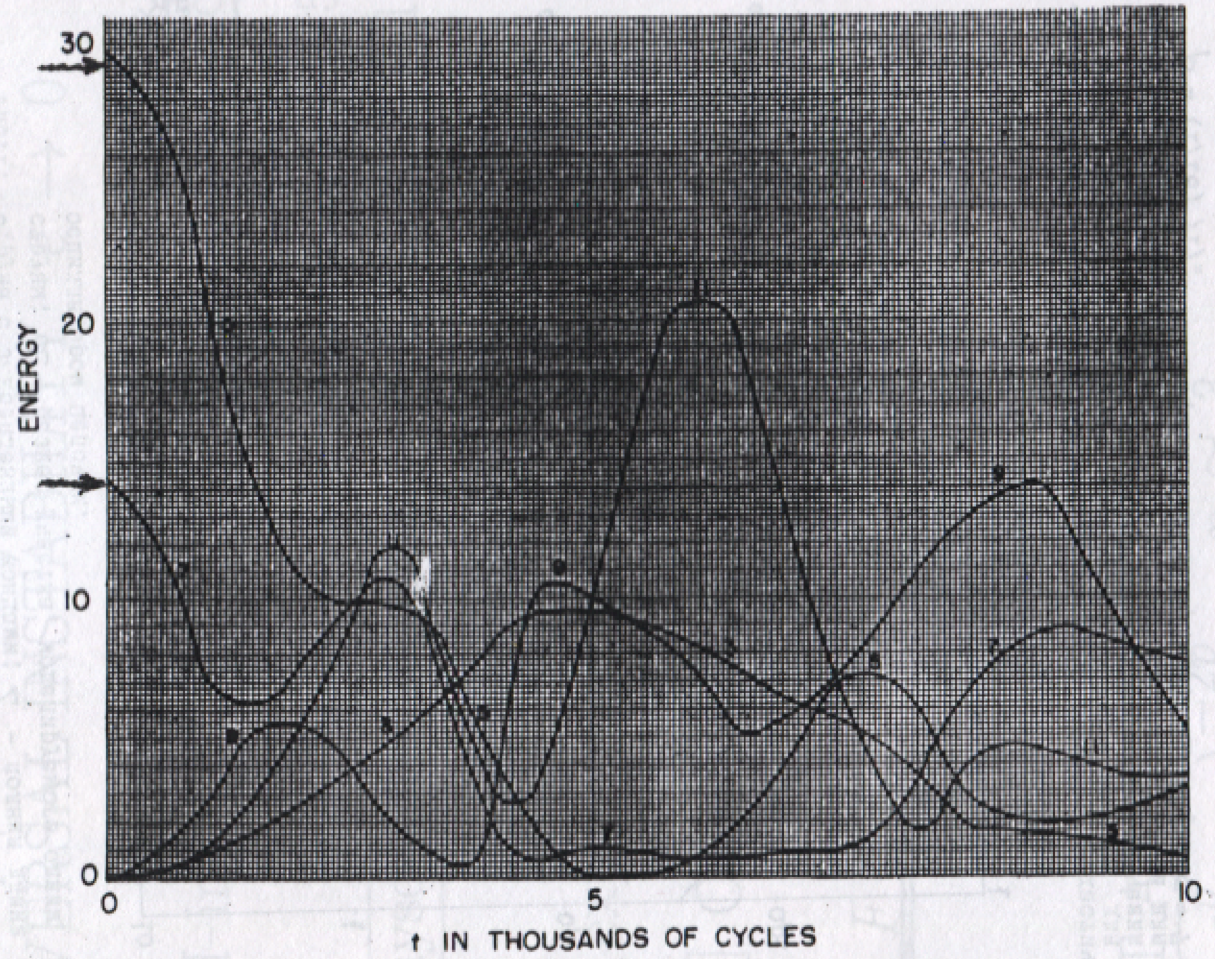


Fig. 5. $N = 32$; $\delta t^2 = 1/64$; $\beta = 1/16$. The initial configuration was a combination of 2 modes. The initial energy was chosen to be 2/3 in mode 5 and 1/3 in mode 7.

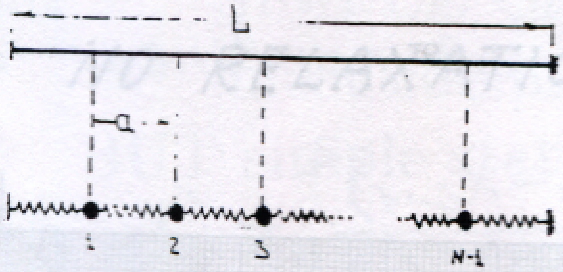


Рис.I. Струна с закрепленными концами; L - полная длина струны; a - размер участка, моделируемого одним осциллятором цепочки.

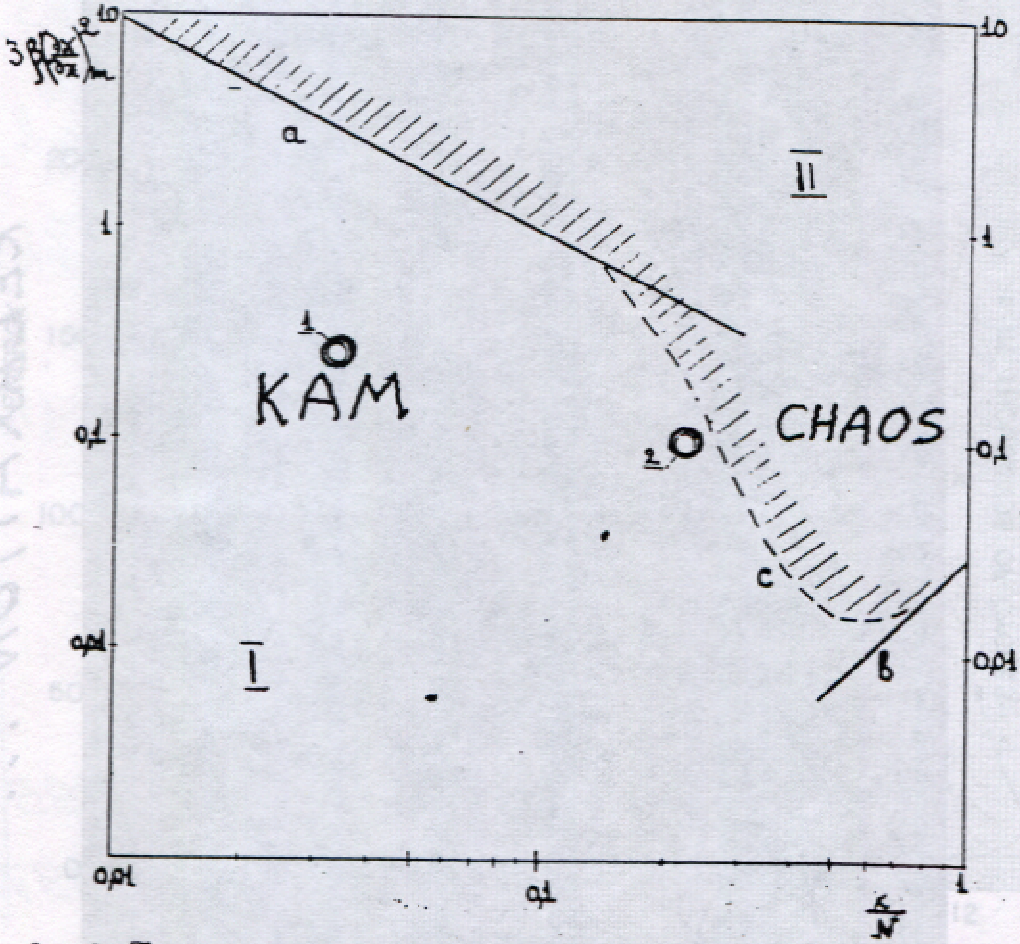


Рис.2. I - область колмогоровской устойчивости; II -область стохастичности; a - граница стохастичности для $k \ll N$ (2.II) b - граница для $k \approx N$ (3.5); c - качественная интерполяция; численные значения прямых a, b даны для $N = 32$; 1 - результат численного счёта для $N = 32$; $x_m = I$; $k = I$; $\beta = 8$ /I/; 2 - тоже самое для $k = ?$; $\beta = (I/I6)$ /I/.

COMPLEX CHAOS:

most complex dynamics AND statistics

- ARNOLD DIFFUSION, 1964

  $F > 2$ everywhere dense web of measure

  $\mu \sim \exp\left(-A\,\epsilon^{-1/2}\right), \quad D \sim \mu^2, \quad \lambda \sim \epsilon^{-1/2}$

  UNIVERSAL INSTABILITY $\epsilon \to 0$

  and

  KAM–integrability (= <u>adiabatic invariance</u>)

– Chirikov, Gadiyak, Izrailev, 1975

– $\boxed{\text{Tennyson}}$, Lichtenberg and Lieberman, 1979

– Nekhoroshev, 1977

$\mu \sim \exp\left(-A\,\epsilon^{-1/2L}\right), 1 < L(F) = ?$

– $\boxed{\text{Ford}}$, Chirikov and Vivaldi, 1979

– Lochak, 1990

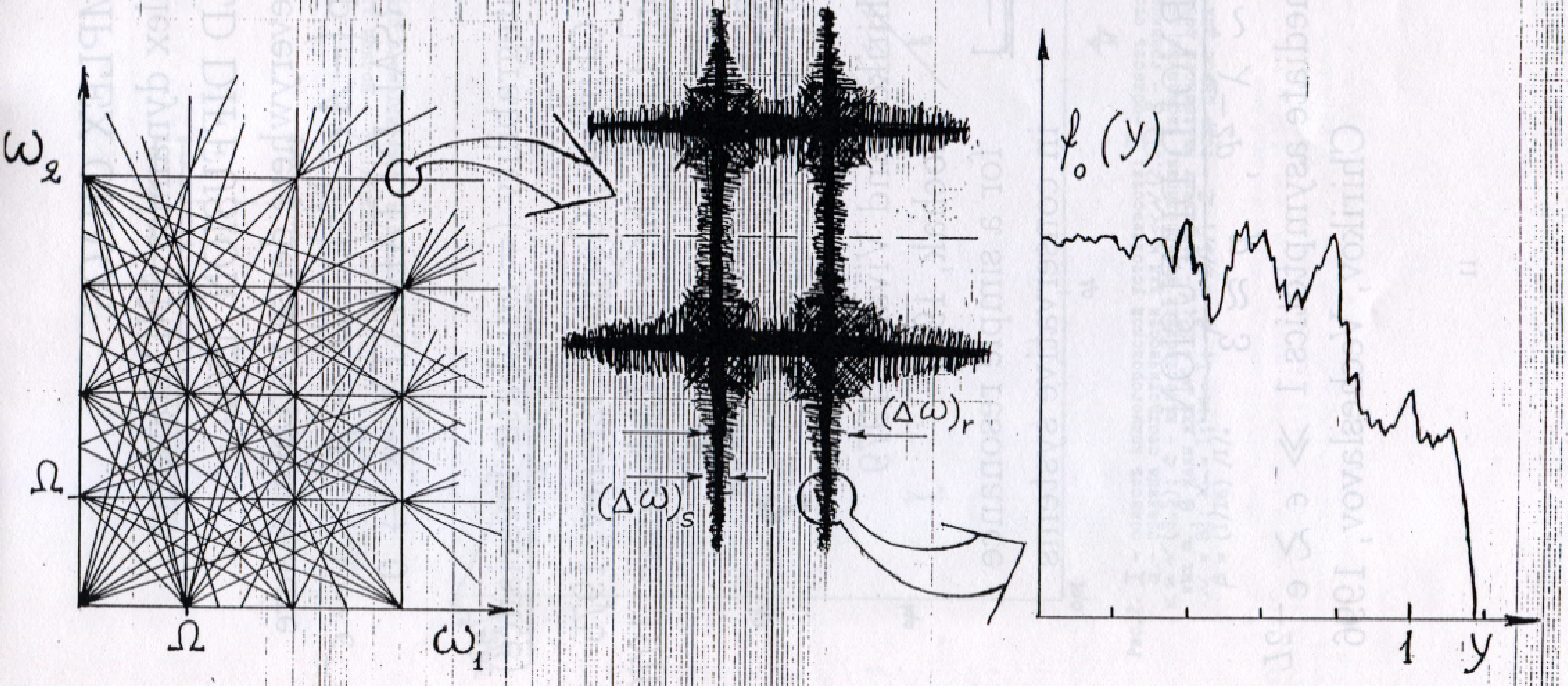
$\underline{L = F - 1}$ for a simple resonance in conservative systems

<u>FAST ARNOLD DIFFUSION</u>

$\mu \sim \epsilon^p \sim \lambda^{-2p}, \quad p \approx 3$

in intermediate asymptotics $1 \gg \epsilon \gtrsim \mathrm{e}^{-2L}$

– Chirikov, Vecheslavov, 1996

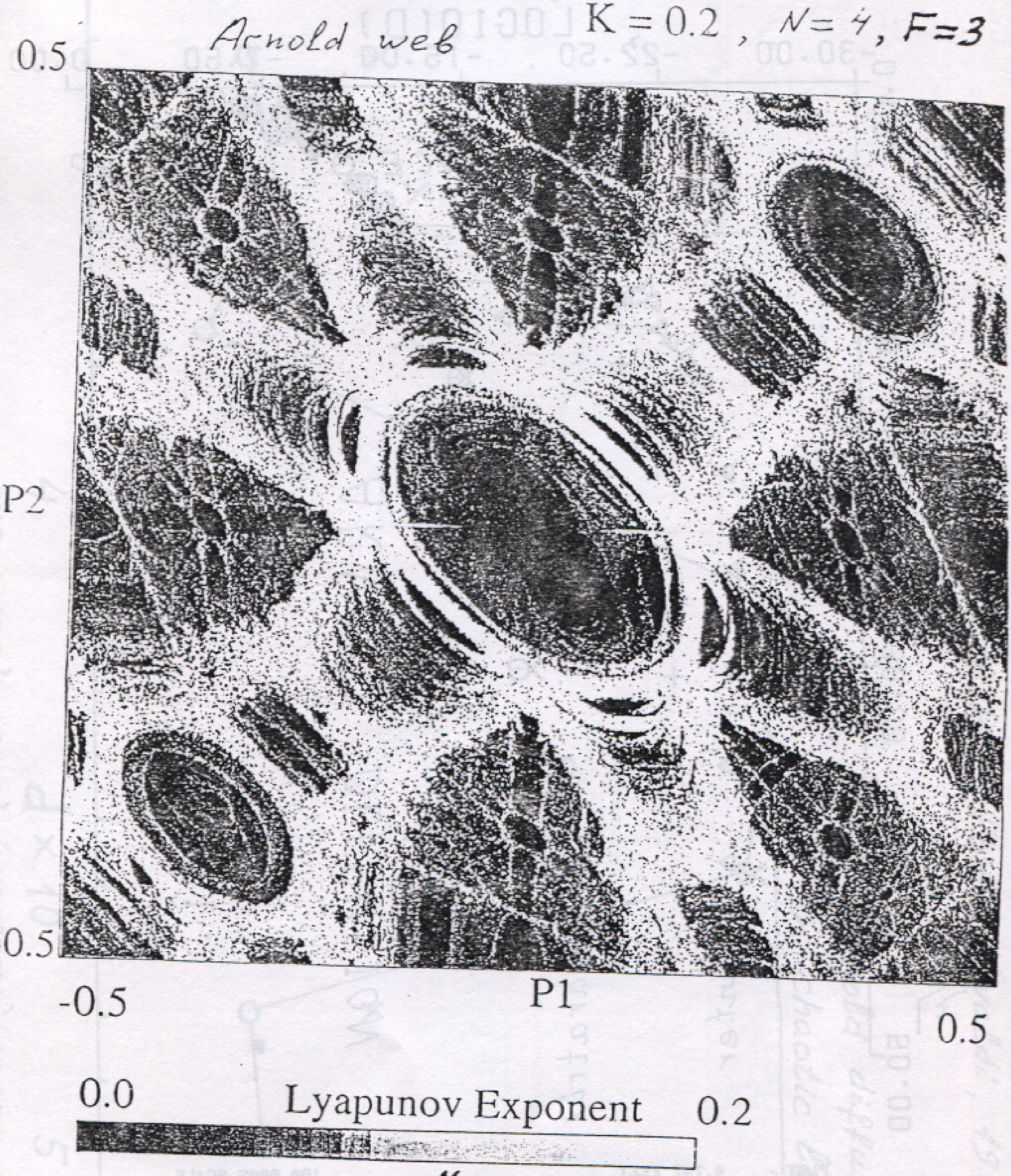


a

b

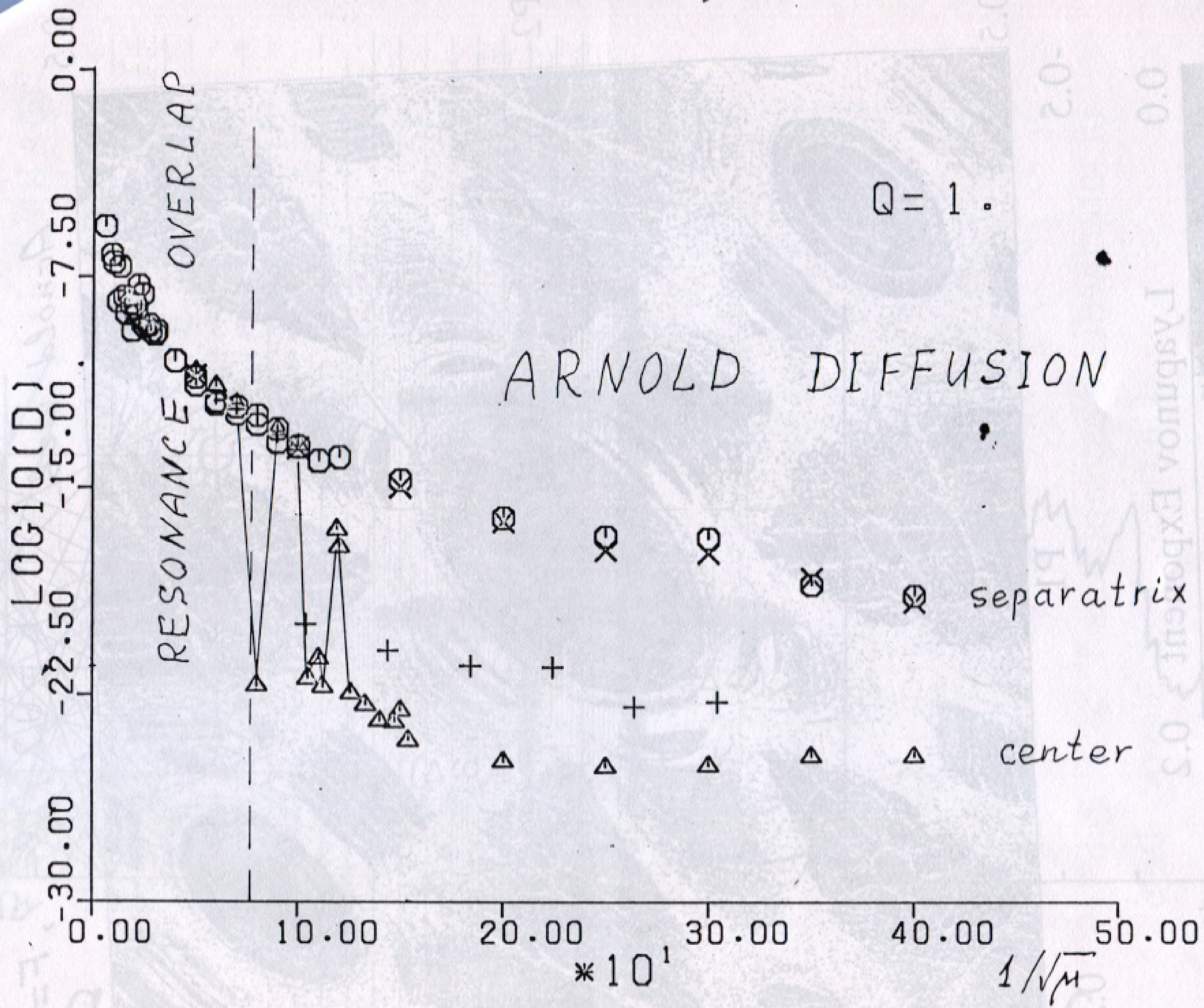
c

Kaneko, Konishi, 1993



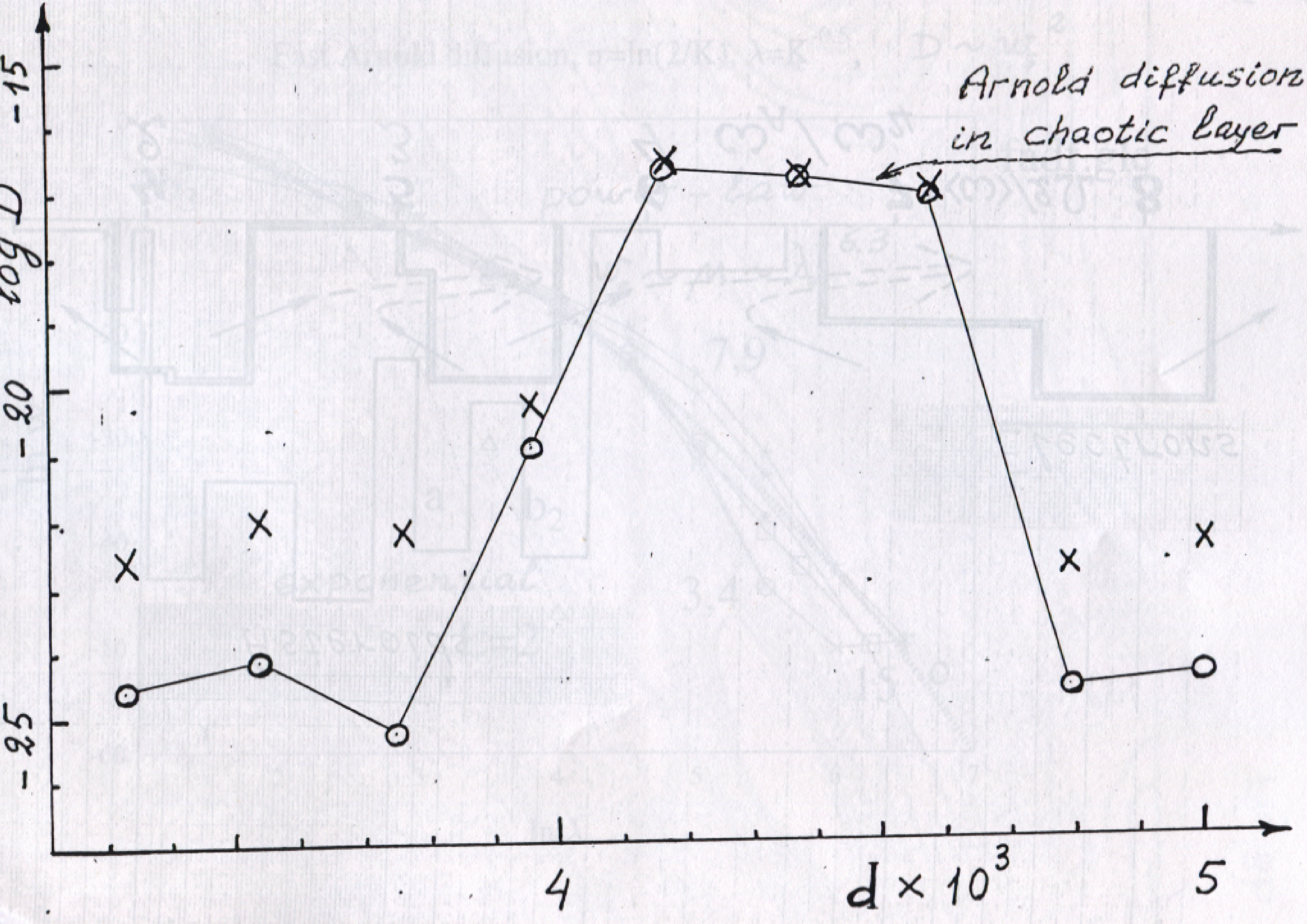
$$H = \frac{|p|^2}{2} - K\sum_{i=1}^{N} \cos(x_{i+1} - x_i)\cdot \delta_1(t)$$

Chirikov, Ford, Vivaldi, 1979



PLOT: 0.747 FEET. 100.000% SCALE.

Chirikov, Ford, Vivaldi, 1979



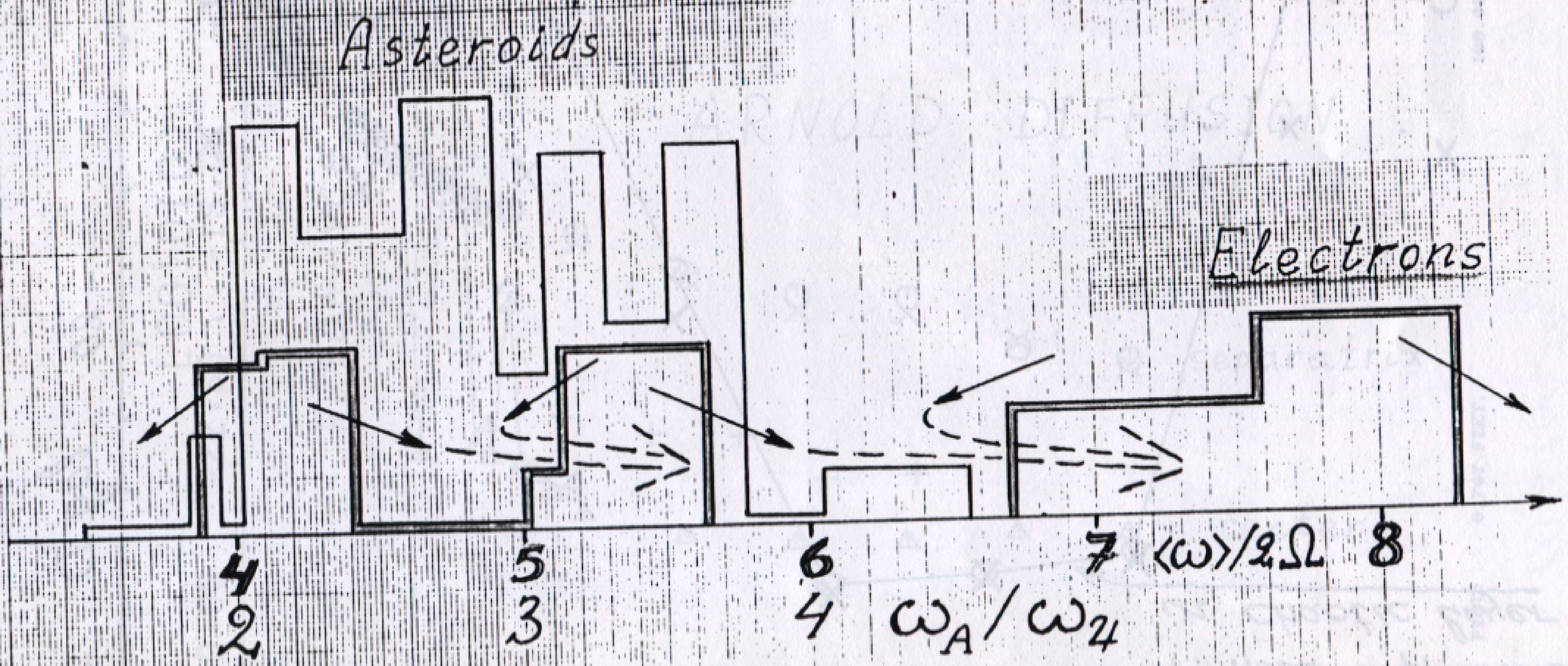
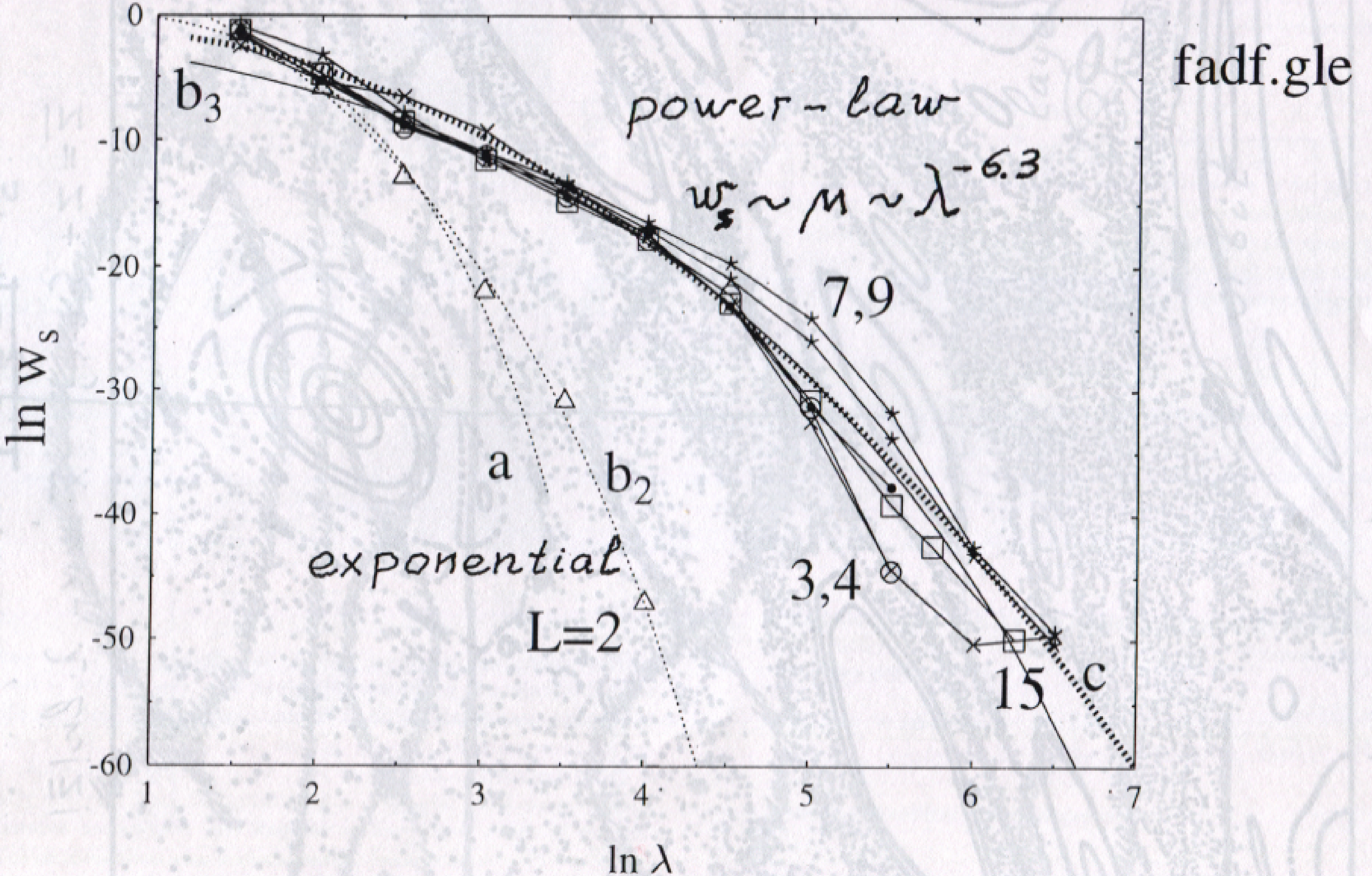

Asteroids
Electrons
4
2
5
3
6
4
7
8
ω_A / ω_4
$\langle\omega\rangle / 2\Omega$

Fig. 7

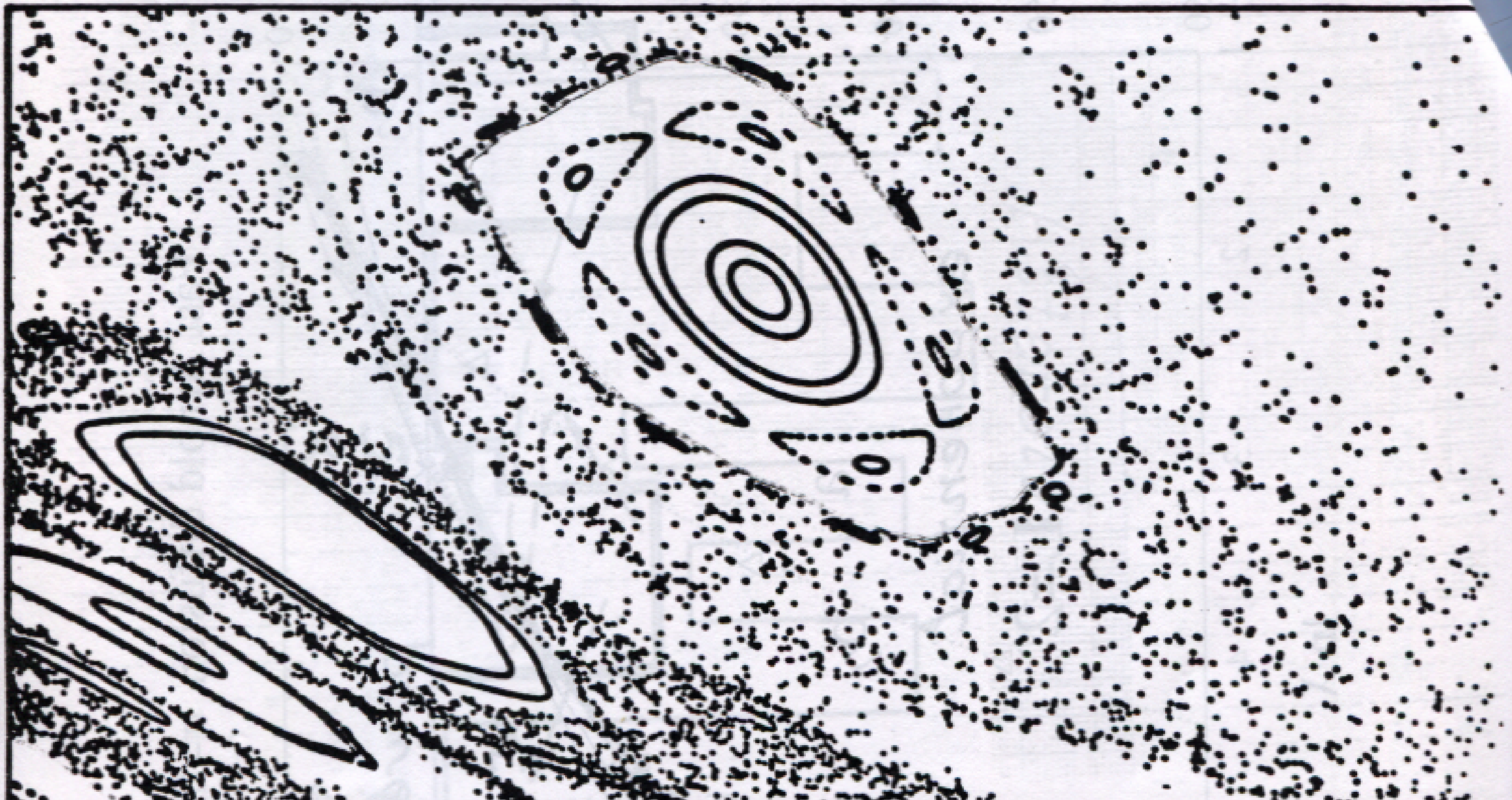
Chirikov, Vecheslavov, 1996

Fast Arnold diffusion, $\sigma=\ln(2/K)$, $\lambda=K^{-0.5}$, $D \sim w_s^{\,2}$



$\boxed{\lambda = 5}$ Vivaldi

$\bar{z} = z + \sin x ; \quad \bar{x} = x - \lambda \cdot \ln|\bar{z}|$

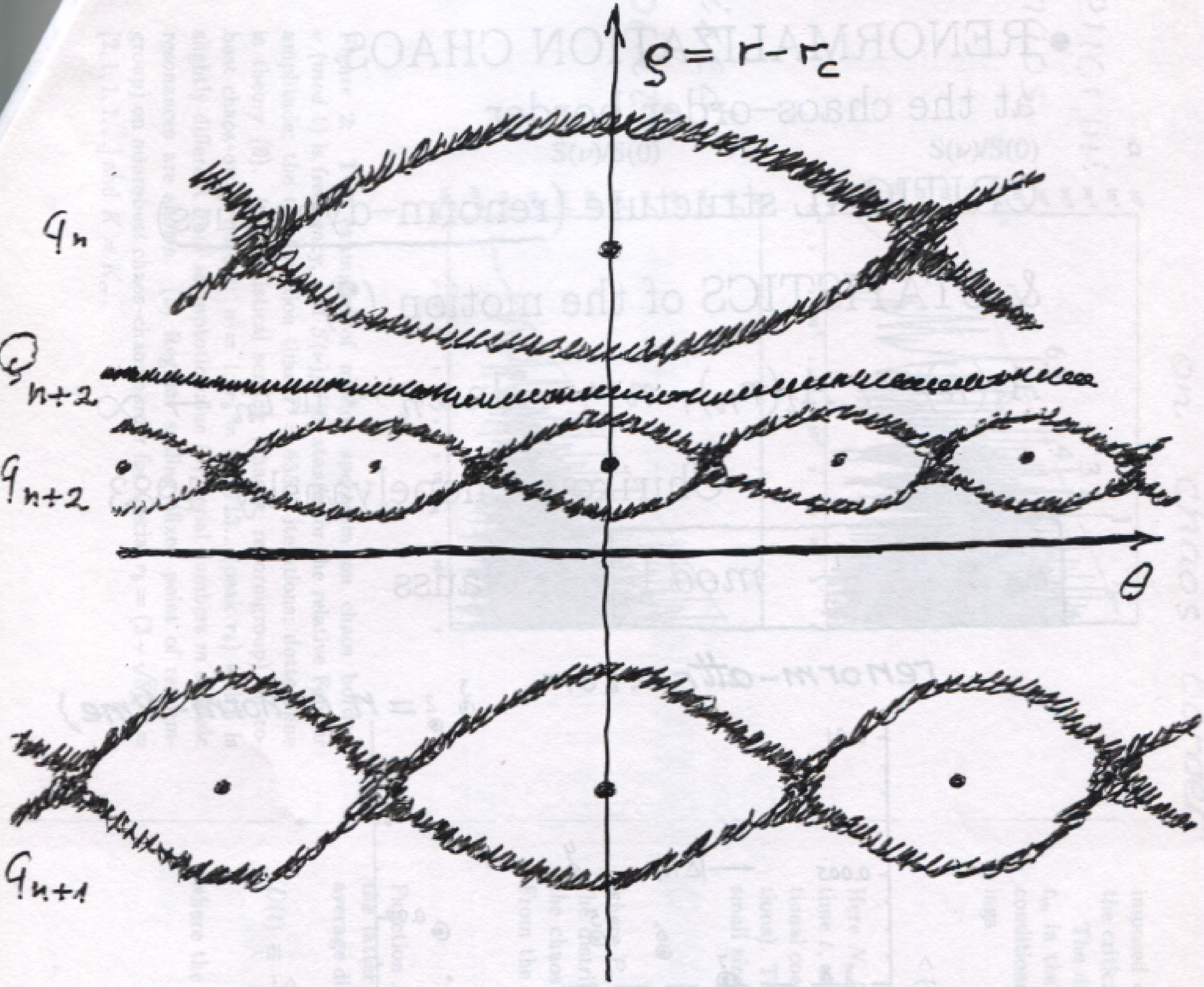


main chaos border

X

Fig 3b

Critical structure



stable periodic orbits

separatrix chaotic layers

? a bottleneck ?

critical orbit

- RENORMALIZATION CHAOS
  at the chaos–order border

CRITICAL structure (renorm–dynamics)

& STATISTICS of the motion (**?**)

$$A_i(n) \to A_i(r_n),\ n \sim -\ln x_n \sim \ln t_n \to \infty$$

Chirikov, Shepelyansky, 1983

$$\boxed{r_{n+1} = \frac{1}{r_n} \quad mod\, 1}$$ Gauss

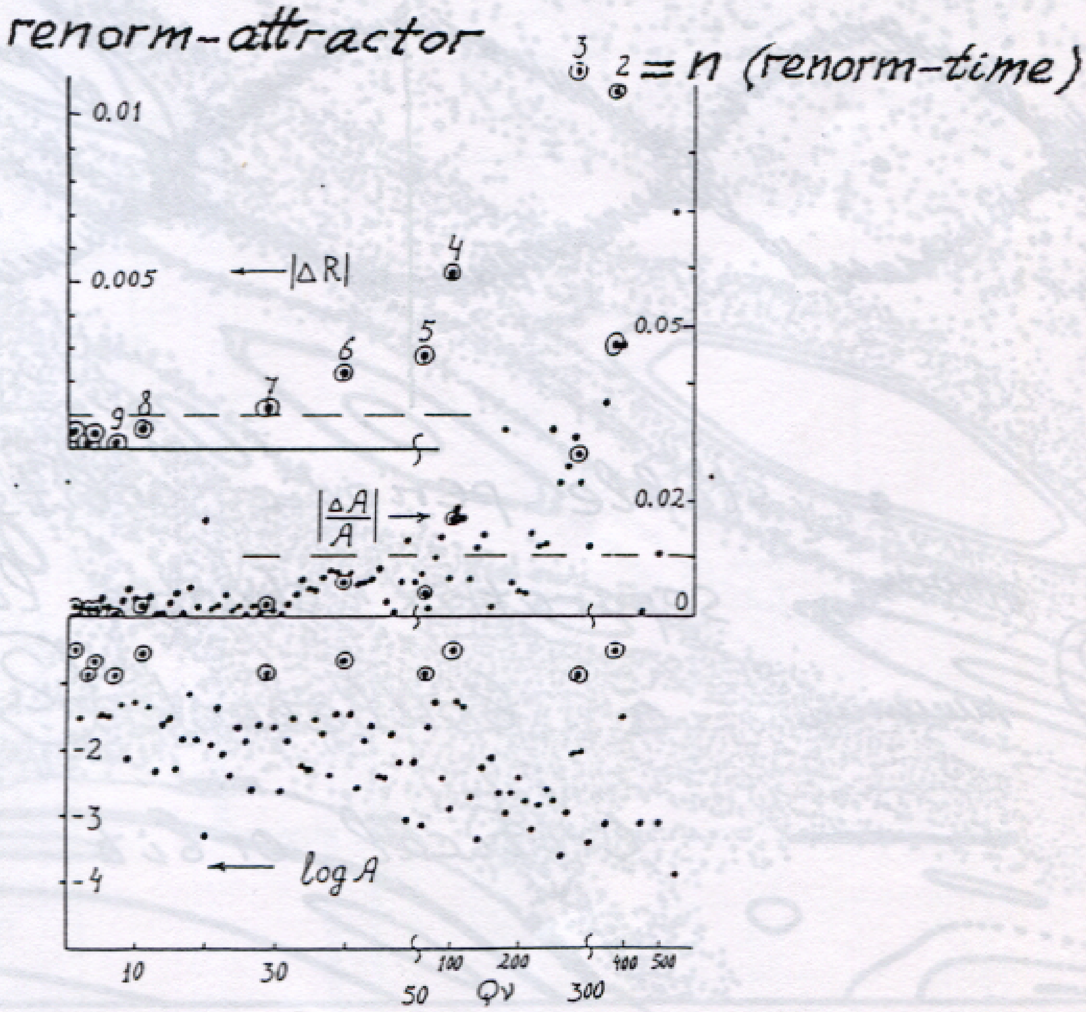


Fig. 4. Chaotic "renormalization attractor" in standard map (see the text). Arrows show the related scales. Notice changes in scale along $Q\nu$ axis. Numbers at upper points are the values of renormalization time n. The principal scales are marked by circles.

motion spectrum on chaos border

a renorm chaos

b fixed point

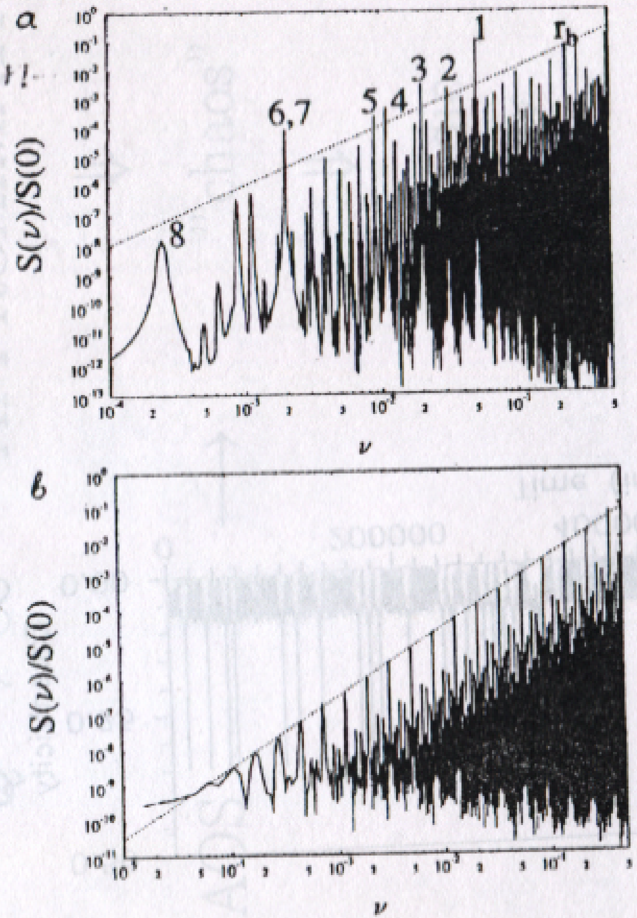


Figure 2: Two examples of motion spectrum on chaos border: ν (*mod* 1) is frequency, and $S(\nu)/S(0)$ stands for the relative Fourier amplitude; the total motion time $T = 65536$ iterations; dotted line is theory (8). (a) Statistical scaling (chaotic renormgroup) on robust chaos–order border: $n = 1$; $r_b = 0.23713...$ (peak r_b) which is slightly different from asymptotic value (6); serial numbers m of basic resonances are shown. (b) Regular scaling (fixed point of renormgroup) on nonrobust chaos–chaos border for special $r_b = (3-\sqrt{5})/2 = [2,1,1,1,...]$ and $K = K_{cr}$.

imposed on strong fluctuations which are generally characteristic for the critical phenomena.

The diffusion rate is determined by the statistic of sticking times t_m in the corresponding scale m. In average over time or the initial conditions (ergodicity), and assuming statistical independence of stickings

$$< (\Delta p)^2 > \approx \sum_m (\Delta p)_m^2 \approx K^2 \sum_m t_m^2 N_m + \frac{K^2}{2} C_0(K)\, t \quad (9)$$

Here $N_m(t)$ is the number of entries into scale m for the total motion time t, and the latter term describes the normal diffusion (with additional coefficient $C_0(K) \sim 1$ which accounts for the short–time correlations). The normal diffusion occupies the most of the time owing to a small size of regular domains. In turn, the number of entries

$$N_m = t \cdot P_m\,, \qquad P_m \sim \frac{A}{t_m^{c_P}} \quad (10)$$

where $P_m = P(t_m)$ stands for the Poincare recurrence statistic that is the distribution of the delays during the reflection (scattering) from the chaos border which is characterized by the critical exponent c_P. From the motion ergodicity

$$\frac{t_m N_m}{t} = t_m P_m = A_m \sim \frac{A}{q_m^2} \sim \frac{A}{t_m^{c_A}} \quad (11)$$

Function $A_m = A(t_m)$ plays a role of the sticking correlation, and from the latter estimate (scaling) $c_P = c_A + 1$. Whence, the asymptotic average diffusion rate

$$D(t) \equiv \frac{< (\Delta p)^2 >}{t} \to K^2 \sum_m t_m A_m \sim AK^2 t_{max}^{1-c_A} \sim AK^2 \cdot (At)^{c_D} \quad (12)$$

where the critical diffusion exponent

$$c_D = \frac{1-c_A}{1+c_A} \quad (13)$$

, 1996

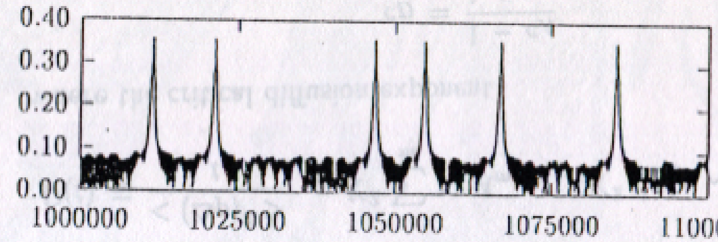
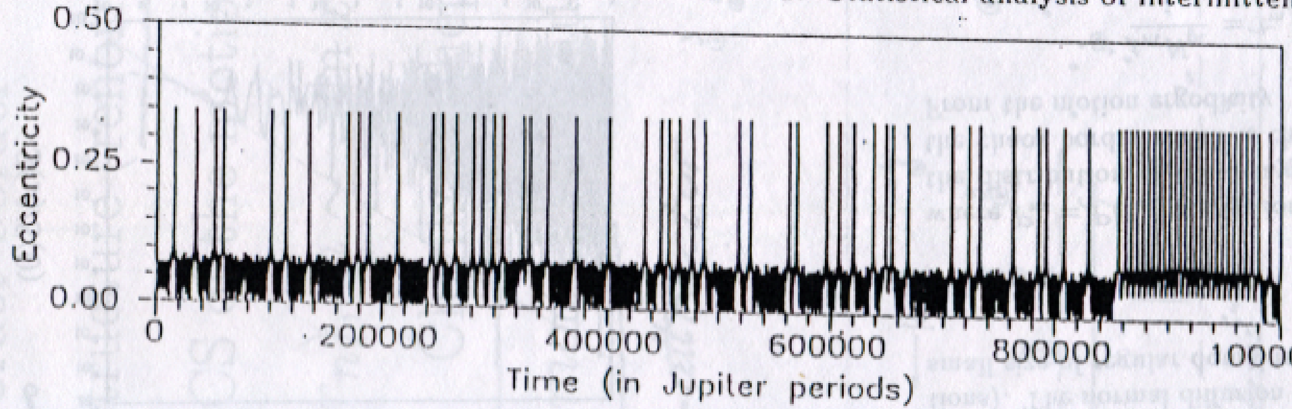


Figure 1. Eccentricity versus time for an intermittent trajectory.

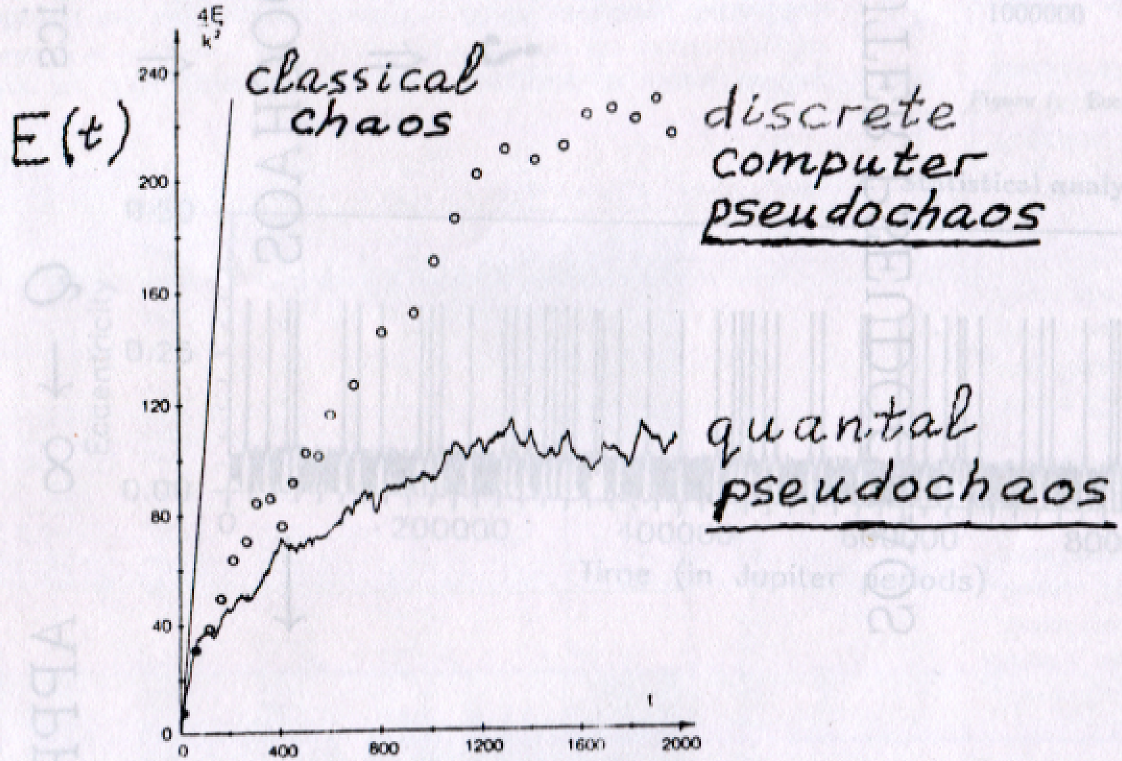
2. Statistical analysis of intermittent trajectories



asteroid in 3/1 resonance

$$\begin{array}{ccc} \text{QUANTUM} & \longrightarrow & \text{classical} \\ \text{mechanics} & Q \to \infty & \text{APPROXIMATION} \\ \Downarrow & & \Downarrow \\ \text{PSEUDOCHAOS} & \longrightarrow & \text{"chaos"} \\ \Downarrow & & \Downarrow \\ ? & & \text{ergodic} \\ & & \text{theory} \end{array}$$

COMPUTER PSEUDOCHAOS



$$\bar{n} = n + [k \cdot \sin \theta]$$
$$\bar{\theta} = \theta + T \cdot \bar{n}.$$

Fig. 21. "Quantum localization" in the classical discrete map (7.13) [38]: the straight line represents classical diffusion, the wiggly curve shows quantum localization, and circles are for the discrete model; t is the number of iterations; $k = 20$, $K = 5$.