Localization and Spreading of Diseases in Networks

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www2012:

Tim Berners-Lee:
"A Google search shows you the eigenvectors of society".

More correctly:
it shows you only the principal vector of society.
$\longrightarrow$ "mostly"

The standard SIS model:

Infected vertices become susceptible with unit rate:

$$I \xrightarrow{1} S,$$

and each susceptible vertex becomes infected by its infective neighbour with the infection rate λ:

$$\begin{array}{c} S \xrightarrow{\lambda} I. \\ \Uparrow I_{nn} \end{array}$$

Seminal papers:

Pastor-Satorras and Vespignani (2001):

$$\lambda_c = \langle q \rangle / \langle q^2 \rangle.$$

If $\langle q^2 \rangle \to \infty$, then $\lambda_c = 0$.
If $\langle q^2 \rangle < \infty$, then $\lambda_c > 0$.

:(

[For the SIR model, $\lambda_c = \langle q \rangle / (\langle q^2 \rangle - \langle q \rangle)$.]

Their approximations:

(1) Correlations between infectives and susceptibles were neglected.

(2) A random graph is substituded with its annealed counterpart !!!

(3) $N \longrightarrow \infty$.

Without approximation 2, for an individual graph:

Y. Wang, D. Chakrabarti, C. Wang, and C. Faloutsos (2003):

$$\lambda_c = 1/\Lambda_1$$

Λ_1 is the eigenvalue of the principal eigenvector of the adjacency matrix.
$\Lambda_1 \sim \sqrt{q_{max}}$, $q_{max}(N \to \infty) \to \infty$ for the ER graphs, and so $\lambda_c(N \to \infty) \to 0$.

SIS: the transition is not thermodynamic

The Ising model, percolation:
a thermodynamic phase transition — it exists only if $N \to \infty$.

Sync, neural networks, the SIS model:
a sharp transition is even in finite systems, even for two pendulums.

The SIS on an individual graph,

(but not on a statistical ensemble!)

The evolution equation:

$$\frac{d\rho_i(t)}{dt} = -\rho_i(t) + \lambda[1-\rho_i(t)]\sum_{j=1}^{N} A_{ij}\rho_j(t).$$

The steady state:

$$\rho_i = \frac{\lambda\sum_j A_{ij}\rho_j}{1+\lambda\sum_j A_{ij}\rho_j}.$$

$$\rho_i = \sum_{\Lambda} c(\Lambda) f_i(\Lambda).$$

$$c(\Lambda) = \lambda \sum_{\Lambda'} \Lambda' c(\Lambda') \sum_{i=1}^{N} \frac{f_i(\Lambda) f_i(\Lambda')}{1 + \lambda \sum_{\widetilde{\Lambda}} \widetilde{\Lambda} c(\widetilde{\Lambda}) f_i(\widetilde{\Lambda})}.$$

If $\tau \equiv \lambda\Lambda_1 - 1 \ll 1$, then

$$\rho \equiv \sum_{i=1}^{N} \rho_i / N \approx \alpha_1 \tau,$$

$$\alpha_1 = [\sum_{i=1}^{N} f_i(\Lambda_1)] / [N \sum_{i=1}^{N} f_i^3(\Lambda_1)].$$

Localized and delocalized eigenvectors:

Normalization: $\sum_i^N f_i^2(\Lambda) = 1$
The inverse participation ratio:

$$IPR(\Lambda) \equiv \sum_{i=1}^{N} f_i^4(\Lambda).$$

A localized state: $IPR(\Lambda) \overset{N\to\infty}{\longrightarrow} 0$.

A delocalized state: $IPR(\Lambda) \overset{N\to\infty}{\longrightarrow} const$.

It depends on a particular network realization.

A delocalized principal eigenvector: $f_i(\Lambda) = O(1/\sqrt{N})$, so

$$\alpha_1 = O(1)$$

A localized principal eigenvector:

$$\alpha_1 = O(1/N)$$

— the disease is localized on a finite number of vertices in contrast to "localization" within k-cores (a finite fraction of vertices), see Kitsak (2010), Castellano and Pastor-Satorras (2012).

Unweighted networks:

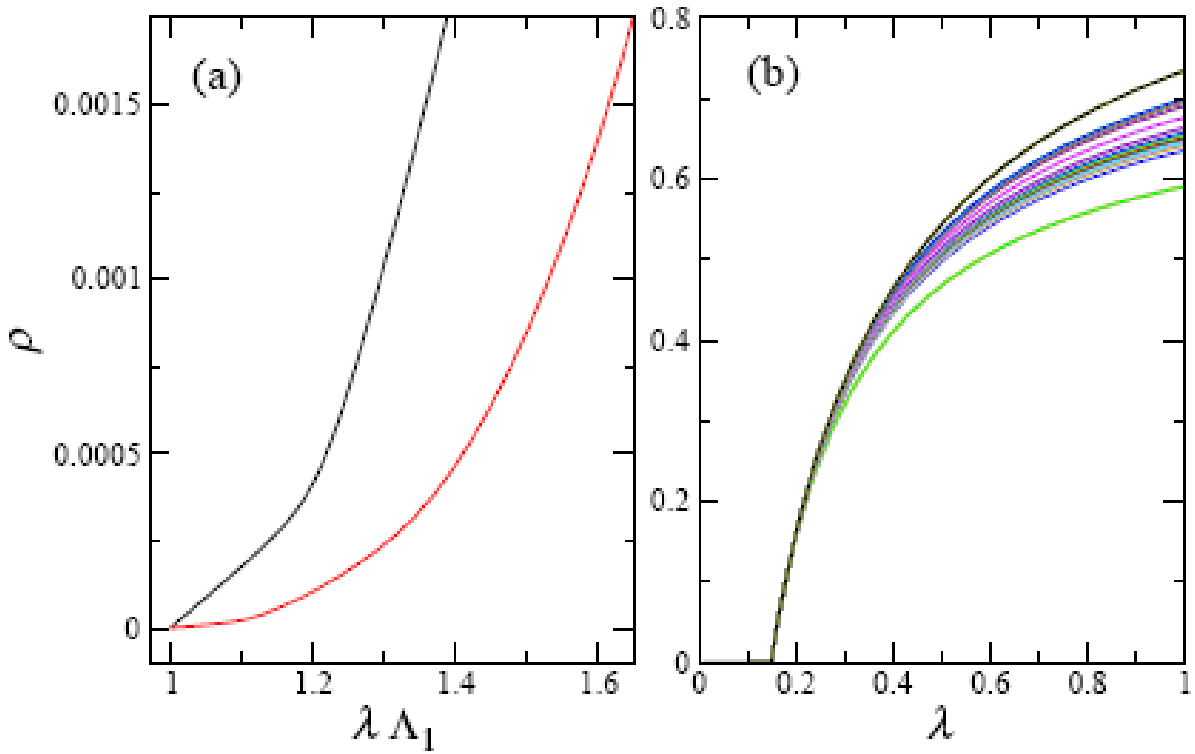
The first iteration ($\mathbf{g}^{(0)}$=1):

$$\Lambda_1^{(1)} = \frac{1}{\langle q^2 \rangle N} \sum_{i,j} q_i A_{ij} q_j = \Lambda_{MF} + \frac{\langle q \rangle \sigma^2 r}{\langle q^2 \rangle},$$

$$IPR_{MF} = \langle q^4 \rangle / [N \langle q^2 \rangle^2] \sim O(1/N)$$

where r is the Pearson coefficient, $\Lambda_{MF} \equiv \langle q^2 \rangle / \langle q \rangle$.
So assortative degree–degree correlations increase Λ_1 and decrease λ_c.

Weighted and unweighted real-world nets:



(a) astro-phys (upper) and cond-mat-2005 (lower) weighted networks. (b) Karate-club network (unweighted). The lower curve accounts for only the eigenstate Λ_1. The most upper curve is the "exact" ρ.

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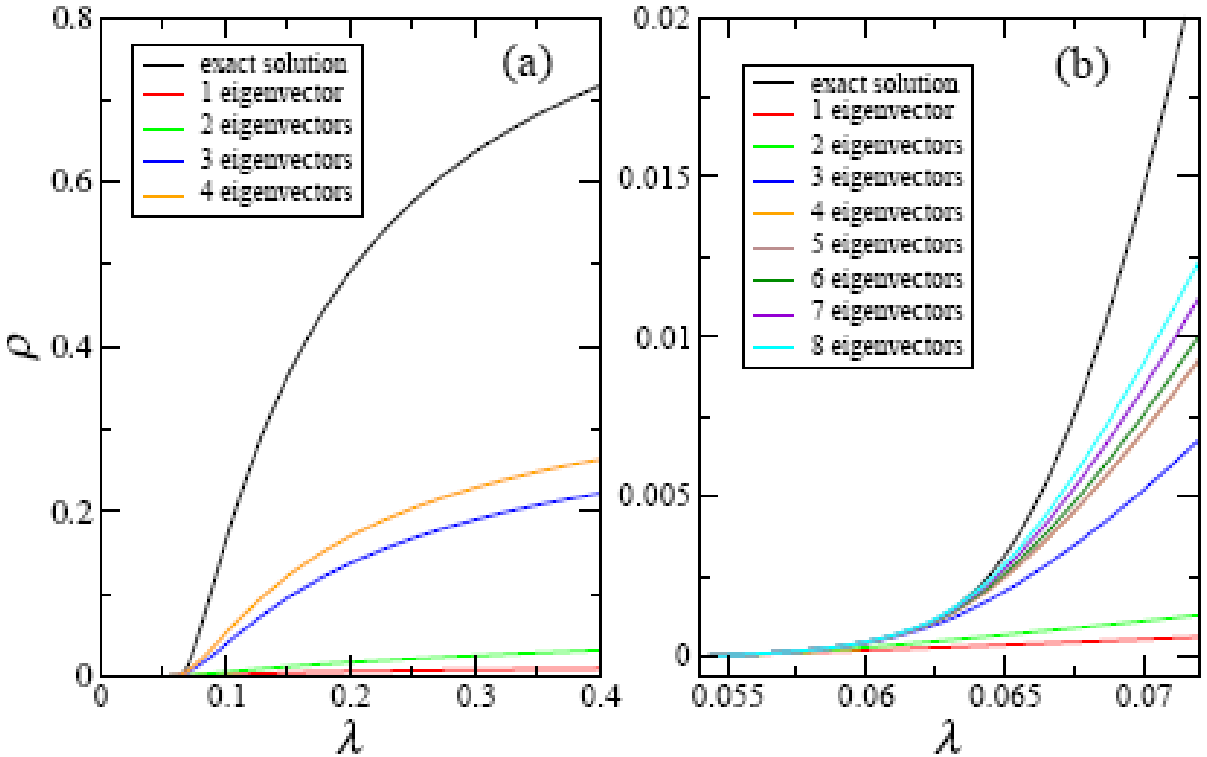
An uncorrelated scale-free net:



A scale-free network of 10^5 vertices generated by the static model with $\gamma = 4$, $\langle q \rangle = 10$. (b) Zoom of the prevalence at λ near $\lambda_c = 1/\Lambda_1$. I, II eigenvectors are localized, III is delocalized.

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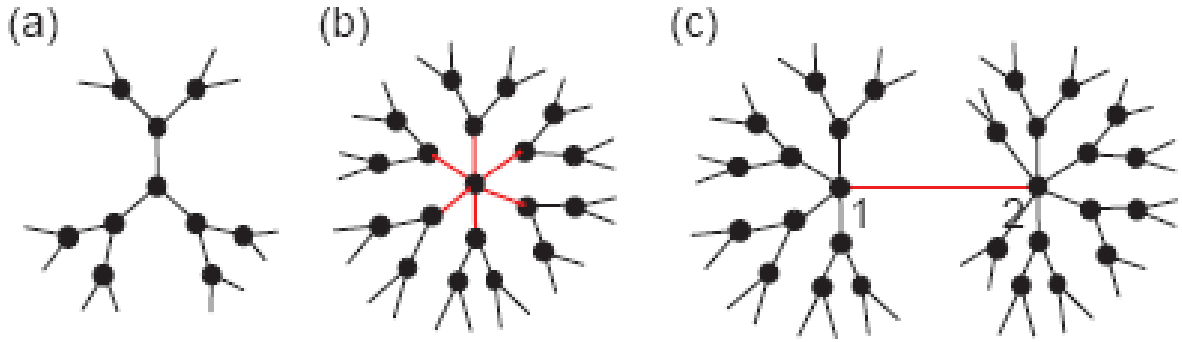
A Bethe lattice with a hub:



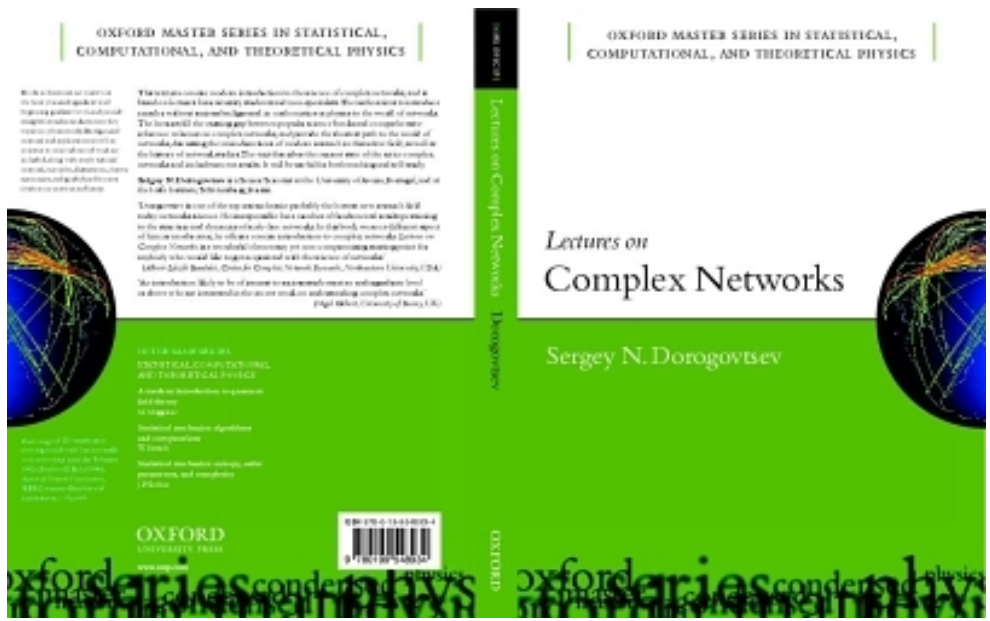
Find Λ_1 if $B = k - 1$ is the branching and then B substitute with $\langle B \rangle$.
The localization of the principal eigenvector at a hub with q_{max} occurs if

$$\Lambda_1 = q_{max}/\sqrt{q_{max} - B} \geq \Lambda_d > \approx \Lambda_{MF}$$

Conclusion

If the principal eigenvector is localized,
then right above the threshold $1/\Lambda_1$,
the disease is localized on a finite number of vertices.
In this case, a real epidemic affecting a finite fraction of vertices occurs after a smooth crossover, and
the notion of the epidemic threshold is meaningless.

Lectures on
Complex Networks



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