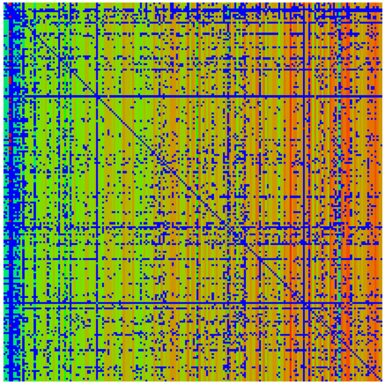

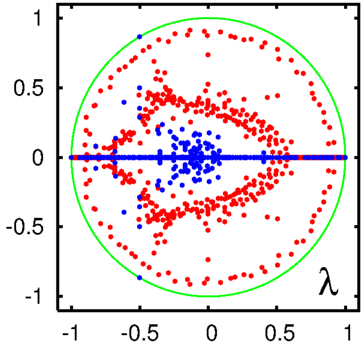

Google matrix of directed networks

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Perron-Frobenius operators

discrete ***Markov process***:

$$p_i(t+1) = \sum_j G_{ij}\, p_j(t)$$

with probabilities $p_i(t) \geq 0$ and the ***Perron-Frobenius*** matrix G such that:

$$\sum_i G_{ij} = 1 \quad , \quad G_{ij} \geq 0 \; .$$

For any vector v :

$$\Rightarrow \qquad \|Gv\|_1 \leq \|v\|_1$$

$\Rightarrow$ complex eigenvalues $|\lambda_j| \leq 1$ and (at least) one eigenvalue $\lambda_1 = 1$ and its right eigenvector P is the stationary distribution:

$$P = \lim_{t\to\infty} p(t)$$

provided λ_1 is <u>not degenerate</u> !

Google matrix for directed networks

Define the ***adjacency matrix*** A by $A_{ij} = 1$ if there is a link from the node j to i in the network (of size N) and $A_{ij} = 0$ otherwise. Let $S_{ij} = A_{ij} / \sum_i A_{ij}$ and $S_{ij} = 1/N$ if $\sum_i A_{ij} = 0$ (dangling nodes). S is of Perron-Frobenius type but for many networks the eigenvalue $\lambda_1 = 1$ is highly degenerate [$\Rightarrow$ convergence problem to arrive at the stationary limit of $p(t+1) = S\, p(t)$].

Therefore define the ***Google matrix***:

$$G(\alpha) = \alpha S + (1-\alpha)\, \frac{1}{N}\, e e^T$$

where $e = (1, \ldots, 1)^T$ and $\alpha = 0.85$ is a typical damping factor. Here there is unique eigenvector for $\lambda_1 = 1$ called the ***PageRank*** P and the convergence goes with α^t.

(***CheiRank*** P^* by replacing: $A \to A^* = A^T$).

Ulam Method

(Ermann, Shepelyansky (2010), KF, Shepelyansky (2010))

to construct a ***Perron-Frobenius matrix*** as discrete approximation for the PF operator of dynamical systems with mixed phase space:

- Subdivide phase space in discrete cells.

- Iterate (for a very long time) a classical trajectory and attribute a new number to each new cell which is entered for the first time. At the same time count the number of transitions from cell i to cell j ($\Rightarrow\ n_{ji}$).

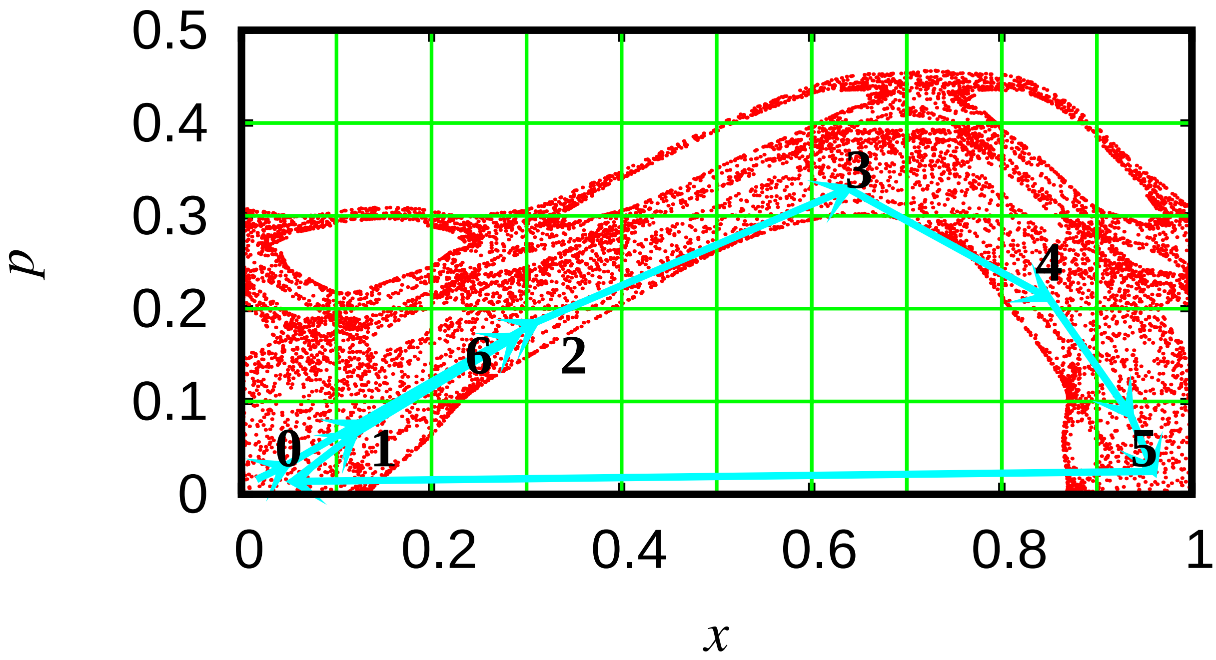
- $\Rightarrow$ The matrix

$$G_{ji} = \frac{n_{ji}}{\sum_l n_{li}}$$

is of Perron-Frobenius type : $G_{ji} \geq 0$, $\sum_j G_{ji} = 1$.

Chirikov Standard map

$$p_{n+1} \;=\; p_n + \frac{k}{2\pi}\sin(2\pi\, x_n)$$
$$x_{n+1} \;=\; x_n + p_{n+1} \quad , \quad k = k_c = 0.971635406$$



Arnoldi method

to (partly) diagonalize large sparse non-symmetric $d \times d$ matrices:

- choose an initial normalized vector ξ_0 (random or "otherwise")

- determine the ***Krylov space*** of dimension n (typically: $1 \ll n \ll d$) spanned by the vectors: $\xi_0,\ G\,\xi_0,\ \ldots,\ G^{n-1}\xi_0$

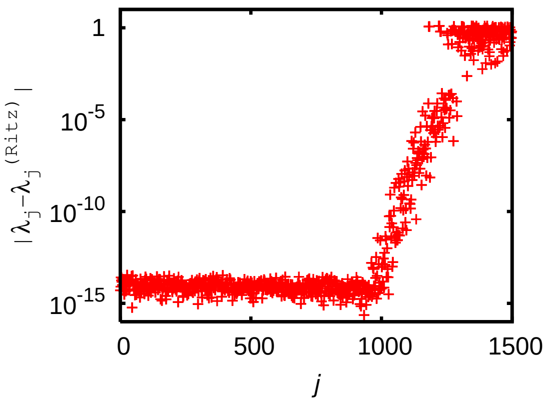
- determine by ***Gram-Schmidt*** orthogonalization an orthonormal basis $\{\xi_0,\ \ldots,\xi_{n-1}\}$ and the representation of G in this basis:

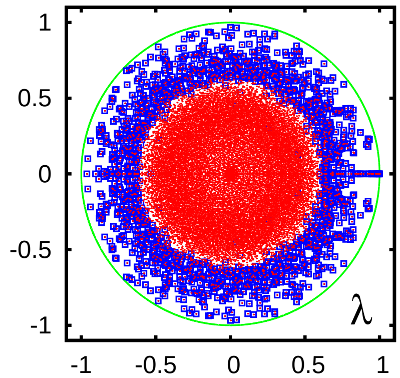
$$G\,\xi_k = \sum_{j=0}^{k+1} H_{jk}\,\xi_j$$

- diagonalize the ***Arnoldi matrix*** H which has ***Hessenberg*** form:

$$H = \begin{pmatrix} * & * & \cdots & * & * \\ * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & * & * \\ \hline 0 & 0 & \cdots & 0 & * \end{pmatrix}$$

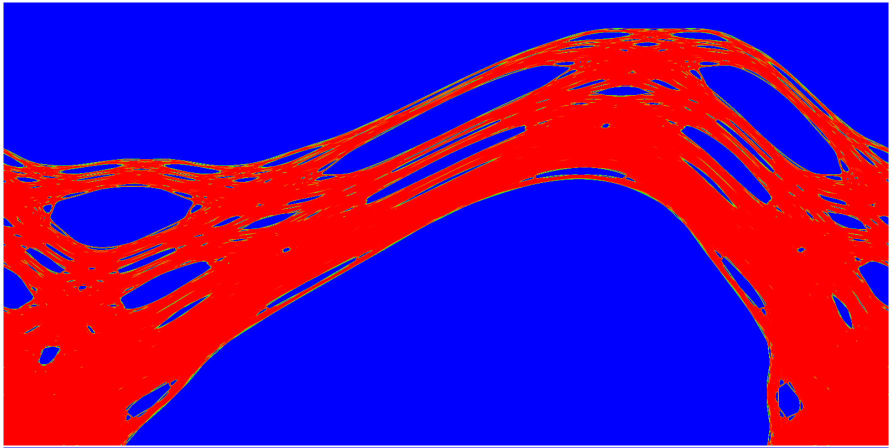
which provides the ***Ritz eigenvalues*** that are very good aproximations to the “largest” eigenvalues of A.





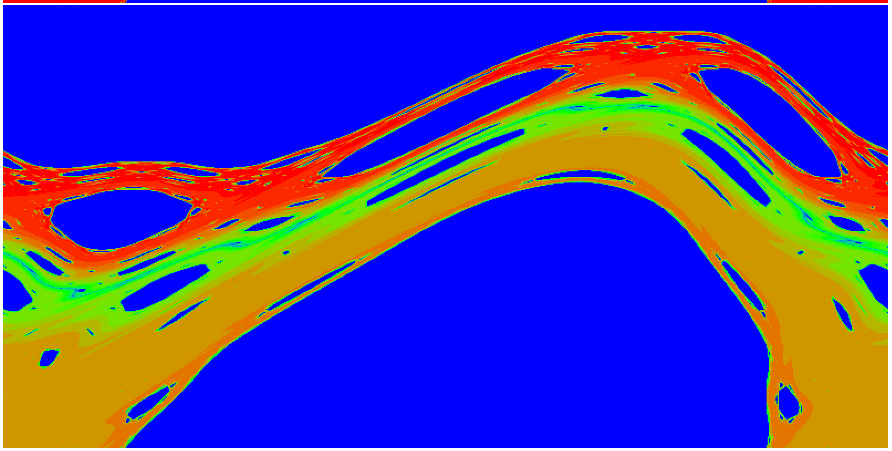
$M \times M/2$ cells, $M = 280$, $d = 16609$, $n = 1500$

Eigenvectors



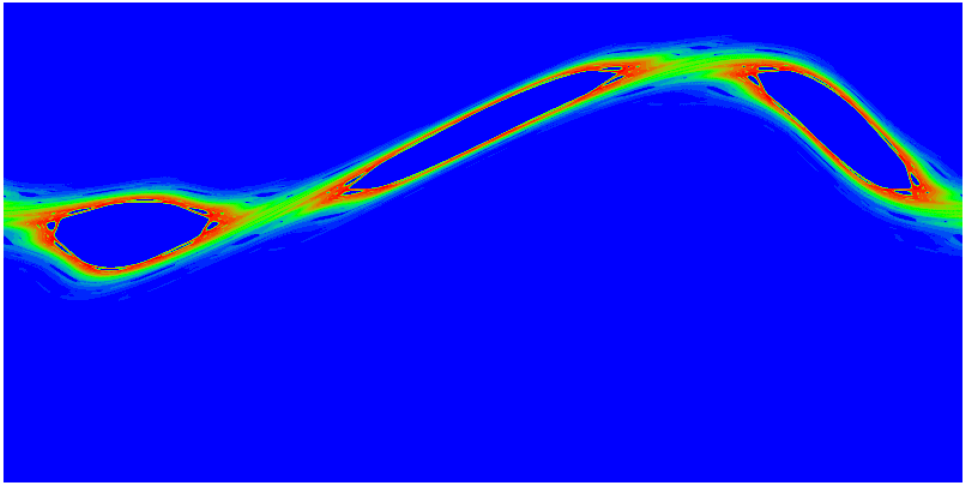
$\lambda_1 = 1$

$M = 1600$
$d = 494964$
$n = 3000$



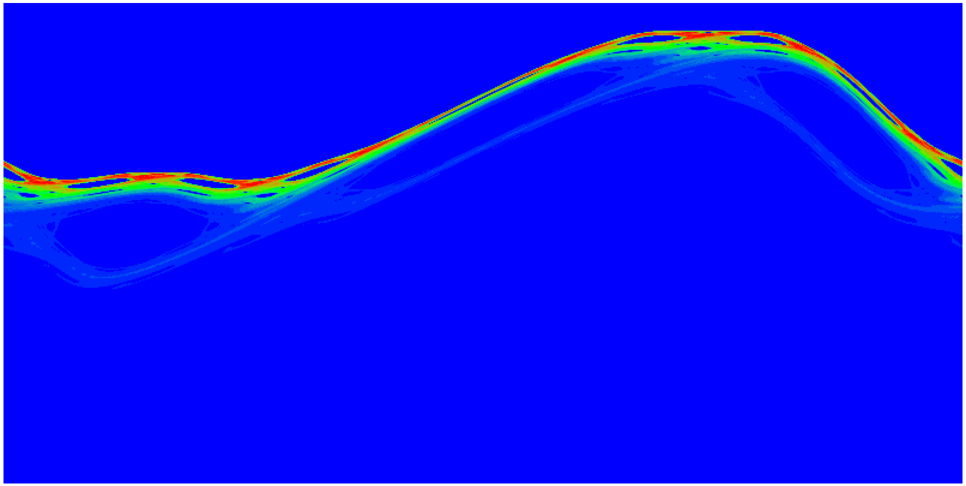
$\lambda_1 = 0.99980431$

$M = 800$
$d = 127282$
$n = 2000$



λ_6
-0.49699831
$+i\, 0.86089756$
$\approx |\lambda_6|\, e^{i\, 2\pi/3}$

$M = 800$
$d = 127282$
$n = 2000$



$\lambda_{19} =$
-0.71213331
$+i\, 0.67961609$
$\approx |\lambda_{19}|\, e^{i\, 2\pi(3/8)}$

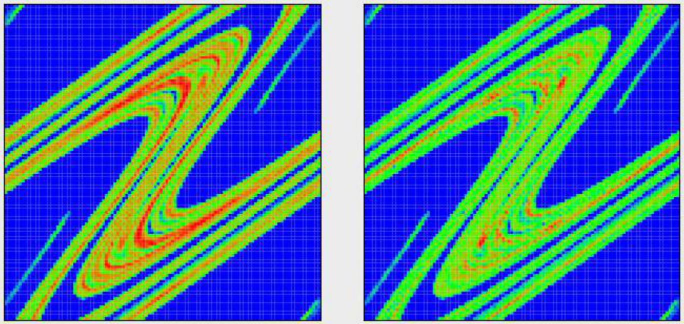
$M = 800$
$d = 127282$
$n = 2000$

Ulam method for dissipative systems

(Ermann, Shepelyansky (2010))

Scattering

$$\begin{cases} \bar{y} &= y + K\sin(x + y/2) \\ \bar{x} &= x + (y + \bar{y})/2 \pmod{2\pi} \end{cases}$$

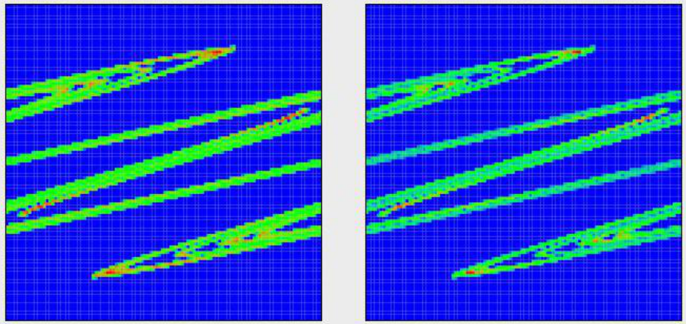


$N = 110 \times 110,\ K = 7, a = 2$

$\lambda_1 = 0.756 \quad \lambda_3 = -0.01 + i0.513$

Dissipation

$$\begin{cases} \bar{y} &= \eta y + K\sin x \\ \bar{x} &= x + \bar{y} \pmod{2\pi} \end{cases}$$



$N = 110 \times 110,\ K = 7, \eta = 0.3$

$\lambda_1 = 1 \quad \lambda_3 = -0.258 + i0.445$

Fractal Weyl law

$N_\gamma =$ number of Gamow eigenstates that have escape rates $\gamma_j = -2\ln|\lambda_j|$ in a finite bandwidth $0 \le \gamma_j \le \gamma_b$.

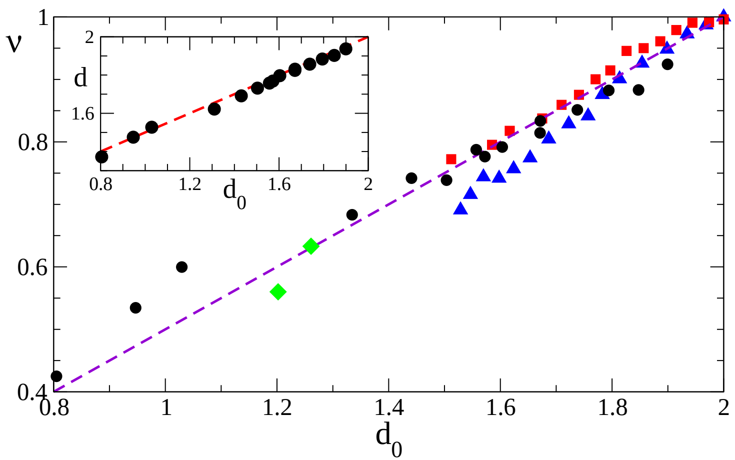
Fractal Weyl law for open quantum systems :

(e.g. Shepelyansky (2008))

$N_\gamma \propto N^{d-1} \propto \hbar^{-(d-1)}$ where d is a fractal dimension of a strange invariant set formed by orbits non-escaping in the future.

Fractal Weyl law for Ulam networks : $N_\gamma \propto N^\nu \propto N^{d_0/2}$

(Ermann, Shepelyansky (2010))



d_0 = dimension of invariant set of strange repeller (formed by orbits non-escaping in the future **and** in the past).

$\nu = d_0/2$

$d =$ dimension of orbits non-escaping in the future

$d = d_0/2 + 1$ (inset)

University networks

(KF, Georgeot, Shepelyansky (2011))

In realistic WWW networks invariant subspaces of nodes create large degeneracies of λ_1 (or λ_2 if $\alpha < 1$) which is very problematic for the Arnoldi method.

Therefore determine the ***invariant subspaces*** as follows:

Let $N_c = bN$ a certain fraction of the network size N (e.g. $b = 0.1$).

- For a given initial node i_0 determine a sequence of node sets S_n by $S_0 = \{i_0\}$ and S_{n+1} is the set containing all nodes of S_n and those which can be reached by a link from a node in S_n.

- If $S_n = S_{n+1}$ with at most N_c elements for some $n \Rightarrow S_n$ is an ***invariant subspace***.

- If for some n the set S_n contains a dangling node (connected by construction to any other node) or if S_n contains more than N_c elements $\Rightarrow i_0$ is identified as a node belonging to the ***core space*** (space of nodes not belonging to an invariant subspace).
- Repeat the procedure for every network node as potential initial node except for those nodes which are already identified as subspace nodes. If for some n the set S_n contains a previously found core space node $\Rightarrow i_0$ also belongs to the core space.
- Merge all subspaces with common members. In this way one obtains a decomposition of the network in many ***separate subspaces*** with N_s nodes and a "big" ***core space***.

This procedure can be efficiently implemented as a computer program. It turns out that for most networks the exact choice of b is not important (e.g. $b = 0.1$ or $b = 0.9$) as long as $b = \mathcal{O}(1)$. Note that a core space node may have a link to an invariant subspace but a subspace node may not have a link to another subspace or the core space.

The decomposition in subspaces and a core space implies a block structure of the matrix S:

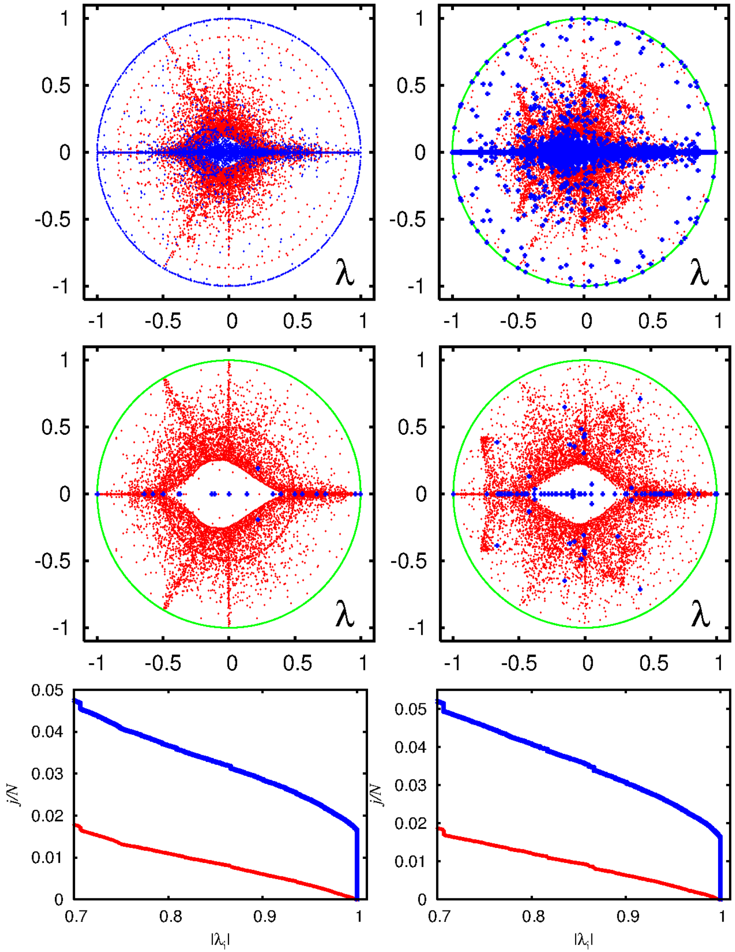
$$S = \begin{pmatrix} S_{ss} & S_{sc} \\ 0 & S_{cc} \end{pmatrix}$$

where S_{ss} is block diagonal according to the subspaces. The subspace blocks of S_{ss} are all matrices of PF type with at least one eigenvalue $\lambda_1 = 1$ explaining the high degeneracies.

To determine the spectrum of S apply:

- Exact (or Arnoldi) diagonalization on each subspace.

- The Arnoldi method to S_{cc} to determine the largest core space eigenvalues λ_j (note: $|\lambda_j| < 1$). The largest eigenvalues of S_{cc} are no longer degenerate but other degeneracies are possible (e.g. $\lambda_j = 0.9$ for Wikipedia).



Cambridge 2006 (left), $N = 212710$, $N_s = 48239$

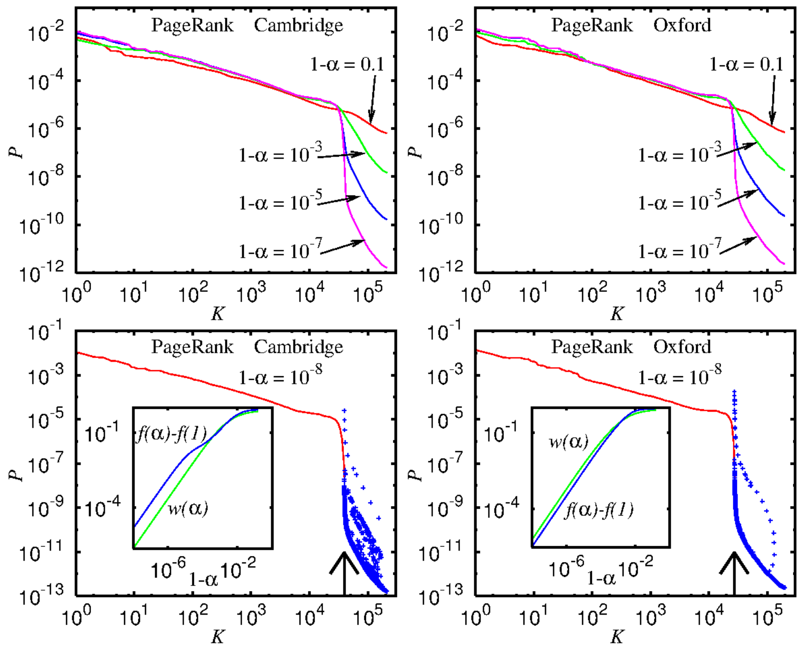
Oxford 2006 (right), $N = 200823$, $N_s = 30579$

Spectrum of S (upper panels), S^* (middle panels) and dependence of rescaled level number on $|\lambda_j|$ (lower panels).

Blue: subspace eigenvalues

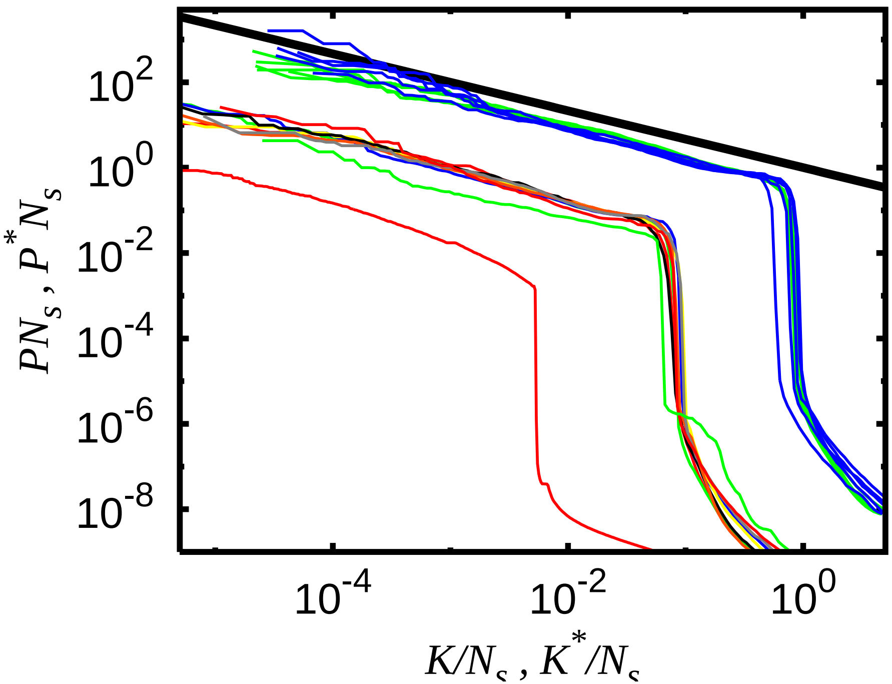
Red: core space eigenvalues (with Arnoldi dimension $n_A = 20000$)

PageRank for $\alpha \to 1$:



$$P = \underbrace{\sum_{\lambda_j=1} c_j\,\psi_j}_{\text{subspace contributions}} + \sum_{\lambda_j\neq 1} \frac{1-\alpha}{(1-\alpha)+\alpha(1-\lambda_j)}\, c_j\,\psi_j\ .$$

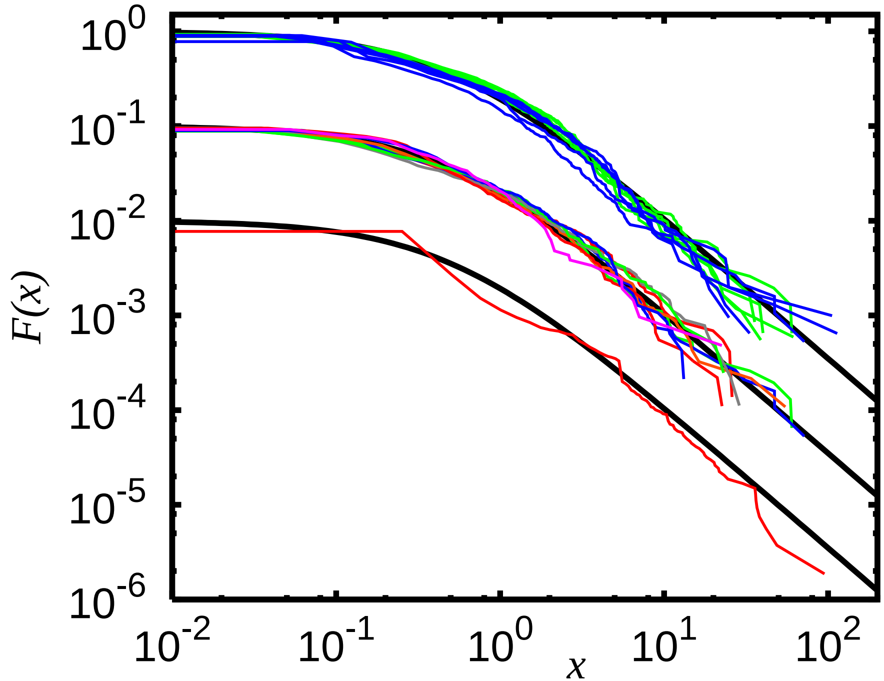
Rescaled PageRank at $\alpha = 1 - 10^{-8}$:



Top: Cambridge, Oxford 2002-2006; middle: other universities; bottom: Wikipedia*; black line $\propto K^{-2/3}$; N_s = sum of all subspace dimensions.

Distribution of dimensions of invariant subspaces

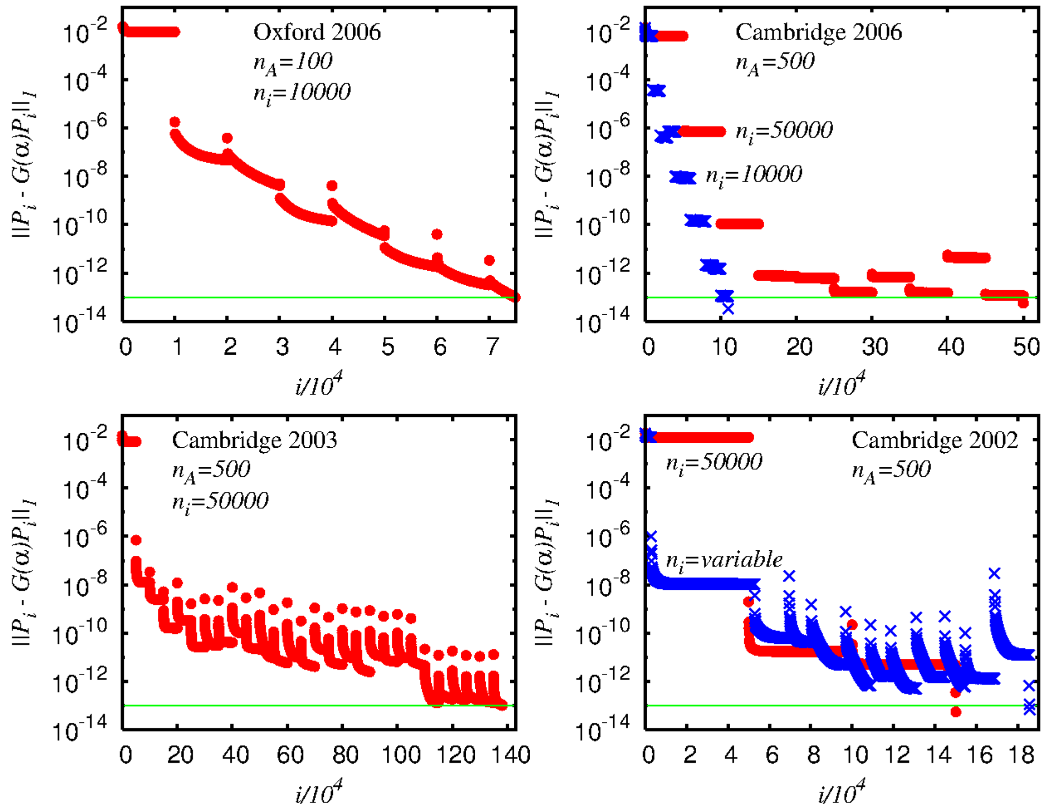
$F(x) =$ fraction of invariant subspaces with dimension larger than $x\langle d\rangle$ where $\langle d\rangle =$ average subspace dimension.



Top: Cambridge, Oxford 2002-2006; middle: other universities; bottom: Wikipedia*; black line: $F(x) = 1/(1+2x)^{3/2}$.

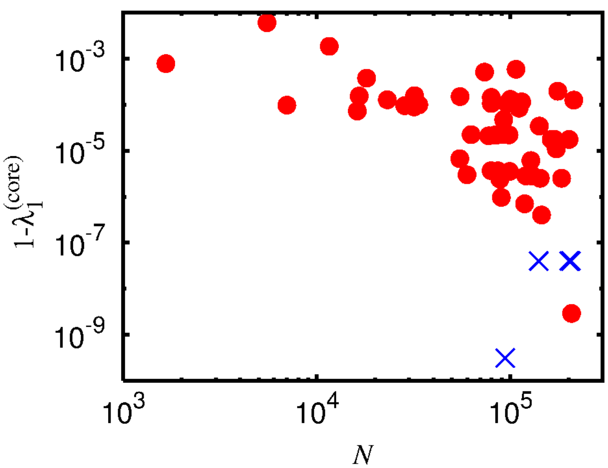
Numerical PageRank method for $\alpha \to 1$

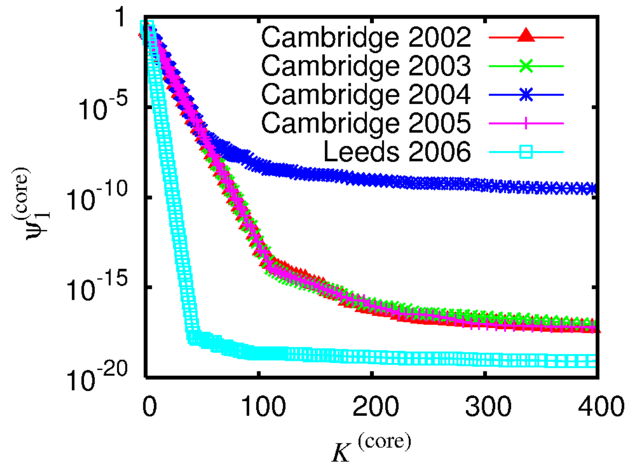
Combination of power method and Arnoldi diagonalization :



Here: $\alpha = 1 - 10^{-8}$

Core space gap and quasi-subspaces





Left: Core space gap $1 - \lambda_1^{(core)}$ vs N for certain british universities.

Red dots for gap $> 10^{-9}$; blue crosses (moved up by 10^9) for gap $< 10^{-16}$.

Right: first core space eigenvecteur for universities with gap $< 10^{-16}$ or gap $= 2.91 \times 10^{-9}$ for Cambridge 2004.

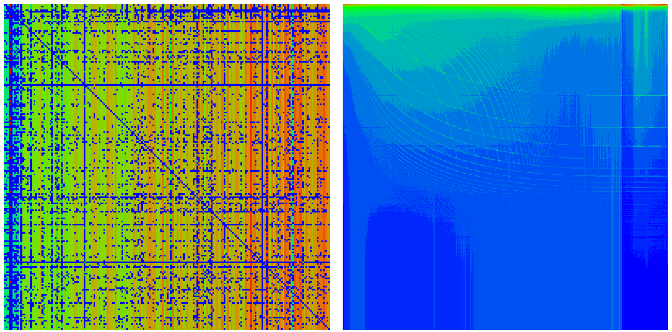
Core space gaps $< 10^{-16}$ correspond to ***quasi-subspaces*** where it takes quite many "iterations" to reach a dangling node.

Twitter network

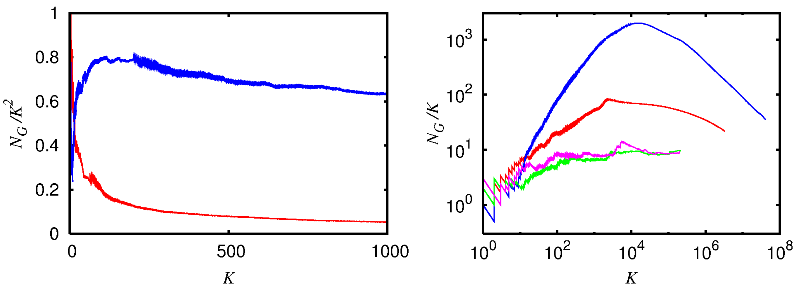
(KF, Shepelyansky (2012), preprint)

Twitter 2009 : $N = 41652230$ nodes, $N_\ell = 1468365182$ network links.

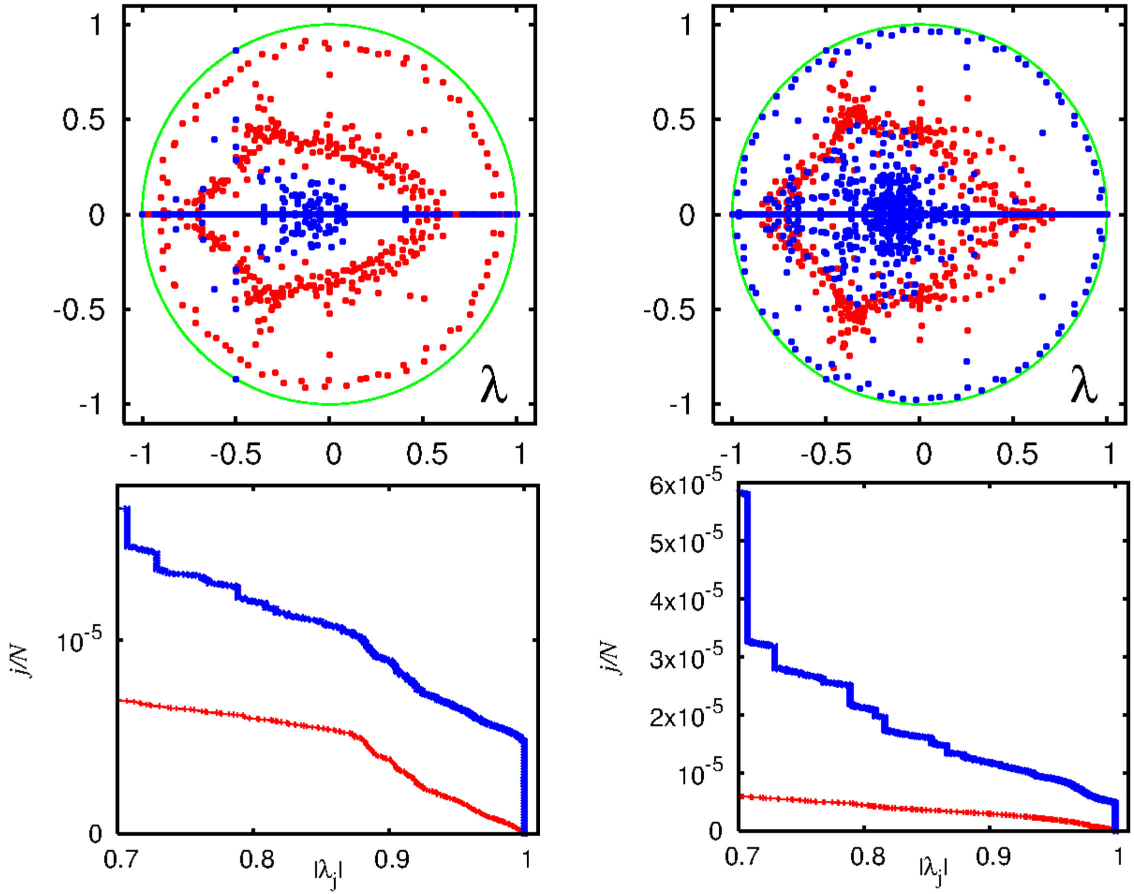
Matrix structure in K-rank order:



Number N_G of non-empty matrix elements in $K \times K$-square:

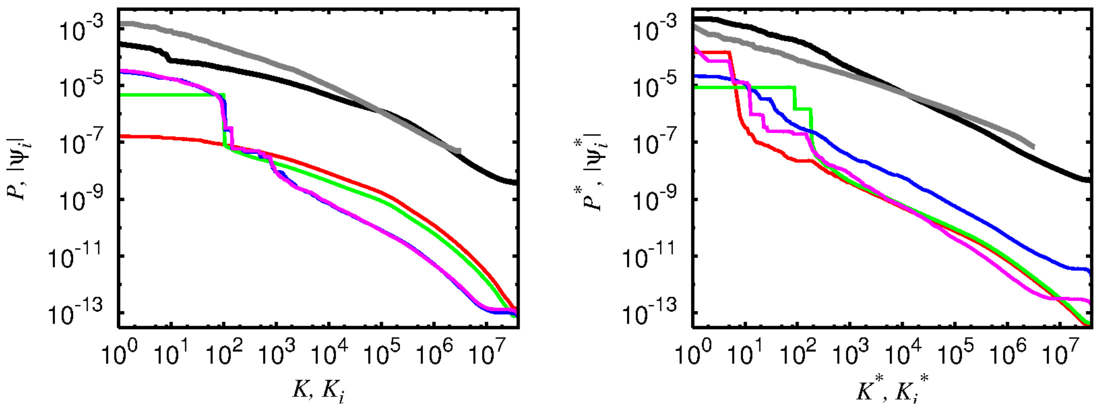


Spectrum

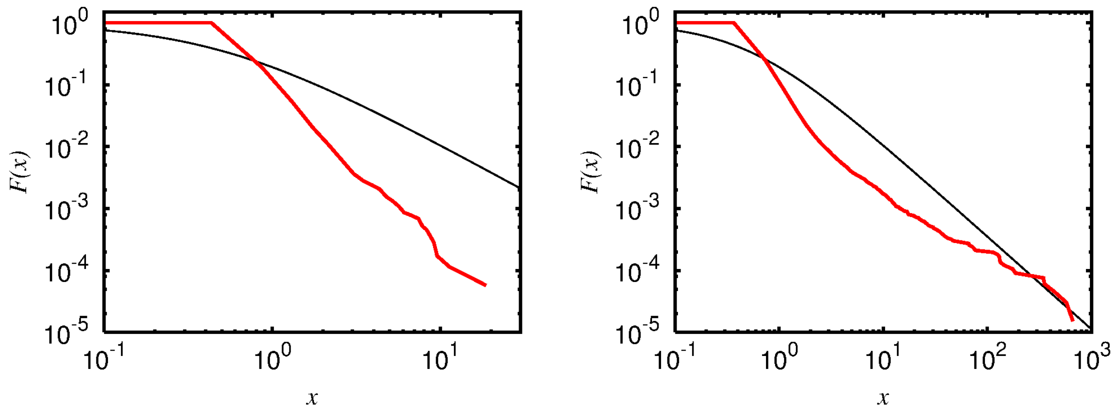


$n_A = 640 \quad \Rightarrow \quad 250$ GB of RAM memory.

PageRank, CheiRank, eigenvectors



Subspace distribution

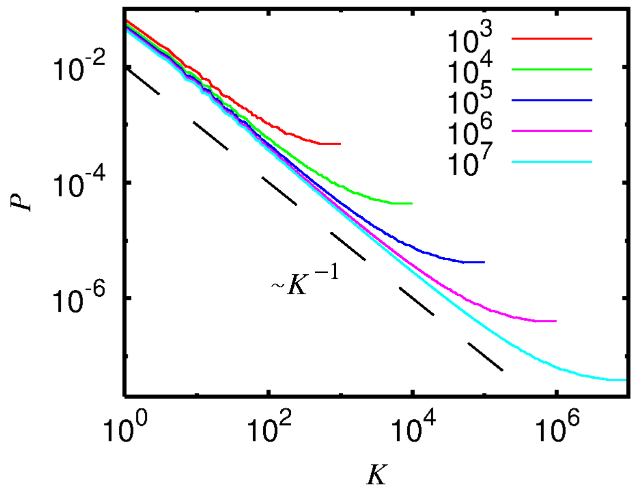


Black line: $F(x) = 1/(1+2x)^{3/2}$.

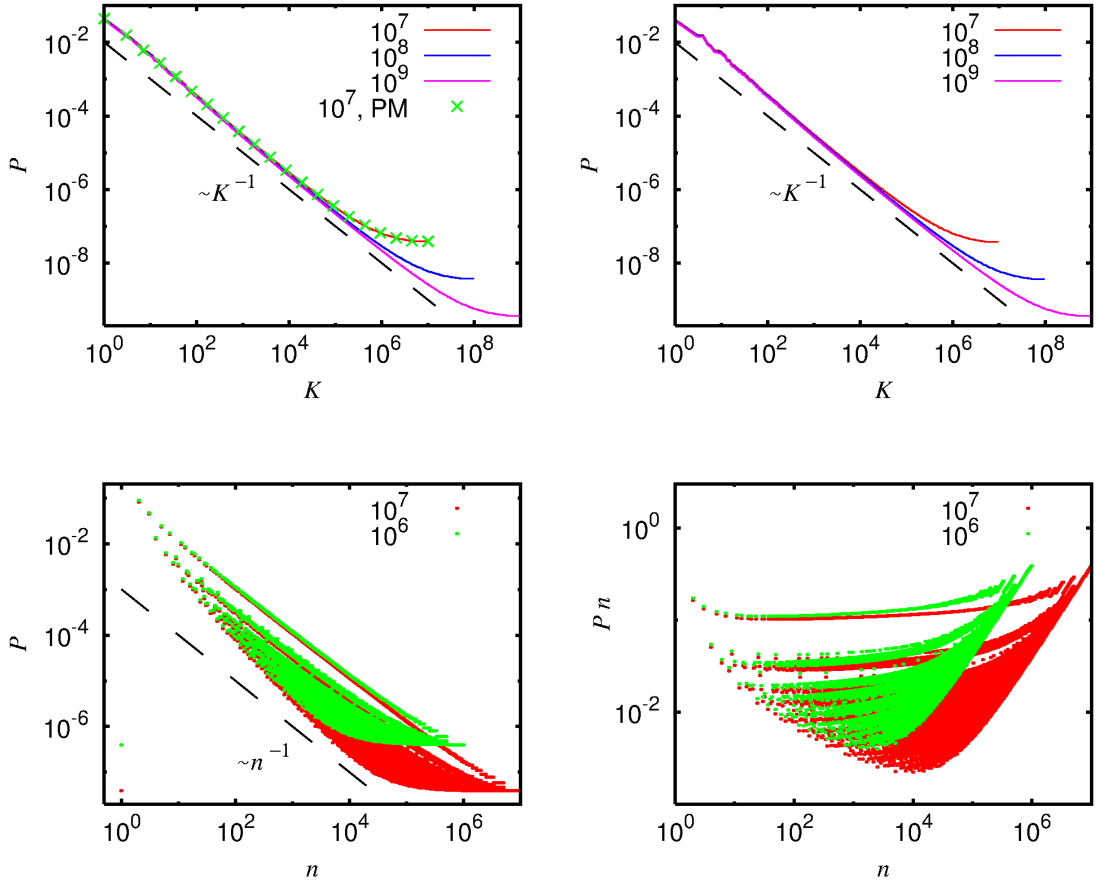
Integer network

(KF, Chepelianskii, Shepelyansky (2012), preprint)

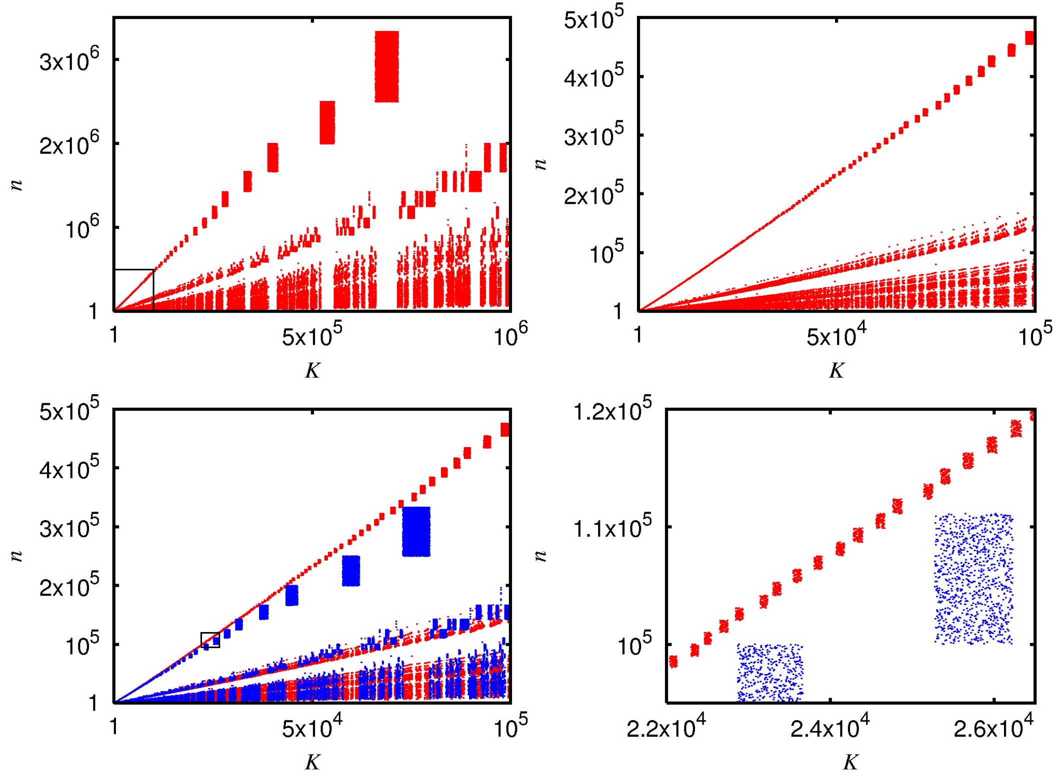
Consider the integers $n \in \{1, \ldots, N\}$ and construct an adjacency matrix by $A_{mn} = k$ where k is the largest integer such that m^k is a divisor of n if $1 < m < n$ and $A_{mn} = 0$ if $m = 1$ or $m = n$ (note $A_{mn} = k = 0$ if m is not a divisor of n). Construct S and G in the usual way from A.



PageRank



Dependence of n on K-index



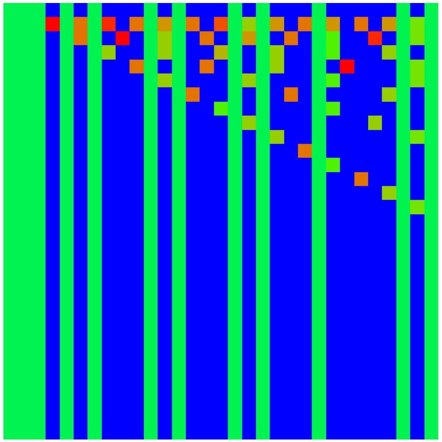
red: $N = 10^7$

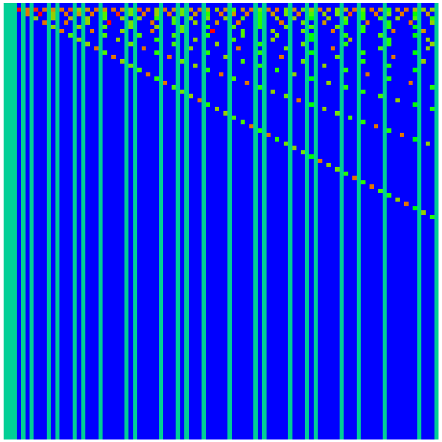
blue: $N = 10^6$

"New order" of integers: $K = 1, 2, \ldots, 32 \Rightarrow n = 2, 3, 5, 7, 4, 11, 13, 17, 6, 19, 9, 23, 29, 8, 31, 10, 37, 41, 43, 14, 47, 15, 53, 59, 61, 25, 67, 12, 71, 73, 22, 21.$

Semi-analytical determination of spectrum, PageRank and eigenvectors

Matrix structure:





$$\boxed{S = S_0 + v\, d^T}$$

where $v = e/N$, $d_j = 1$ for dangling nodes (primes and 1) and $d_j = 0$ otherwise. S_0 is the pure link matrix which is ***nil-potent***:

$$\boxed{S_0^l = 0}$$

with $l = [\log_2(N)] \ll N$.

Let ψ be an eigenvector of S with eigenvalue λ and $C = d^T\psi$.

- If $C = 0 \;\Rightarrow\; \psi$ eigenvector of $S_0 \;\Rightarrow\; \lambda = 0$ since S_0 nil-potent.

- If $C \neq 0 \;\Rightarrow\; \lambda \neq 0$ since the equation $S_0\psi = -C\,v$ does not have a solution $\Rightarrow\; \lambda\mathbf{1} - S_0$ invertible.

$$\Rightarrow\; \psi = C\,(\lambda\mathbf{1} - S_0)^{-1}\,v = \frac{C}{\lambda}\sum_{j=0}^{l-1}\left(\frac{S_0}{\lambda}\right)^j\,v \quad .$$

$$\text{From } \lambda^l = (d^T\psi/C)\lambda^l \;\Rightarrow\; \boxed{\mathcal{P}_r(\lambda) = 0}$$

with the ***reduced polynomial*** of degree $l = [\log_2(N)]$:

$$\boxed{\mathcal{P}_r(\lambda) = \lambda^l - \sum_{j=0}^{l-1}\lambda^{l-1-j}\,c_j = 0 \quad , \quad c_j = d^T\,S_0^j\,v\;.}$$

$\Rightarrow$ at most l eigenvalues $\lambda \neq 0$ which can be numerically determined as the zeros of $\mathcal{P}_r(\lambda)$. (Note: $l \leq 29$ for $N \leq 10^9$).

Furthermore for $\lambda = 1 \ \Rightarrow$ PageRank:

$$\boxed{P = C \sum_{j=0}^{l-1} S_0^j \, v \; , \; C = d^T P \; .}$$

The subspace of $\lambda \neq 0$ is represented by the vectors $v^{(j)} = S_0^j v$ for $j = 0, \ldots, l-1$

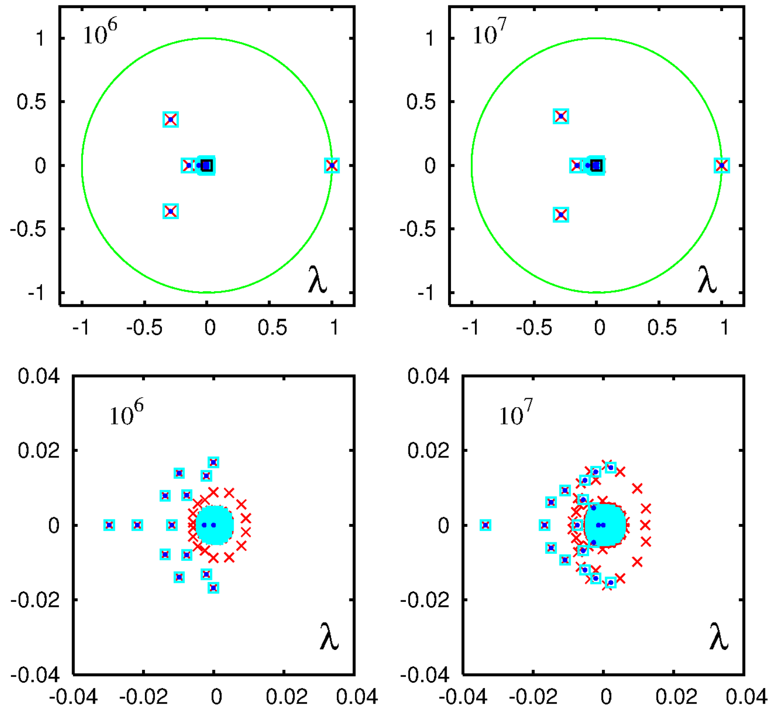
$$\Rightarrow \quad S\, v^{(j)} = c_j\, v^{(0)} + v^{(j+1)} = \sum_{k=0}^{l-1} \bar{S}_{k+1,j+1}\, v^{(k)}$$

"Small" $l \times l$-representation matrix :

$$\bar{S} = \begin{pmatrix} c_0 & c_1 & \cdots & c_{l-2} & c_{l-1} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \quad , \quad \bar{P} = C \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

with $P = \sum_j \bar{P}_{j+1}\, v^{(j)} = C \sum_j v^{(j)}$ and due to sum rule: $\sum_j c_j = 1$.

Spectrum I

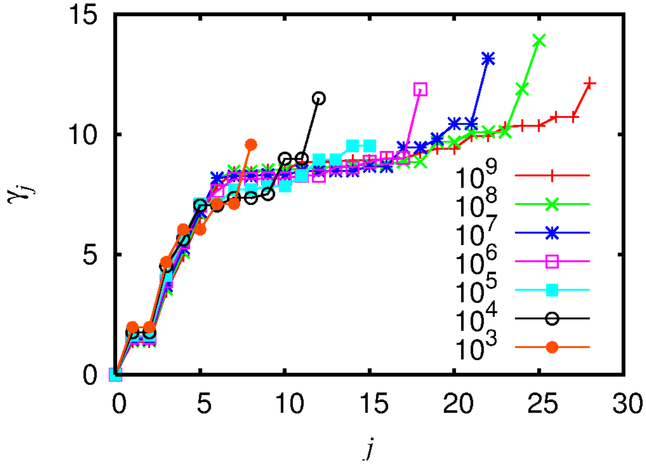


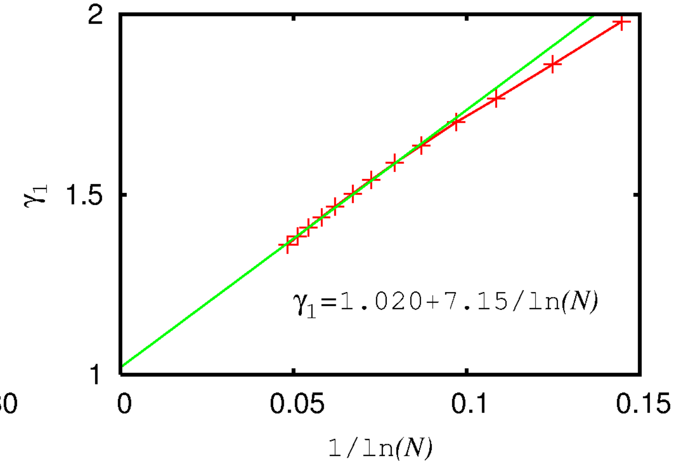
blue dots: semi-analytical eigenvalues as zeros from $\mathcal{P}_r(\lambda)$ (or eigenvalues of $\bar{S}$).

red crosses: Arnoldi method with random initial vector and $n_A = 1000$.

light blue boxes: Arnoldi method with constant initial vector $v = e/N$ and $n_A = 1000$.

Spectrum II





$$\gamma_j = -2 \ln |\lambda_j|$$

Large N limit of γ_1 with the scaling parameter: $1/\ln(N)$.

Note:

$$c_0 = d^T v = \frac{1}{N} \sum_{j=1}^{N} d_j = \frac{1 + \pi(N)}{N} \approx \frac{1}{\ln(N)}$$

where $\pi(N)$ is the number of primes below N.

References

1. D. L. Shepelyansky ***Fractal Weyl law for quantum fractal eigenstates***, Phys. Rev. E **77**, p.015202(R) (2008).
2. L. Ermann and D. L. Shepelyansky, ***Ulam method and fractal Weyl law for Perron-Frobenius operators***, Eur. Phys. J. B **75**, 299 (2010).
3. K. M. Frahm and D. L. Shepelyansky, ***Ulam method for the Chirikov standard map***, Eur. Phys. J. B **76**, 57 (2010).
4. K. M. Frahm, B. Georgeot and D. L. Shepelyansky, ***Universal emergence of PageRank***, J. Phys. A: Math. Theor. **44**, 465101 (2011).
5. K. M. Frahm, A. D. Chepelianskii and D. L. Shepelyansky, ***PageRank of integers***, arxiv:1205.6343[cs.IR] (2012).
6. K. M. Frahm and D. L. Shepelyansky, ***Google matrix of Twitter***, arxiv:1207.3414[cs.SI] (2012).