Cooperative and competitive interactions on random graphs

Cristian Giardina'

Joint work with

- Remco van der Hofstad (TU Eindhoven)
- Sander Dommers (TU Eindhoven)
- Shannon Starr (University of Rochester)
- Pierluigi Contucci (Universita' di Bologna)

Plan of the talk

Plan of the talk

1. Spatial processes on random networks.
    - From empirical complex networks...
    - ... to random graph models...
    - ... and processes.

Plan of the talk

1. Spatial processes on random networks.
   - From empirical complex networks...
   - ... to random graph models...
   - ... and processes.
2. Two examples:
   - Ferromagnetic Ising model on power law random graphs, Dommers, G., van der Hofstad, JSP **141**, 638-660 (2010) + work in progress on crit. exp.
   - Antiferromagnetic Potts model on Erdös-Rényi random graphs, Contucci, Dommers, G., Starr, arXiv:1106.4714

Empirical networks

Two emerging properties (among others)

- Scale free

- Small-world

Empirical networks

Two emerging properties (among others)

- Scale free

  Number of vertices with degree k is proportional to $k^{-\alpha}$

- Small-world

  distance between most pairs of vertices are small

Empirical networks

	network	type	n	m	z	ℓ	α
social	film actors	undirected	449 913	25 516 482	113.43	3.48	2.3
	company directors	undirected	7 673	55 392	14.44	4.60	–
	math coauthorship	undirected	253 339	496 489	3.92	7.57	–
	physics coauthorship	undirected	52 909	245 300	9.27	6.19	–
	biology coauthorship	undirected	1 520 251	11 803 064	15.53	4.92	–
	telephone call graph	undirected	47 000 000	80 000 000	3.16		2.1
	email messages	directed	59 912	86 300	1.44	4.95	1.5/2.0
	email address books	directed	16 881	57 029	3.38	5.22	–
	student relationships	undirected	573	477	1.66	16.01	–
	sexual contacts	undirected	2 810				3.2
information	WWW `nd.edu`	directed	269 504	1 497 135	5.55	11.27	2.1/2.4
	WWW Altavista	directed	203 549 046	2 130 000 000	10.46	16.18	2.1/2.7
	citation network	directed	783 339	6 716 198	8.57		3.0/–
	Roget's Thesaurus	directed	1 022	5 103	4.99	4.87	–
	word co-occurrence	undirected	460 902	17 000 000	70.13		2.7
technological	Internet	undirected	10 697	31 992	5.98	3.31	2.5
	power grid	undirected	4 941	6 594	2.67	18.99	–
	train routes	undirected	587	19 603	66.79	2.16	–
	software packages	directed	1 439	1 723	1.20	2.42	1.6/1.4
	software classes	directed	1 377	2 213	1.61	1.51	–
	electronic circuits	undirected	24 097	53 248	4.34	11.05	3.0
	peer-to-peer network	undirected	880	1 296	1.47	4.28	2.1
biological	metabolic network	undirected	765	3 686	9.64	2.56	2.2
	protein interactions	undirected	2 115	2 240	2.12	6.80	2.4
	marine food web	directed	135	598	4.43	2.05	–
	freshwater food web	directed	92	997	10.84	1.90	–
	neural network	directed	307	2 359	7.68	3.97	–

M.E.J. Newman, *The structure and function of complex networks* (2003)

Random Graph models for empirical networks

- Inhomogeneous random graph
- Configuration model
- Preferential attachment model

Random Graph models for empirical networks

- Inhomogeneous random graph
  Static random graph, independent edges with inhomogeneous edge occupation probability
- Configuration model
  Static random graph, with prescribed degree sequence
- Preferential attachment model
  Dynamic random graph, attachment proportional to degree plus constant

Networks functions

- Social networks (friendship, sexual, collaboration,..)
- Information networks (WWW, citation, ..)
- Technological networks (internet, airlines, roads, power grids,..)
- Biological networks (protein, neural, ...)

Networks functions

- Social networks (friendship, sexual, collaboration,..)
  spread of disease, opinion formation,..
- Information networks (WWW, citation, ..)
  email, routing, reputation,..
- Technological networks (internet, airlines, roads, power grids,..)
  communication, robustness to attack,..
- Biological networks (protein, neural, ...)
  metabolic pathways, reactions,..

Statistical Mechanics

Configurations $\sigma \in \Omega_n = \{-1, +1\}^n$

Hamiltonian $H(\sigma) : \Omega_n \to \mathbb{R}$, depending on a few parameters (temperature, external field,..)

Boltzmann-Gibbs measure $\mu_n(\sigma) = \frac{1}{Z_n} e^{-H(\sigma)}$

Statistical Mechanics

Configurations $\sigma \in \Omega_n = \{-1, +1\}^n$

Hamiltonian $H(\sigma) : \Omega_n \to \mathbb{R}$, depending on a few parameters (temperature, external field,..)

Boltzmann-Gibbs measure $\mu_n(\sigma) = \frac{1}{Z_n} e^{-H(\sigma)}$

Aim

Study the means $\langle \sigma_i \rangle_{\mu_n}$, correlations $\langle \sigma_i \sigma_j \rangle_{\mu_n}$,...

It is useful to compute the pressure

$$\psi_n = \frac{1}{n} \ln Z_n = \frac{1}{n} \ln \sum_{\sigma \in \Omega_n} e^{-H(\sigma)}$$

Statistical Mechanics

Configurations $\sigma \in \Omega_n = \{-1, +1\}^n$

Hamiltonian $H(\sigma) : \Omega_n \to \mathbb{R}$, depending on a few parameters (temperature, external field,..)

Boltzmann-Gibbs measure $\mu_n(\sigma) = \frac{1}{Z_n} e^{-H(\sigma)}$

Aim

Study the means $\langle \sigma_i \rangle_{\mu_n}$, correlations $\langle \sigma_i \sigma_j \rangle_{\mu_n}$,...

It is useful to compute the pressure

$$\psi_n = \frac{1}{n} \ln Z_n = \frac{1}{n} \ln \sum_{\sigma \in \Omega_n} e^{-H(\sigma)}$$

Outcome

In the thermodynamic limit $n \to \infty$, phase transitions may occur.

Statistical Mechanics on Random Graphs

(At least) Two level of randomness

$$H(\sigma) = -\beta \sum_{(i,j)\in E_n} J_{i,j}\sigma_i\sigma_j - B\sum_{i\in V_n}\sigma_i$$

- Randomness of the graph $G_n = (V_n, E_n)$
- Randomness of the couplings $\{J_{i,j}\}$
  - Ferromagnets, $J_{i,j} > 0$: easy physics, interesting mathematics.
  - Antiferromagnets, $J_{i,j} < 0$: frustration appears.
  - Spin glasses, $J_{i,j}$ i.i.d. random variables with symmetric distribution: order parameter is not self-averaging!

Quenched state $\mathbb{E}(\langle\cdot\rangle_{\mu_n})$ is studied.

Ferromagnetic models

Basic questions

- How does ferromagnetic Ising model behave on random graphs with arbitrary degree distribution?
- What is the effect of scale-free random graphs on the ferromagnetic phase transition? In particular for exponent $2 < \alpha < 3$.

Basic questions

- How does ferromagnetic Ising model behave on random graphs with arbitrary degree distribution?
- What is the effect of scale-free random graphs on the ferromagnetic phase transition? In particular for exponent $2 < \alpha < 3$.

Previous answers

- Physics: Leone et al (2002), Dorogotsev et al (2002),...
- Mathematics: Dembo & Montanari (2010), restricted to degree distributions with finite variance.

Basic questions

- How does ferromagnetic Ising model behave on random graphs with arbitrary degree distribution?
- What is the effect of scale-free random graphs on the ferromagnetic phase transition? In particular for exponent $2 < \alpha < 3$.

Previous answers

- Physics: Leone et al (2002), Dorogotsev et al (2002),...
- Mathematics: Dembo & Montanari (2010), restricted to degree distributions with finite variance.

Our results

- Rigorous analysis for degree distribution with finite mean degree

Local convergence to homogeneous trees

$\{G_n\}_{n\geq 1}$ is *locally tree-like* with asymptotic degree distribution P if

$$\lim_{n\to\infty} \mathbb{P}_n[B_i(t) \simeq \mathcal{T}] = \mathbb{P}[\mathcal{T}(P,\rho,t) \simeq \mathcal{T}].$$

$B_i(t)$ = ball in G_n centered at a uniformly chosen vertex $i \in V$

$\mathcal{T}(P,\rho,t)$ = rooted random tree with t generations (offspring distribution P in the first generation, size-biased law ρ in the further generation)

$$\rho_k = \frac{(k+1)P_{k+1}}{\sum_k kP_k}$$

Example: the configuration model

- Fix the degree distribution P. Assign D_i half-edges to each vertex $i \in V_n$, where D_i are i.i.d. with distribution P ($\mathbb{E}(D_i) < \infty$, also make sure $\sum_i D_i$ is even).
- Choose pairs of stubs at random and connect them together.

Example: the configuration model

- Fix the degree distribution P. Assign D_i half-edges to each vertex $i \in V_n$, where D_i are i.i.d. with distribution P ($\mathbb{E}(D_i) < \infty$, also make sure $\sum_i D_i$ is even).
- Choose pairs of stubs at random and connect them together.

Local Structure

- The degree distribution of a random vertex is P.
- The probability that the neighbor of a random vertex has degree $k+1$ equals the probability that a random stub is attached to a vertex with $k+1$ stubs:

$$\frac{(k+1)\sum_{i\in V_n} \mathbb{I}_{\{D_i=k+1\}}}{\sum_{i\in V_n} D_i} \longrightarrow \frac{(k+1)P_{k+1}}{\mathbb{E}(D)} = \rho_k$$

Strongly finite mean degree distribution

There exist constants $\alpha > 2$ and $c > 0$ such that

$$\sum_{i=k}^{\infty} P_i \leq ck^{-(\alpha-1)}$$

Remark: Empirical networks with infinite variance degree distribution are included.

Strongly finite mean degree distribution

There exist constants $\alpha > 2$ and $c > 0$ such that

$$\sum_{i=k}^{\infty} P_i \leq ck^{-(\alpha-1)}$$

Remark: Empirical networks with infinite variance degree distribution are included.

Uniform sparsity

$$\lim_{n\to\infty} \frac{|E_n|}{n} = \lim_{n\to\infty} \frac{1}{2n} \sum_{i\in V_n} \sum_{k=1}^{\infty} k\mathbb{I}_{\{D_i=k\}} = \frac{\mathbb{E}(D)}{2} < \infty.$$

Theorem

Assume $\{G_n\}_{n\geq 1}$ is uniformly sparse and locally tree-like with asymptotic degree distribution P, where P has strongly finite mean. Let $D \sim P$ and $K \sim \rho$. Then:

Theorem

Assume $\{G_n\}_{n\geq 1}$ is uniformly sparse and locally tree-like with asymptotic degree distribution P, where P has strongly finite mean. Let $D \sim P$ and $K \sim \rho$. Then:

$$\lim_{n\to\infty} \psi_n = \phi$$

Theorem

Assume $\{G_n\}_{n\geq 1}$ is uniformly sparse and locally tree-like with asymptotic degree distribution P, where P has strongly finite mean. Let $D \sim P$ and $K \sim \rho$. Then:

$$\lim_{n\to\infty} \psi_n = \phi$$

$$\begin{aligned}\phi(\beta, B) &= \frac{\mathbb{E}(D)}{2}\log\cosh(\beta) - \frac{\mathbb{E}(D)}{2}\mathbb{E}[\log(1+\tanh(\beta)\tanh(h_1)\tanh(h_2))] \\ &+ \mathbb{E}\left[\log\left(e^B\prod_{i=1}^{D}\{1+\tanh(\beta)\tanh(h_i)\} + e^{-B}\prod_{i=1}^{D}\{1-\tanh(\beta)\tanh(h_i)\}\right)\right]\end{aligned}$$

Theorem

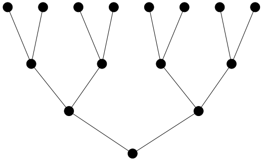
Assume $\{G_n\}_{n\geq 1}$ is uniformly sparse and locally tree-like with asymptotic degree distribution P, where P has strongly finite mean. Let $D \sim P$ and $K \sim \rho$. Then:

$$\lim_{n\to\infty} \psi_n = \phi$$

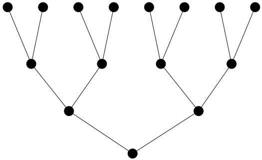
$$\begin{aligned}\phi(\beta, B) &= \frac{\mathbb{E}(D)}{2} \log\cosh(\beta) - \frac{\mathbb{E}(D)}{2} \mathbb{E}[\log(1 + \tanh(\beta)\tanh(h_1)\tanh(h_2))] \\ &+ \mathbb{E}\left[\log\left(e^B \prod_{i=1}^{D}\{1+\tanh(\beta)\tanh(h_i)\} + e^{-B}\prod_{i=1}^{D}\{1-\tanh(\beta)\tanh(h_i)\}\right)\right]\end{aligned}$$

$$h_1 \stackrel{d}{=} B + \sum_{i=1}^{K} \operatorname{arctanh}(\tanh(\beta)\tanh(h_i)) := B + \sum_{i=1}^{K} \xi(h_i)$$

Proof I: Recursion on the random tree

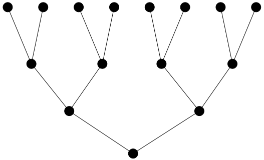


Proof I: Recursion on the random tree



$p^{(t)}(\sigma) = \mathbb{P}(S_\emptyset = \sigma)$ marginal at the root $\emptyset$ of $\mathcal{T}(P, \rho, t)$

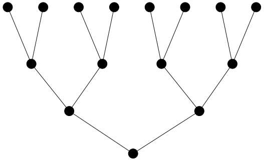
Proof I: Recursion on the random tree



$p^{(t)}(\sigma) = \mathbb{P}(S_\emptyset = \sigma)$ marginal at the root $\emptyset$ of $\mathcal{T}(P, \rho, t)$

$$p^{(t+1)}(\sigma) = C \sum_{\sigma_1,\dots,\sigma_k} p^{(t)}(\sigma_1)\cdots p^{(t)}(\sigma_k) e^{\beta\sigma(\sigma_1+\cdots\sigma_k)+B\sigma}$$

Proof I: Recursion on the random tree

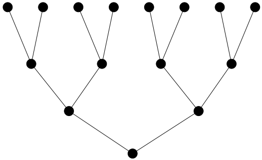


$$p^{(t)}(\sigma) = \mathbb{P}(S_\emptyset = \sigma) \qquad \text{marginal at the root } \emptyset \text{ of } \mathcal{T}(P, \rho, t)$$

$$p^{(t+1)}(\sigma) = C \sum_{\sigma_1,\dots,\sigma_k} p^{(t)}(\sigma_1)\cdots p^{(t)}(\sigma_k) e^{\beta\sigma(\sigma_1+\cdots\sigma_k)+B\sigma}$$

$$p^{(t)}(\sigma) = \frac{e^{\sigma h^{(t)}}}{\sum_{\sigma=\pm 1} e^{\sigma h^{(t)}}} \qquad h^{(t+1)} = B + \sum_{i=1}^{K} \operatorname{arctanh}(\tanh(\beta)\tanh(h_i^t))$$

Proof I: Recursion on the random tree



$p^{(t)}(\sigma) = \mathbb{P}(S_{\emptyset} = \sigma)$ marginal at the root $\emptyset$ of $\mathcal{T}(P, \rho, t)$

$$p^{(t+1)}(\sigma) = C \sum_{\sigma_1,\ldots,\sigma_k} p^{(t)}(\sigma_1) \cdots p^{(t)}(\sigma_k) e^{\beta\sigma(\sigma_1 + \cdots \sigma_k) + B\sigma}$$

$$p^{(t)}(\sigma) = \frac{e^{\sigma h^{(t)}}}{\sum_{\sigma=\pm 1} e^{\sigma h^{(t)}}} \qquad h^{(t+1)} = B + \sum_{i=1}^{K} \text{arctanh}(\tanh(\beta)\tanh(h_i^t))$$

Unique fixed point when $t \to \infty$

Proof II: Internal energy

$$\frac{\partial \psi_n}{\partial \beta}$$

Proof II: Internal energy

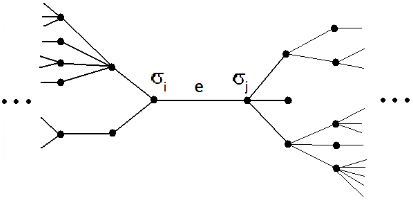
$$\frac{\partial \psi_n}{\partial \beta} = \frac{1}{n} \sum_{(i,j)\in E_n} \langle \sigma_i \sigma_j \rangle_{\mu_n}$$

Proof II: Internal energy

$$\frac{\partial \psi_n}{\partial \beta} = \frac{1}{n} \sum_{(i,j)\in E_n} \langle \sigma_i \sigma_j \rangle_{\mu_n} = \frac{|E_n|}{n} \frac{\sum_{(i,j)\in E_n} \langle \sigma_i \sigma_j \rangle_{\mu_n}}{|E_n|}$$

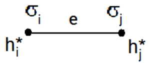
Proof II: Internal energy

$$\frac{\partial \psi_n}{\partial \beta} = \frac{1}{n} \sum_{(i,j)\in E_n} \langle \sigma_i \sigma_j \rangle_{\mu_n} = \frac{|E_n|}{n} \frac{\sum_{(i,j)\in E_n} \langle \sigma_i \sigma_j \rangle_{\mu_n}}{|E_n|} \to \frac{\mathbb{E}(D)}{2} \mathbb{E}(\langle \sigma_i \sigma_j \rangle_{\mu})$$



Proof II: Internal energy

$$\frac{\partial \psi_n}{\partial \beta} = \frac{1}{n} \sum_{(i,j)\in E_n} \langle \sigma_i \sigma_j \rangle_{\mu_n} = \frac{|E_n|}{n} \frac{\sum_{(i,j)\in E_n} \langle \sigma_i \sigma_j \rangle_{\mu_n}}{|E_n|} \to \frac{\mathbb{E}(D)}{2} \mathbb{E}(\langle \sigma_i \sigma_j \rangle_{\mu})$$



$$\to \frac{\mathbb{E}(D)}{2} \mathbb{E}(\langle \sigma_i \sigma_j \rangle_{\text{tree}}) = \frac{\partial \phi}{\partial \beta}$$

Antiferromagnetic models

Antiferromagnetic models

- The long loops of locally tree-like random graphs do matter .
- They induce frustration.
- Rather than compare to the tree, better to compare to the spin-glass.

Model: Potts Antiferromagnet on Erdös-Rényi random graphs

$$H_n(\sigma) = \sum_{i,j=1}^{n} J_{i,j}\delta(\sigma_i, \sigma_j)$$

- $J_{i,j}$ i.i.d. Poisson$(c/2n)$, $c > 1$ and $\sigma_i \in \{1, 2, \ldots, q\}$
- At $\beta = \infty$ it gives the coloring problem.

Model: Potts Antiferromagnet on Erdös-Rényi random graphs

$$H_n(\sigma) = \sum_{i,j=1}^{n} J_{i,j}\delta(\sigma_i, \sigma_j)$$

- $J_{i,j}$ i.i.d. Poisson$(c/2n)$, $c > 1$ and $\sigma_i \in \{1, 2, \ldots, q\}$
- At $\beta = \infty$ it gives the coloring problem.

Previous results

- Physics: Krzakala-Zdeborova (2007) conjectured the critical point for the ER Potts AF = ER Potts SG.
- Mathematics: Achlioptas and Naor (2005) found a formula for $q_{AN}(c)$ such that $q^*(c) \in \{q_{AN}, q_{AN} + 1\}$.

Model: Potts Antiferromagnet on Erdös-Rényi random graphs

$$H_n(\sigma) = \sum_{i,j=1}^{n} J_{i,j}\delta(\sigma_i, \sigma_j)$$

- $J_{i,j}$ i.i.d. Poisson$(c/2n)$, $c > 1$ and $\sigma_i \in \{1, 2, \ldots, q\}$
- At $\beta = \infty$ it gives the coloring problem.

Previous results

- Physics: Krzakala-Zdeborova (2007) conjectured the critical point for the ER Potts AF = ER Potts SG.
- Mathematics: Achlioptas and Naor (2005) found a formula for $q_{AN}(c)$ such that $q^*(c) \in \{q_{AN}, q_{AN} + 1\}$.

Our results

- We rigorously prove the existence of a phase transition and confirm KZ conjecture for $q = 2$.

Theorem

Given $q \in \mathbb{N}$ and $c > \min\left\{(q-1)^2, \frac{2\ln q}{|\ln(1-q^{-1})|}\right\}$, the AF model on the ER random graph has a critical temperature $\beta_{crit}(c,q)$ with

$$\beta_{2^{nd}}(c,q) \leq \beta_{crit}(c,q) \leq \min\{\beta_{RS}(c,q), \beta_{ent}(c,q)\}$$

where

$$\beta_{RS} = -\ln\left(1 - \frac{q}{1+\sqrt{c}}\right), \quad \beta_{entr} = \inf\{\beta : \mathcal{S}(\beta,c,q) < 0\}$$

$$\beta_{2^{nd}} = -\ln\left(1 - \frac{q}{q-1+\sqrt{c/(2q\ln q)}}\right) \text{ if } q > 2, \quad \beta_{2^{nd}} = \beta_{RS} \text{ if } q = 2$$

Theorem

Given $q \in \mathbb{N}$ and $c > \min\left\{(q-1)^2, \frac{2\ln q}{|\ln(1-q^{-1})|}\right\}$, the AF model on the ER random graph has a critical temperature $\beta_{crit}(c,q)$ with

$$\beta_{2nd}(c,q) \le \beta_{crit}(c,q) \le \min\{\beta_{RS}(c,q), \beta_{ent}(c,q)\}$$

where

$$\beta_{RS} = -\ln\left(1 - \frac{q}{1+\sqrt{c}}\right), \quad \beta_{entr} = \inf\{\beta : \mathcal{S}(\beta,c,q) < 0\}$$

$$\beta_{2nd} = -\ln\left(1 - \frac{q}{q-1+\sqrt{c/(2q\ln q)}}\right) \text{ if } q > 2, \quad \beta_{2nd} = \beta_{RS} \text{ if } q = 2$$

A phase transition is a non-analyticity in β:

$$\psi(\beta,c) \begin{cases} = \mathcal{P}(\beta,c) := \ln q + \frac{c}{2}\ln\left(1 - \frac{1-e^{-\beta}}{q}\right) & \text{if } \beta \le \beta_{2nd}, \\ < \mathcal{P}(\beta,c) & \text{if } \beta \ge \min\{\beta_{RS}, \beta_{ent}\}. \end{cases}$$

Proof ingredients

1. Interpolation method from spin glasses:
   existence of TD-limit, Extended Variational Principle, pressure upper-bounds

2. (Conditioned) second moment method:
   control of high temperature region.

Interpolation

If $X \sim Poisson(\lambda)$ then

$$\frac{d}{d\lambda}\mathbb{E}[f(X)] = \mathbb{E}[f(X+1) - f(X)]$$

Interpolation

If $X \sim Poisson(\lambda)$ then

$$\frac{d}{d\lambda}\mathbb{E}[f(X)] = \mathbb{E}[f(X+1) - f(X)]$$

Consider $c \mapsto ct$ for $t \in [0,1]$. Then

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}[\psi_n(t)] &= \frac{c}{2n}\sum_{i,j=1}^{n}\mathbb{E}\left[\psi_n(t)|_{J_{i,j}\to J_{i,j}+1} - \psi_n(t)\right] \\
&= \frac{c}{2n^2}\sum_{i,j=1}^{n}\mathbb{E}\left[\ln\left(\frac{Z_n(t)|_{J_{i,j}\to J_{i,j}+1}}{Z_n(t)}\right)\right] \\
&= \frac{c}{2n^2}\sum_{i,j=1}^{n}\mathbb{E}\left[\ln\langle e^{-\beta\delta(\sigma_i,\sigma_j)}\rangle_t\right]
\end{aligned}$$

Interpolation

$$
\begin{aligned}
\frac{d}{dt}\mathbb{E}[\psi_n(t)] &= \frac{c}{2n^2}\sum_{i,j=1}^{n}\mathbb{E}\left[\ln\langle e^{-\beta\delta(\sigma_i,\sigma_j)}\rangle\right] \\
&= \frac{c}{2n^2}\sum_{i,j=1}^{n}\mathbb{E}\left[\ln\left(1-(1-e^{-\beta})\langle\delta(\sigma_i,\sigma_j)\rangle\right)\right]
\end{aligned}
$$

Interpolation

$$\frac{d}{dt}\mathbb{E}[\psi_n(t)] = \frac{c}{2n^2}\sum_{i,j=1}^{n}\mathbb{E}\left[\ln\langle e^{-\beta\delta(\sigma_i,\sigma_j)}\rangle\right]$$

$$= \frac{c}{2n^2}\sum_{i,j=1}^{n}\mathbb{E}\left[\ln\left(1-(1-e^{-\beta})\langle\delta(\sigma_i,\sigma_j)\rangle\right)\right]$$

Assuming $\langle\delta(\sigma_i,\sigma_j)\rangle = \frac{1}{q}$ then

$$\mathbb{E}[\psi_n(\beta,c)] = \mathbb{E}[\psi_n(\beta,0)] + \int_0^1 dt\,\frac{d}{dt}\mathbb{E}[\psi_n(t)]$$

$$= \ln q + \frac{c}{2}\ln\left(1-\frac{1-e^{-\beta}}{q}\right) = \mathcal{P}(\beta,c)$$

Interpolation: consequences

- The sequence $(\mathbb{E}[\psi_n(\beta)])_{n\in\mathbb{N}}$ is superadditive: let $n_1 + n_2 = n$

$$\mathbb{E}[\psi_n] \geq \frac{n_1}{n}\mathbb{E}[\psi_{n_1}] + \frac{n_2}{n}\mathbb{E}[\psi_{n_2}]$$

Interpolation: consequences

- The sequence $(\mathbb{E}[\psi_n(\beta)])_{n\in\mathbb{N}}$ is superadditive: let $n_1 + n_2 = n$

$$\mathbb{E}[\psi_n] \geq \frac{n_1}{n}\mathbb{E}[\psi_{n_1}] + \frac{n_2}{n}\mathbb{E}[\psi_{n_2}]$$

- Extended variational principle

$$\mathbb{E}[\psi_n] \leq \min_{\mathcal{L}} \left[G^{(1)}(\beta, c, q, \mathcal{L}) - G^{(2)}(\beta, c, q, \mathcal{L}) \right]$$

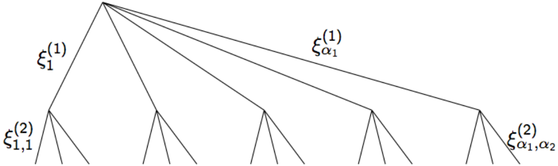
$$G^{(1)} = \mathbb{E}\left[\frac{1}{n}\ln\sum_\alpha \xi_\alpha \sum_\sigma e^{-\beta\tilde{H}(\sigma,\tau_\alpha)}\right]$$

$$G^{(2)} = \mathbb{E}\left[\frac{1}{n}\ln\sum_\alpha \xi_\alpha \; e^{-\beta\sum_{k=1}^{K}\delta(\tau_{\alpha,2k-1},\tau_{\alpha,2k})}\right]$$

$\{\xi_\alpha\}$ random, $\quad \{\tau_\alpha\} = \{\tau_{\alpha,1}, \tau_{\alpha,2}, \ldots\} \in [q]^{\mathbb{N}} \sim \mathcal{L}$ (exchangeable)

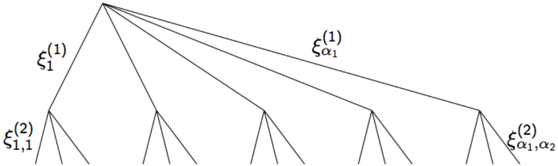
Interpolation: upper bounds

By choosing $\{\xi_\alpha\}$ as Derrida-Ruelle probability cascades



Interpolation: upper bounds

By choosing $\{\xi_\alpha\}$ as Derrida-Ruelle probability cascades

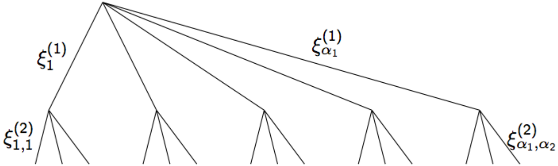


- If $\mathcal{L}$ is product with uniform marginals

$$\mathbb{E}[\psi_n] \leq \mathcal{P}(\beta, c) = \text{ Trivial RS-solution}$$

Interpolation: upper bounds

By choosing $\{\xi_\alpha\}$ as Derrida-Ruelle probability cascades



- If $\mathcal{L}$ is product with uniform marginals

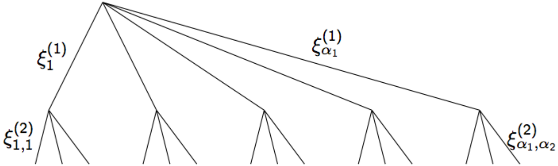
$$\mathbb{E}[\psi_n] \le \mathcal{P}(\beta, c) = \text{ Trivial RS-solution}$$

- If $\mathcal{L}$ is product with non-uniform marginals

$$\mathbb{E}[\psi_n] \le \text{ RS-solution}$$

Interpolation: upper bounds

By choosing $\{\xi_\alpha\}$ as Derrida-Ruelle probability cascades



- If $\mathcal{L}$ is product with uniform marginals

$$\mathbb{E}[\psi_n] \leq \mathcal{P}(\beta, c) = \text{ Trivial RS-solution}$$

- If $\mathcal{L}$ is product with non-uniform marginals

$$\mathbb{E}[\psi_n] \leq \text{ RS-solution}$$

- If $\mathcal{L}$ is hierarchical

$$\mathbb{E}[\psi_n] \leq \text{ RSB-solution}$$

2^{nd} moment method

To conclude existence of a phase transition we need to show that the region $\{\beta : \mathbb{E}[\psi_n] = \mathcal{P}(\beta, c)\}$ is non empty.

2^{nd} moment method

To conclude existence of a phase transition we need to show that the region $\{\beta : \mathbb{E}[\psi_n] = \mathcal{P}(\beta, c)\}$ is non empty.

Let $B = \{\sharp \text{ of bonds} = \frac{nc}{2}\}$

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{E}\left[Z_n \mid B\right] = \mathcal{P}(\beta, c)$$

2^{nd} moment method

To conclude existence of a phase transition we need to show that the region $\{\beta : \mathbb{E}[\psi_n] = \mathcal{P}(\beta, c)\}$ is non empty.

Let $B = \{\sharp \text{ of bonds} = \frac{nc}{2}\}$

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{E}\left[Z_n \mid B\right] = \mathcal{P}(\beta, c)$$

If $\beta < \beta_{2^{nd}}$

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{E}\left[Z_n^2 \mid B\right] = \lim_{n\to\infty} \frac{1}{n} \ln \left(\mathbb{E}\left[Z_n \mid B\right]\right)^2$$

2^{nd} moment method

To conclude existence of a phase transition we need to show that the region $\{\beta : \mathbb{E}[\psi_n] = \mathcal{P}(\beta, c)\}$ is non empty.

Let $B = \{\sharp \text{ of bonds} = \frac{nc}{2}\}$

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{E}\left[Z_n \mid B\right] = \mathcal{P}(\beta, c)$$

If $\beta < \beta_{2^{nd}}$

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{E}\left[Z_n^2 \mid B\right] = \lim_{n\to\infty} \frac{1}{n} \ln \left(\mathbb{E}\left[Z_n \mid B\right]\right)^2$$

Therefore

$$\begin{aligned} \frac{1}{n}\mathbb{E}[\ln Z_n] &= \mathbb{P}(B)\frac{1}{n}\mathbb{E}[\ln Z_n|B] + \mathbb{P}(B^c)\frac{1}{n}\mathbb{E}[\ln Z_n|B^c] \\ &\approx \frac{1}{n}\ln[\mathbb{E}[Z_n|B]] \to \mathcal{P}(\beta, c) \end{aligned}$$

THANK YOU!

ER Potts AF - Replica Symmetric solution

$$
\begin{aligned}
\mathbb{E}[\psi_n] \quad \leq \quad & \mathbb{E} \ln \left[\sum_{s=1}^{q} \prod_{k=1}^{K} (1 - (1 - e^{-\beta}) p_k(s)) \right] \\
& - \frac{c}{2} \mathbb{E} \ln \left[1 - (1 - e^{-\beta}) \sum_{s=1}^{q} p_1(s) p_2(s) \right]
\end{aligned}
$$

ER Potts AF - Replica Symmetric solution

$$\mathbb{E}[\psi_n] \leq \mathbb{E}\ln\left[\sum_{s=1}^{q}\prod_{k=1}^{K}(1-(1-e^{-\beta})p_k(s))\right] - \frac{c}{2}\mathbb{E}\ln\left[1-(1-e^{-\beta})\sum_{s=1}^{q}p_1(s)p_2(s)\right]$$

$$(p_1(s)) \stackrel{d}{=} \left(\frac{\prod_{k=1}^{K}(1-(1-e^{-\beta})p_k(s))}{\sum_{s=1}^{q}\prod_{k=1}^{K}(1-(1-e^{-\beta})p_k(s))}\right) \qquad s=1,\ldots,q$$

$$\sum_{s=1}^{q}p_k(s)=1 \qquad K\sim Poisson(c)$$