

# *Multiplexity in networks*

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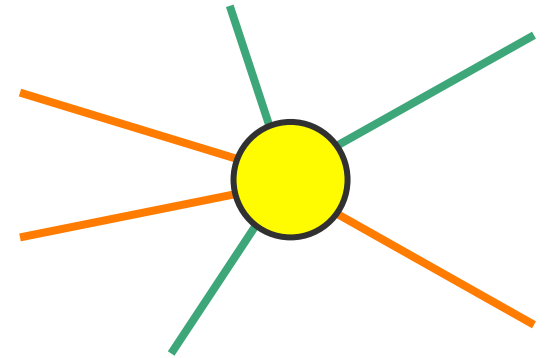
**Thanks to: Kyu-Min Lee, Byungjoon Min, Won-kuk Cho, Jung Yeol Kim,  
Jeehye Choi, In-mook Kim, Charlie Brummitt (UC Davis), NRF Korea**



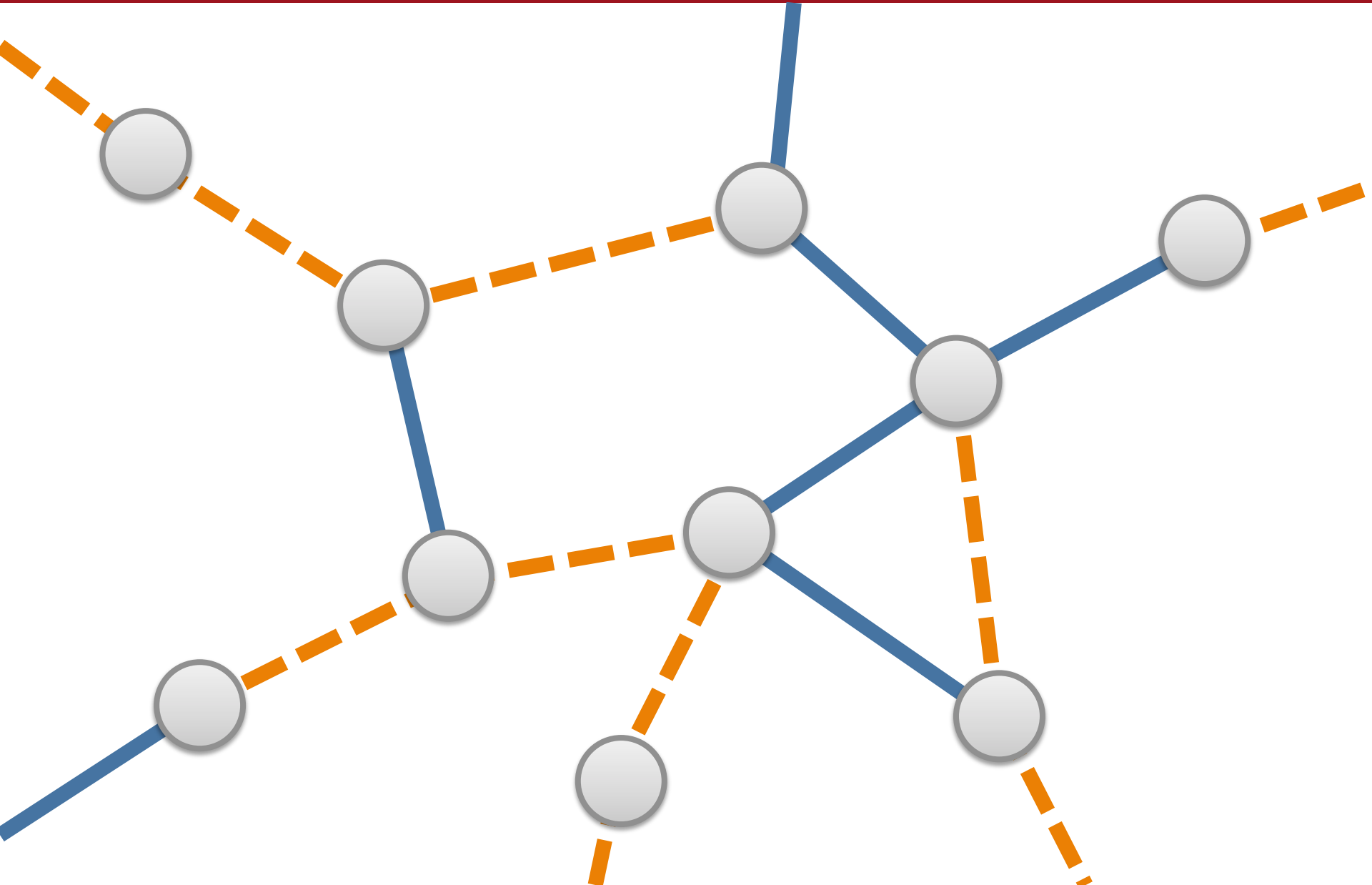
**LIBERTAS  
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VERITAS**

# Complex systems are MULTIPLEX

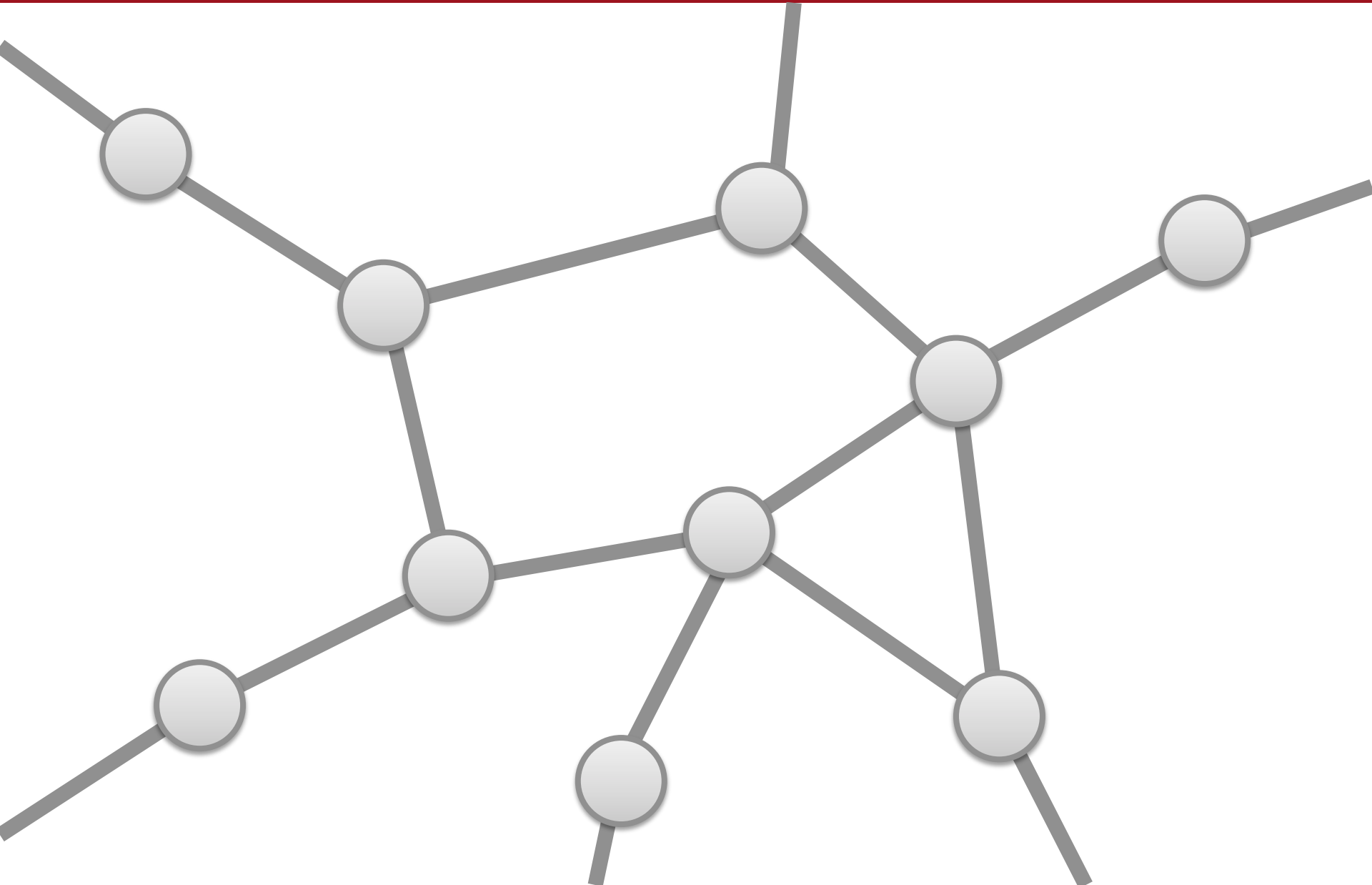
- ***Multiplexity***: Existence of **more than one type of links** whose interplay can affect the **structure and/or function**.
- **Multiplex networks**  
*cf.* **multilayer** networks, **interdependent** networks, **interacting** networks, **coupled** networks, **network of networks**, ...
  - **Multi-relational social networks** [Padgett&Ansell (1993); Szell et al (2010)].
  - **Cellular networks** [Yeast, *M. pneumoniae*, etc]
  - **Interdependent critical infrastructures** [Buldyrev et al (2010)]
  - **Transportation networks** [Parshani et al (2011)].
  - **Economic networks**
- **Single/simplex-network description is incomplete.**



# Multiplex networks



# Simplex networks



# **Multiplexity on Dynamics**

# THRESHOLD CASCADES

- A simple model of global cascades on random networks

Duncan J. Watts\*

Department of Sociology, Columbia University New York, NY 10027

Communicated by Murray Gell-Mann, Santa Fe Institute, Santa Fe, NM, February 14, 2002 (received for review May 29, 2001)

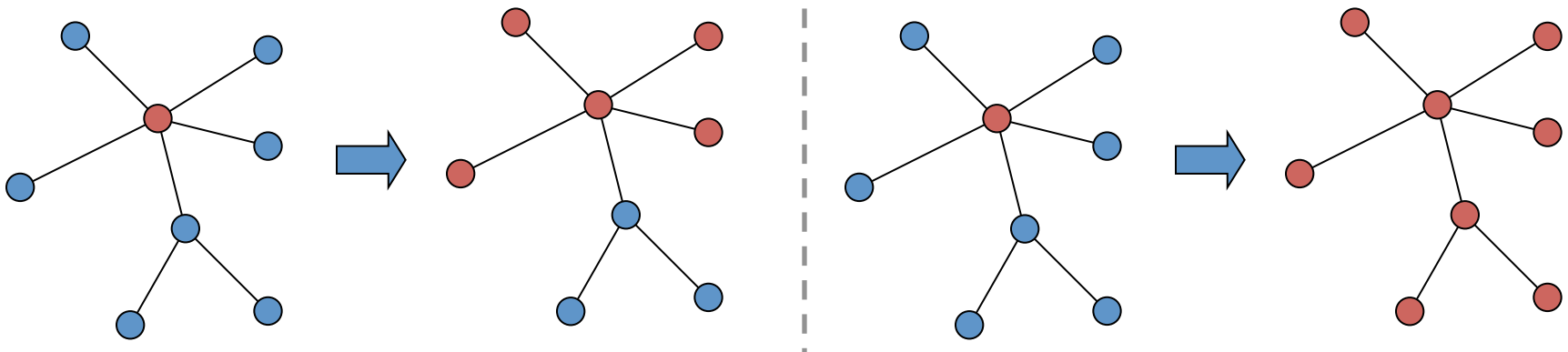
5766-5771 | PNAS | April 30, 2002 | vol. 99 | no. 9

[www.pnas.org/cgi/doi/10.1073/pnas.082090499](http://www.pnas.org/cgi/doi/10.1073/pnas.082090499)

- Model for **behavioral adoption cascade** [Schelling, Granovetter '70s].
- E.g., using a smartphone app, or wearing a hockey helmet.
- **A node gets activated ( $0 \rightarrow 1$ ) if at least a fraction  $R$  of its neighbors are active.**

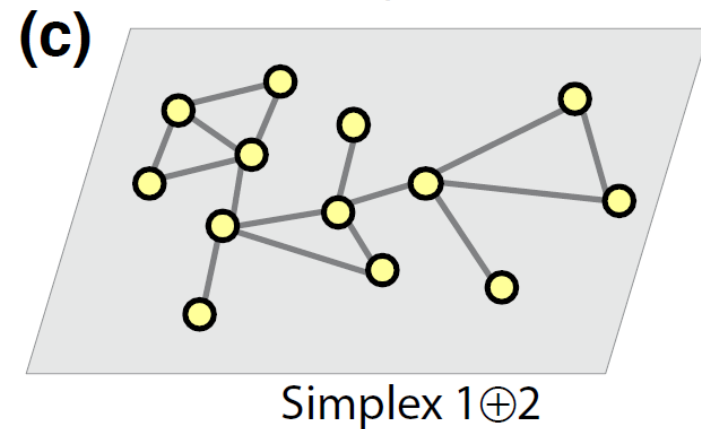
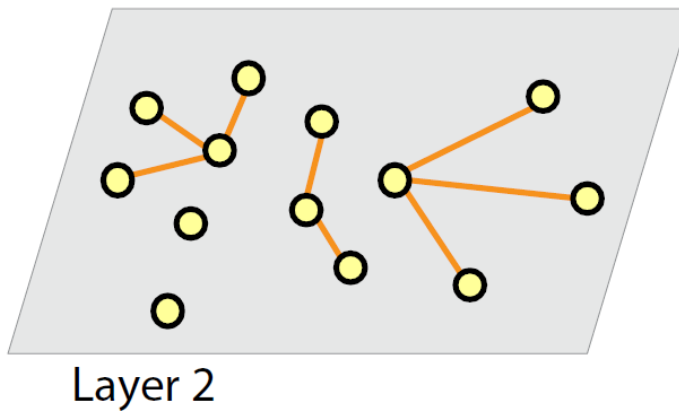
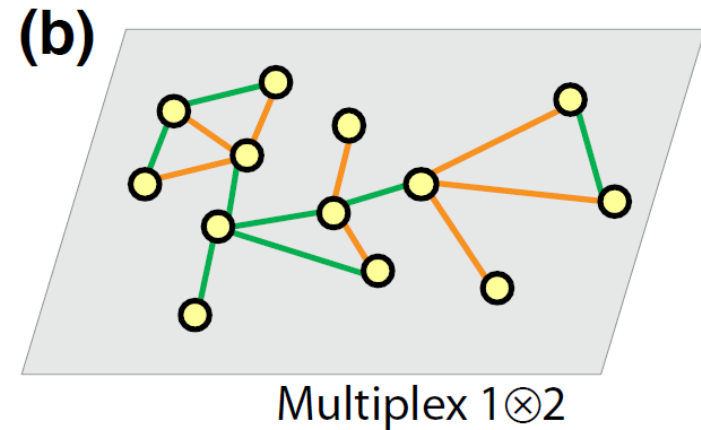
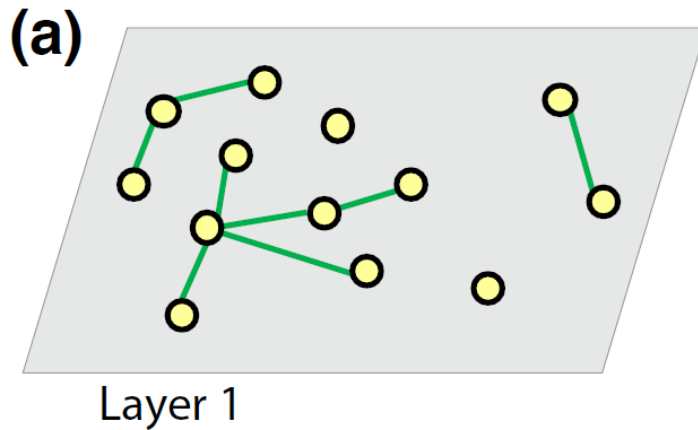
- Ex)  $R = 1/2$

vs.  $R = 1/4$

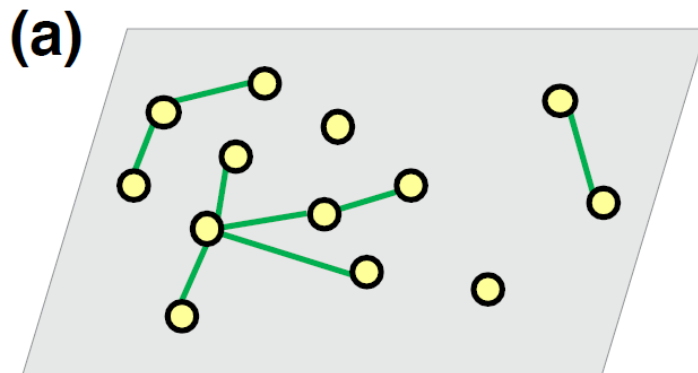


- Interested in the condition for the **global cascades** from small initial active seeds.

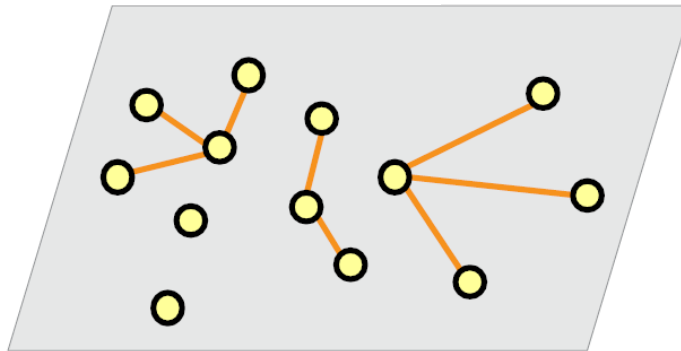
# A duplex network system



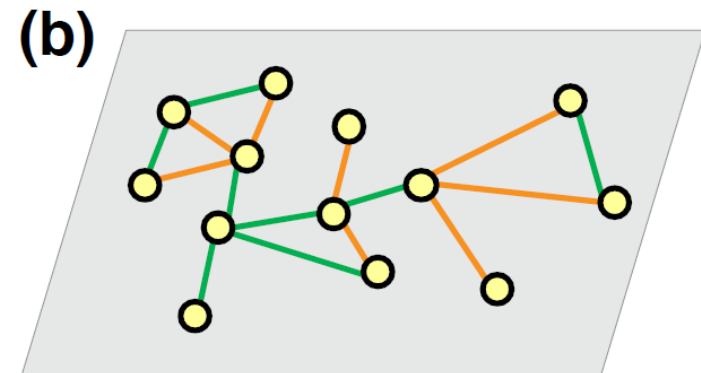
# Multiplex cascade: Multi-layer activation rules



Friendship Layer ↴



Work-colleague Layer ↴



Multiplex  $1 \otimes 2$

- Nodes activate if sufficient fraction of neighbors in **ANY** layer is active (**max**).
- Nodes activate if sufficient fraction of neighbors in **ALL** layers is active (**min**).
- In-between, or mixture rule (**mix**).



# Multiplex Watts model: Analysis

- Generalize Gleeson & Cahalane [PRE 2007] to multiplex networks.
- For the duplex case, we have:

$$\rho = \rho_0 + (1 - \rho_0) \sum_{k_1+k_2 \geq 1} p_{k_1}^{(1)} p_{k_2}^{(2)} \sum_{m_1=0}^{k_1} \sum_{m_2=0}^{k_2} B_{m_1}^{k_1}(q_\infty^{(1)}) B_{m_2}^{k_2}(q_\infty^{(2)}) F_{m_1, m_2}^{k_1, k_2}$$

$$q_{n+1}^{(1)} = \rho_0 + (1 - \rho_0) \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} \frac{k_1 p_{k_1}^{(1)}}{z^{(1)}} p_{k_2}^{(2)} \sum_{m_1=0}^{k_1-1} \sum_{m_2=0}^{k_2} B_{m_1}^{k_1-1}(q_n^{(1)}) B_{m_2}^{k_2}(q_n^{(2)}) F_{m_1, m_2}^{k_1, k_2}$$

- $F_{\text{max/min/mix}}$ : Response functions.

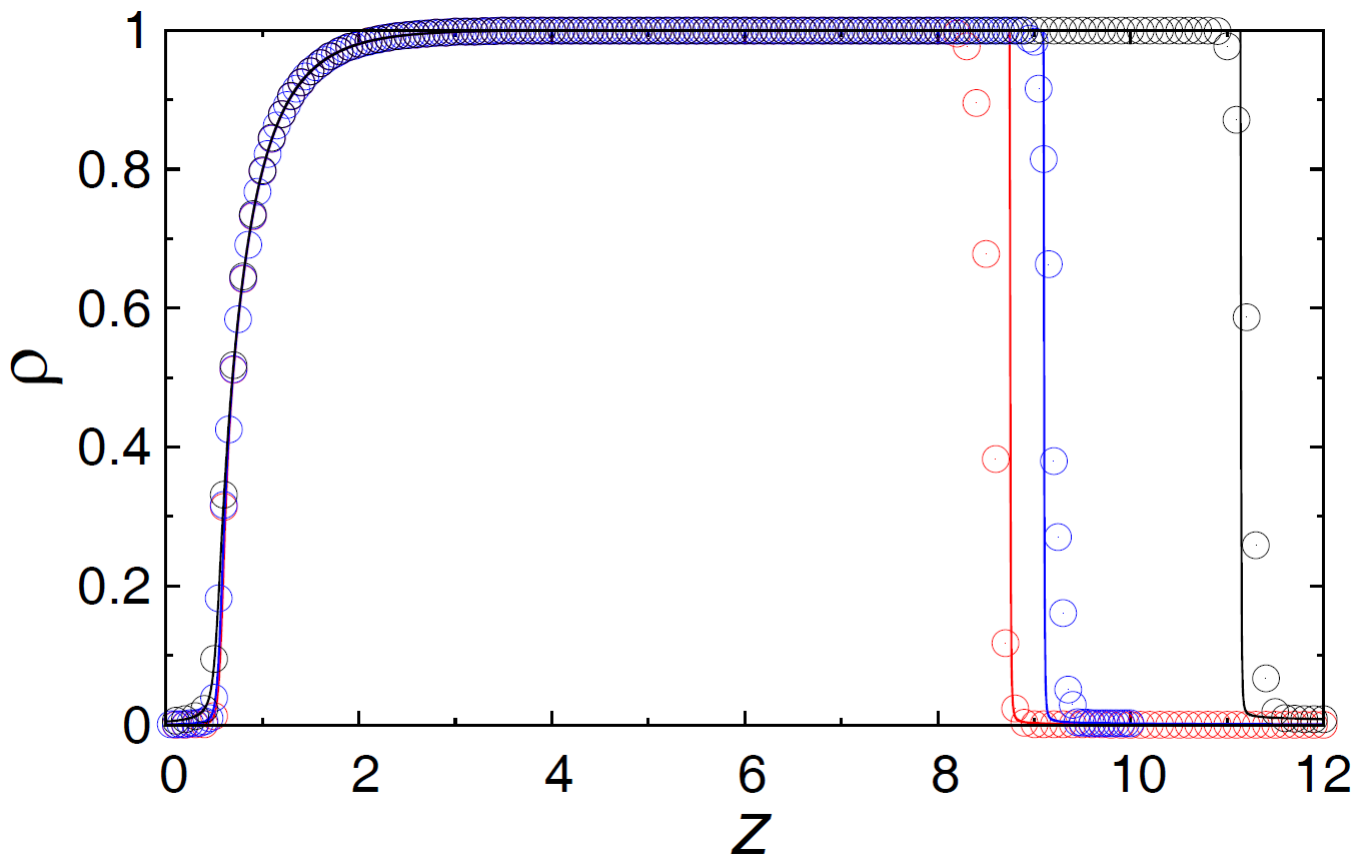
$$F_{m_1, m_2}^{k_1, k_2} = \begin{cases} 0 & \text{if } \max(m_1/k_1, m_2/k_2) \leq r, \\ 1 & \text{if } \max(m_1/k_1, m_2/k_2) > r. \end{cases}$$

$$F_{\text{min}}((m_1, m_2), (k_1, k_2)) = \mathbb{1}_{\{\min(m_1/k_1, m_2/k_2) > r\}}.$$

$$F_{\text{mix}} = \mathcal{E} F_{\text{max}} + (1 - \mathcal{E}) F_{\text{min}}.$$

# Max-model: Theory and simulations agree well

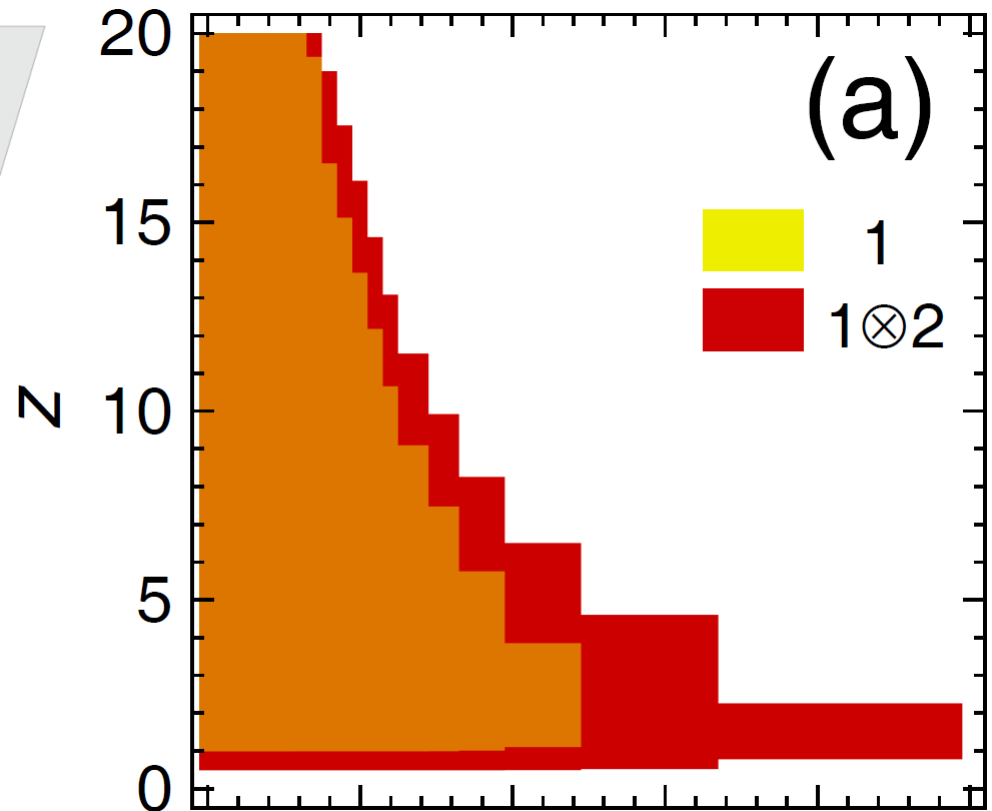
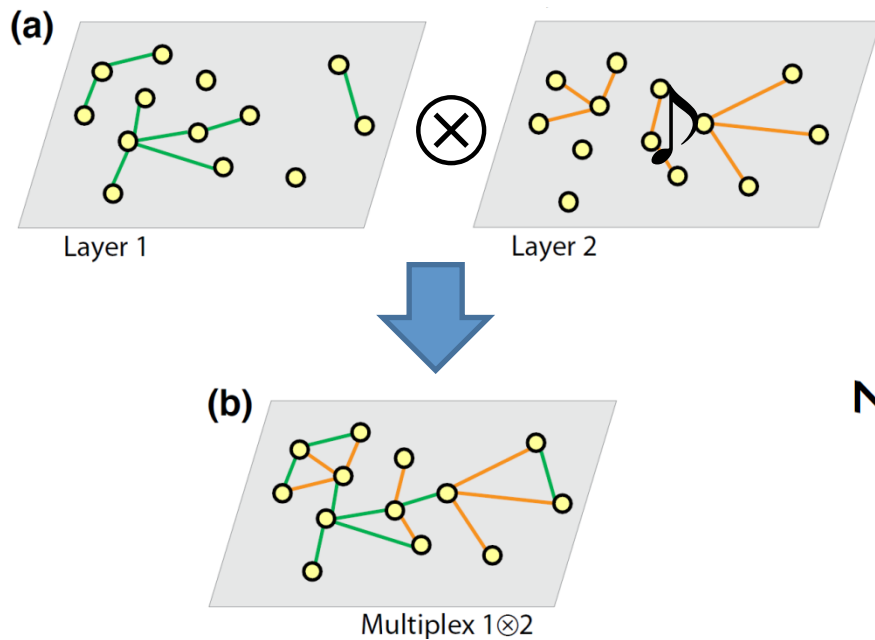
[CD Brummitt, Kyu Min Lee, KIG, *PRE* **85**, 045102(R) (2012)]



E-R network with mean-degree  $z$

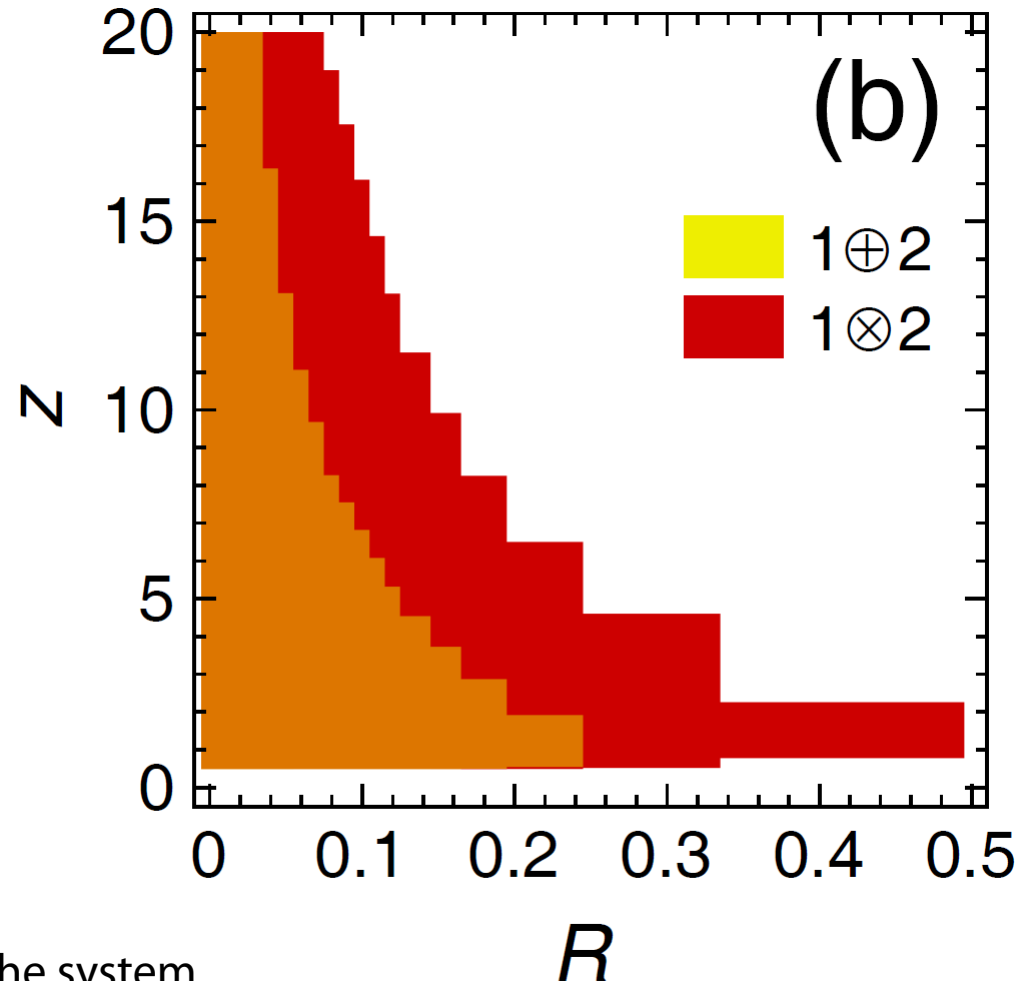
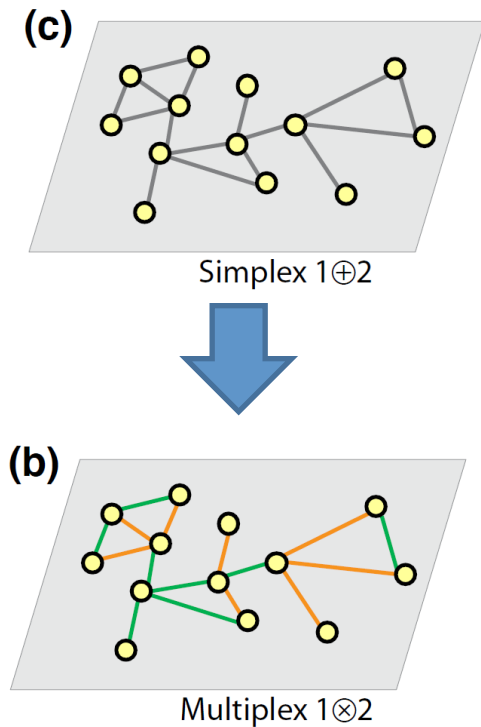
$R=0.18$ ;  $\rho_0=5 \times 10^{-4}$  (○),  $10^{-3}$  (○),  $5 \times 10^{-3}$  (○)

# Multiplexity effect I: Adding layers



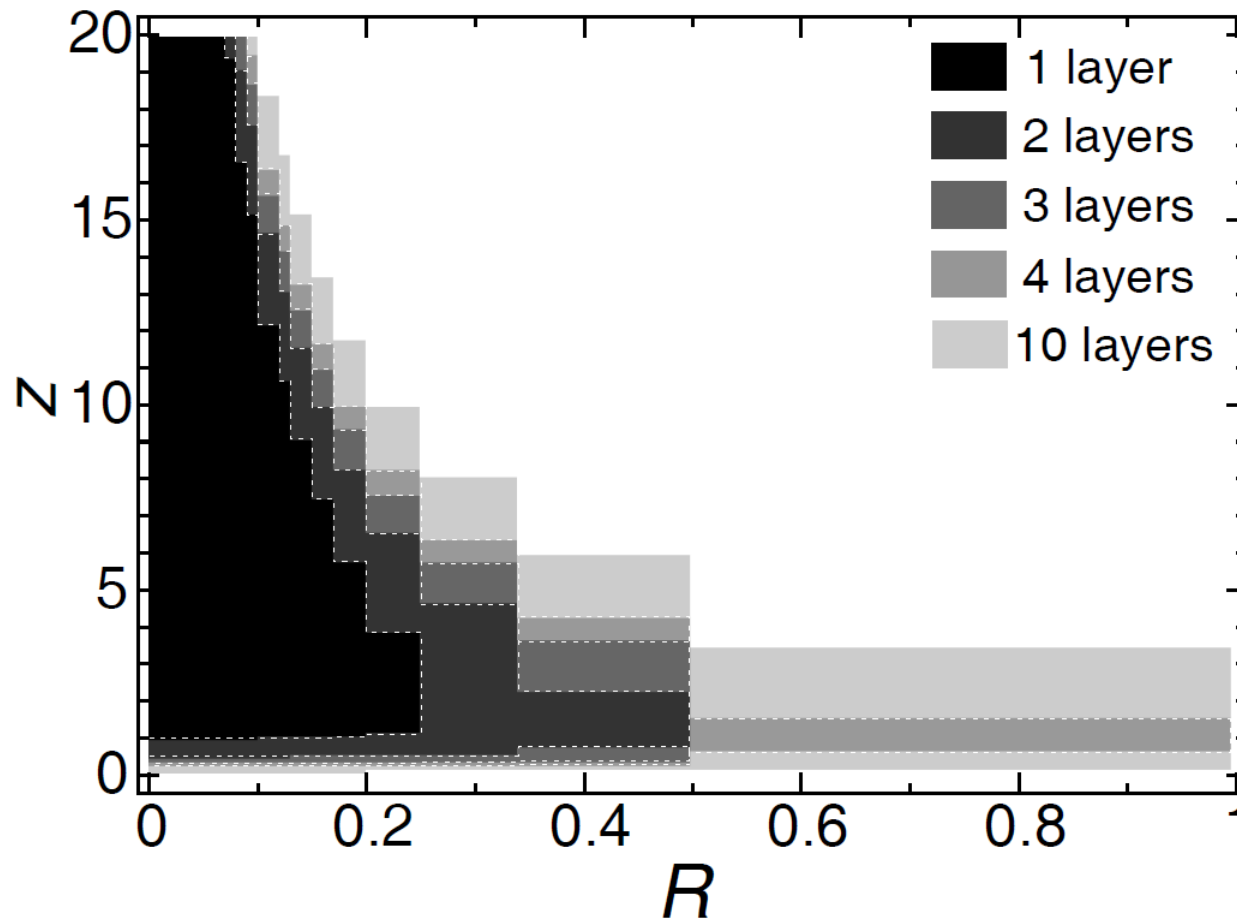
- Adding another layer (i.e. recognizing another type of interaction at play in the system) enlarges the cascade region.
- The **max-dynamics is more vulnerable to global cascades** than single-layer system.

# Multiplexity effect II: Splitting into layers



- Splitting into layers (i.e. recognizing the system in fact consists of multiple channels of interaction) also enlarges the cascade region.
- The **max-rule is more vulnerable to global cascades** than the simplex system.

# More than two layers: $\ell$ -plex networks

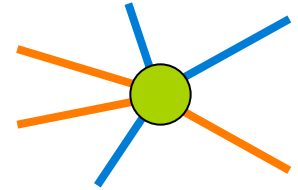


- Cascade possible even for  $R > 1/2$  with enough layers ( $\ell \geq 4$ ).
- Even people extremely difficult to persuade would ride on a bandwagon if she participate a little ( $z \sim 1$ ) in many social spheres ( $\ell \geq 4$ ).

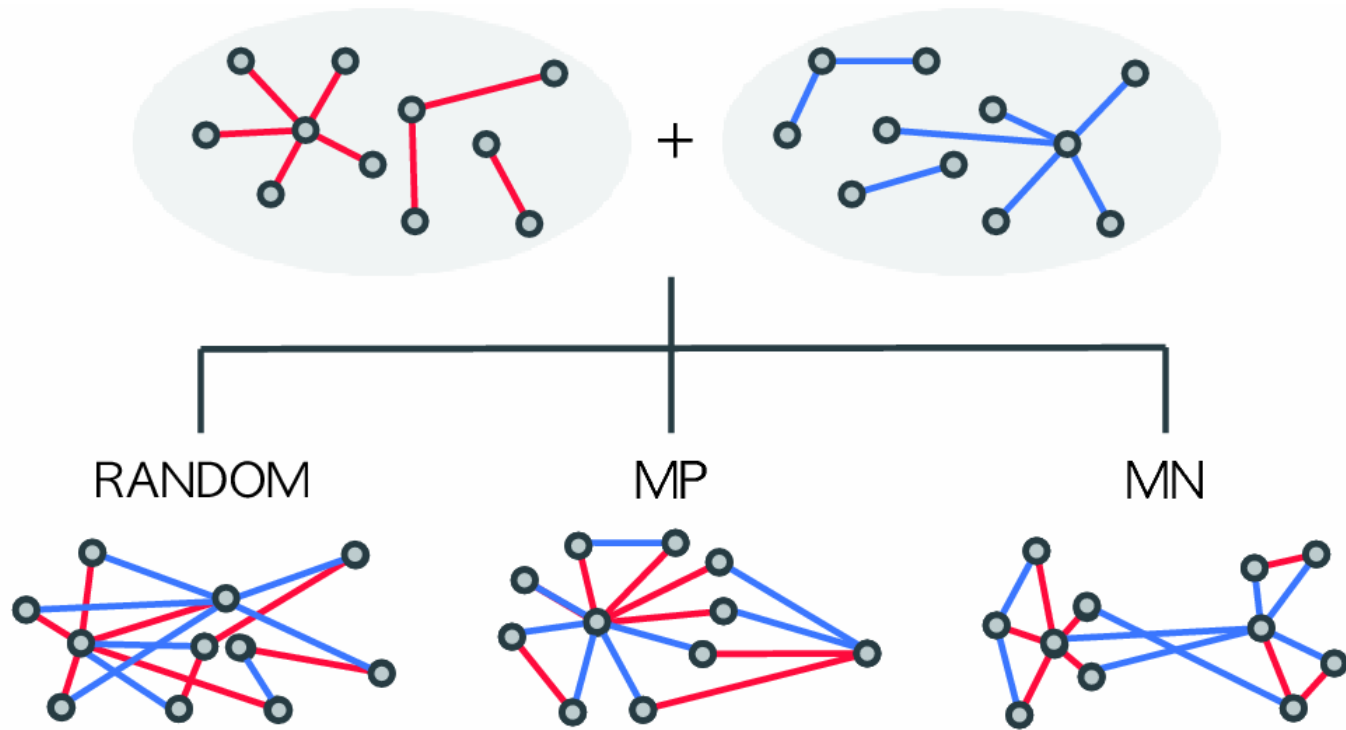
# **Multiplexity on Structure**

# Network couplings are non-random: Correlated multiplexity

- Interlayer couplings are non-random.
  - A node's degree in one layer and those in the others are not randomly distributed.
  - A person with many friends is likely to have many work-related acquaintances.
  - Hub in one layer tends to be hub in another layer.
  - Pair of people connected in one layer is likely to be connected, or at least closer, in another layer.
- Uncorrelated (random) coupling.
- Anti-correlated coupling.



# CORRELATED MULTIPLICITY





# Multiplex ER Networks with Correlated Multiplexity

[Kyu-Min Lee, Jung Yeol Kim, WKCho, KIG, IMKim, *New J Phys* 14, 033027 (2012)]

- **Multiplex network of two ER layers (duplex ER network).**
  - Same set of  $N$  nodes.
  - Generate two ER networks independently, with mean degree  $z_1$  and  $z_2$ .
  - Interlace them in some way:
    - unc (uncorrelated): Match nodes randomly.
    - MP (maximally positive): Match nodes in perfect order of degree-ranks.
    - MN (maximally negative): “ in perfect anti-order of degree-ranks.
  - Obtain the multiplex network.
- Cf. Interacting network model by Leicht & D'Souza arXiv:0907.0894  
Interdependent network model by Buldyrev et al. Nature2010.

# Generating function analysis

$$\pi^{(\ell)}(k) \rightarrow \Pi(k_1, k_2) \rightarrow P(k)$$

$$S = 1 - g_0(u) \quad \left[ g_0(x) = \sum_{k=0}^{\infty} P(k)x^k \right]$$

$$u = g_1(u) \equiv g_0'(u) / g_0'(1) = \frac{1}{\langle k \rangle} \sum_{k=1}^{\infty} kP(k)u^{k-1}$$

$$\langle s \rangle = 1 + \frac{g_0'(1)u^2}{g_0(u)[1 - g_1'(u)]}$$

$$S > 0 \rightarrow u < 1 \rightarrow \sum_k k(k-2)P(k) = \langle k^2 \rangle - 2\langle k \rangle > 0$$

[Newman, Watts, Strogatz (2001); Molloy & Reed (1996)]

- A crucial step is to obtain  $P(k)$  from  $\pi(k)$ .

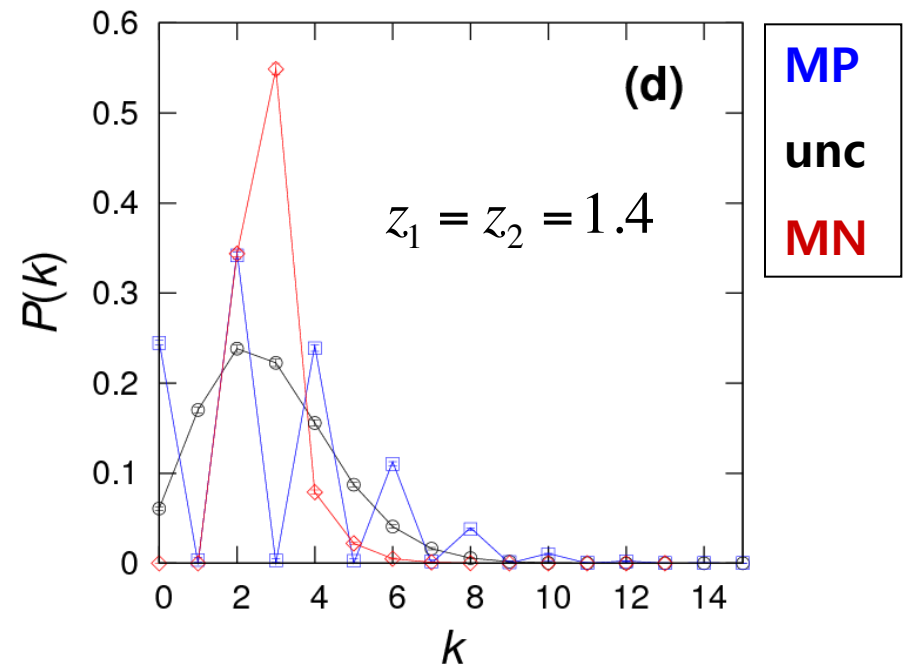
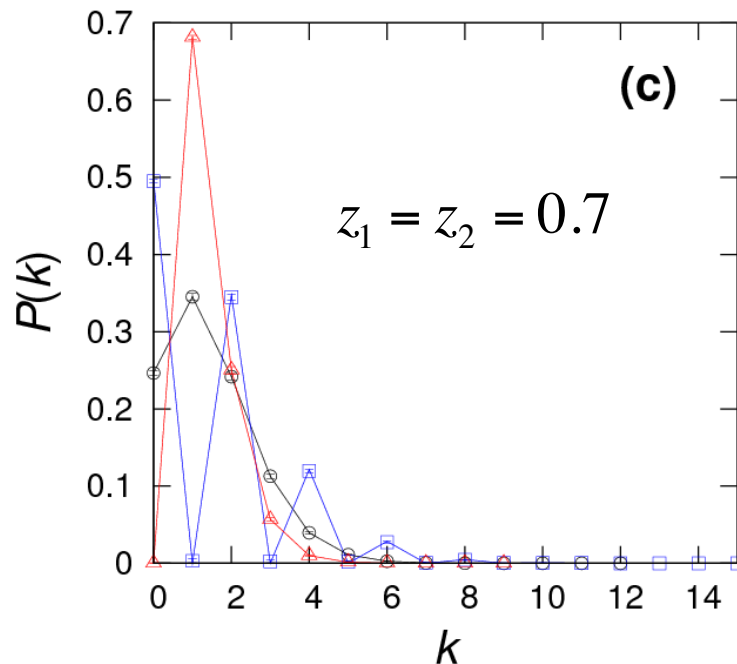
# Superposed degree distributions for $z_1=z_2$

$$\text{i) MP: } P(k) = \begin{cases} 0 & (k \text{ odd}), \\ \frac{e^{-z_1} z_1^{k/2}}{(k/2)!} & (k \text{ even}). \end{cases} \quad \text{ii) unc: } P(k) = \frac{e^{-2z_1} (2z_1)^k}{k!}.$$

$$\text{iii) MN: } z_1 < \ln 2: P(0) = 2\pi(0) - 1, P(k \geq 1) = 2\pi(k).$$

$$\ln 2 < z_1 < z^*: P(0) = 0, P(1) = 2[2\pi(0) + \pi(1) - 1],$$

$$P(2) = 2[\pi(2) - \pi(0)] + 1, P(k \geq 3) = 2\pi(k).$$

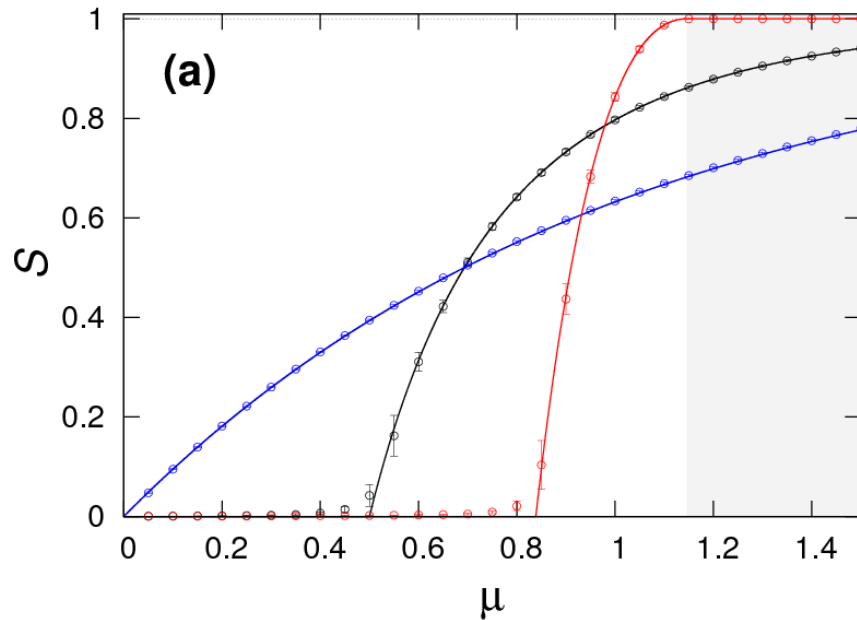


MP

unc

MN

# Giant component sizes for $z_1=z_2$



**MP**

**unc**

**MN**

$$z_c^{MP} = 0$$

$$z_c^{unc} = 0.5$$

$$z_c^{MN} = 0.838587497\dots$$

For MP,

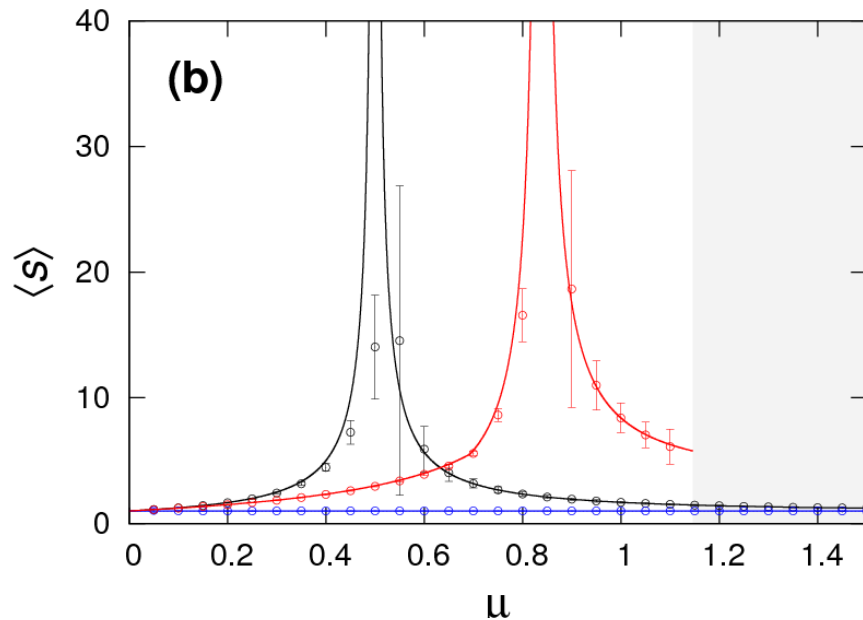
$$S = 1 - e^{-z_1}, \quad \langle s \rangle = 1 \quad \text{for all } z > 0.$$

For MN,

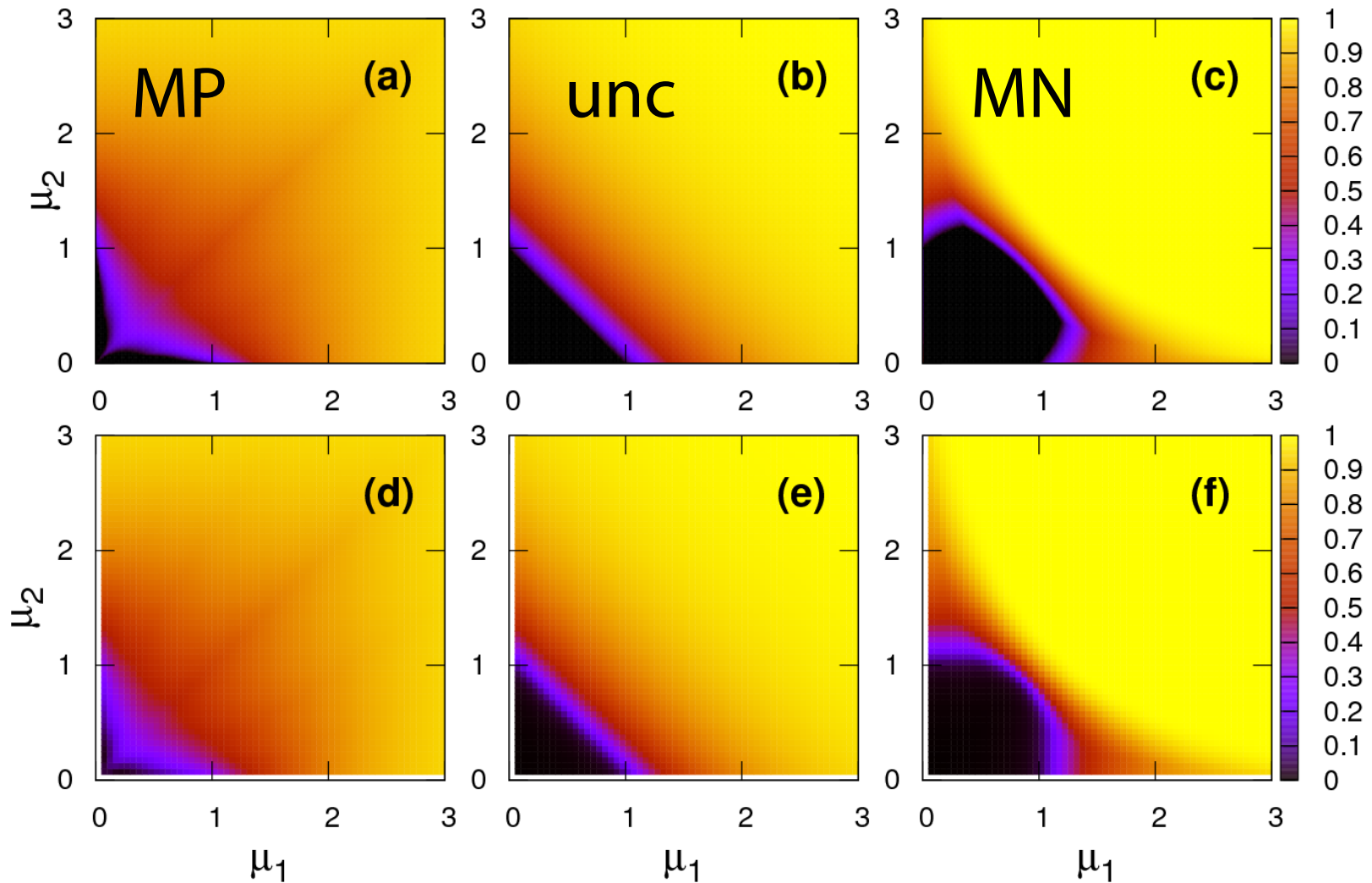
$$S = 1 \quad \text{for } z_1 \geq z^* = 1.14619322\dots$$

$$z_c^{MN} : z^2 - z - e^{-z} + 1 = 0$$

$$z^* : (2+z)e^{-z} = 1$$

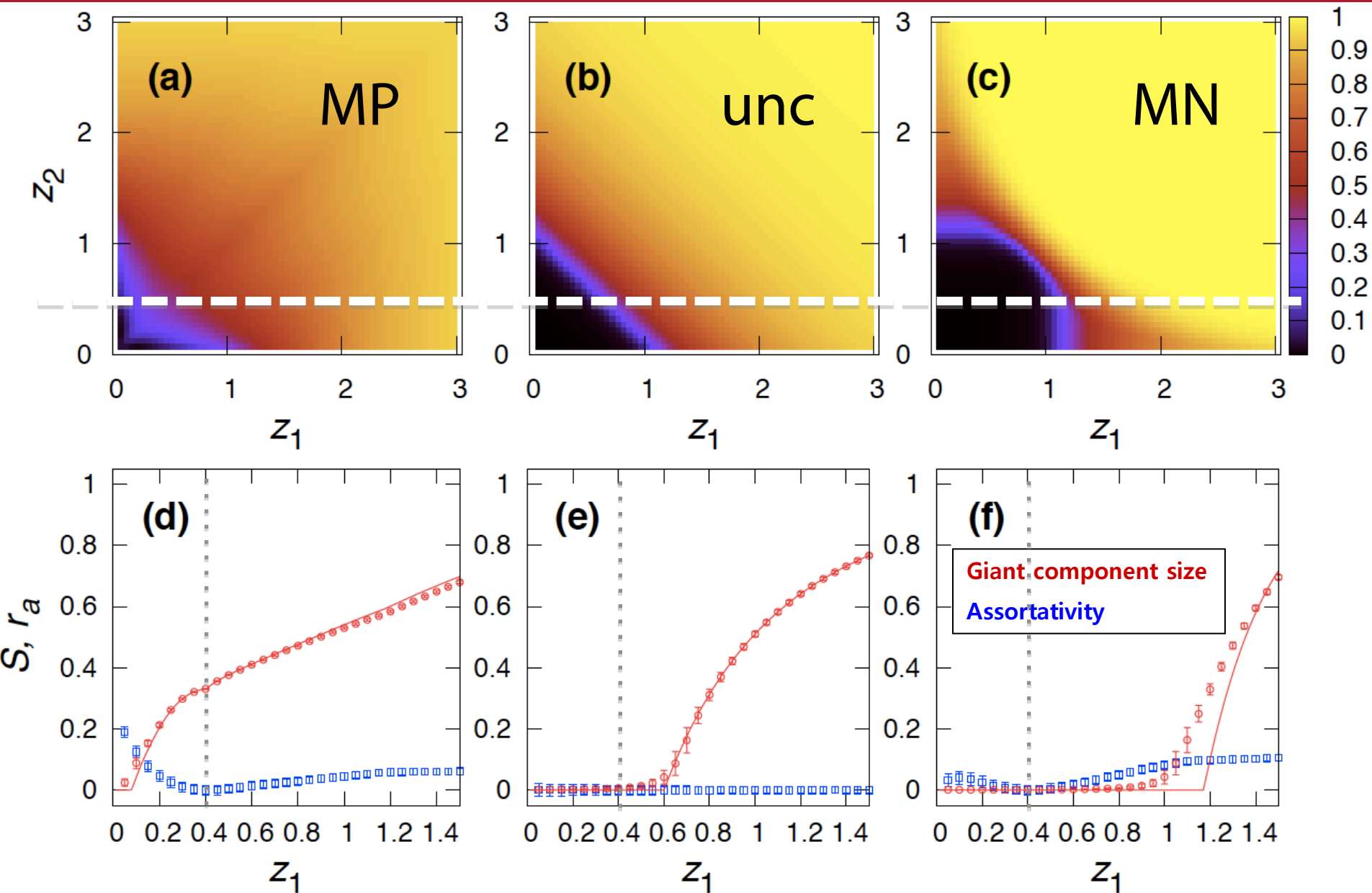


# Giant component sizes with $z_1 \neq z_2$



- Analytics agrees overall but not perfectly – Degree correlations!

# Assortativity via correlated multiplexity



# TAKE-HOME MESSAGE

- ***Think Multiplexity!***
- **Network multiplexity as a new layer of complexity in complex systems' structure and dynamics.**

*Further recent related works...*

- Sandpile dynamics [KM Lee, KIG, IMKim, *J Korean Phys Soc* **60**, 641 (2012)].
- Weighted threshold cascade [Yagan et al. arXiv:1204.0491].
- Boolean network [Arenas/Moreno arXiv:1205.3111].
- Diffusion dynamics [Diaz-Guilera/Moreno/Arenas arXiv:1207.2788]
- *More to come!*

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- On-line Registration** : June 22, 2013
- Abstract Submission** : March 31, 2013
- Abstract Submission Acceptance Notice** : May 15, 2013

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