Multiplexity in networks

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Thanks to: Kyu-Min Lee, Byungjoon Min, Won-kuk Cho, Jung Yeol Kim, Jeehye Choi, In-mook Kim, Charlie Brummitt (UC Davis), NRF Korea

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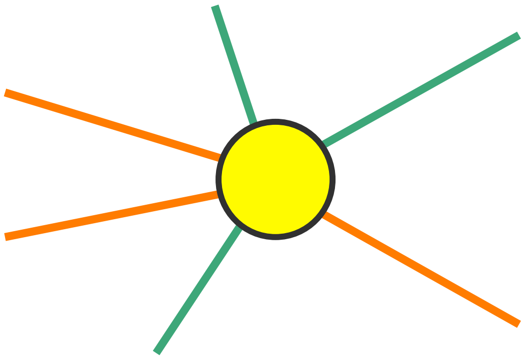
ECT* Workshop, July 23 2012 @ Trento Italy

Complex systems are MULTIPLEX

- ***Multiplexity*:** Existence of **more than one type of links** whose interplay can affect the **structure and/or function**.

- **Multiplex networks**

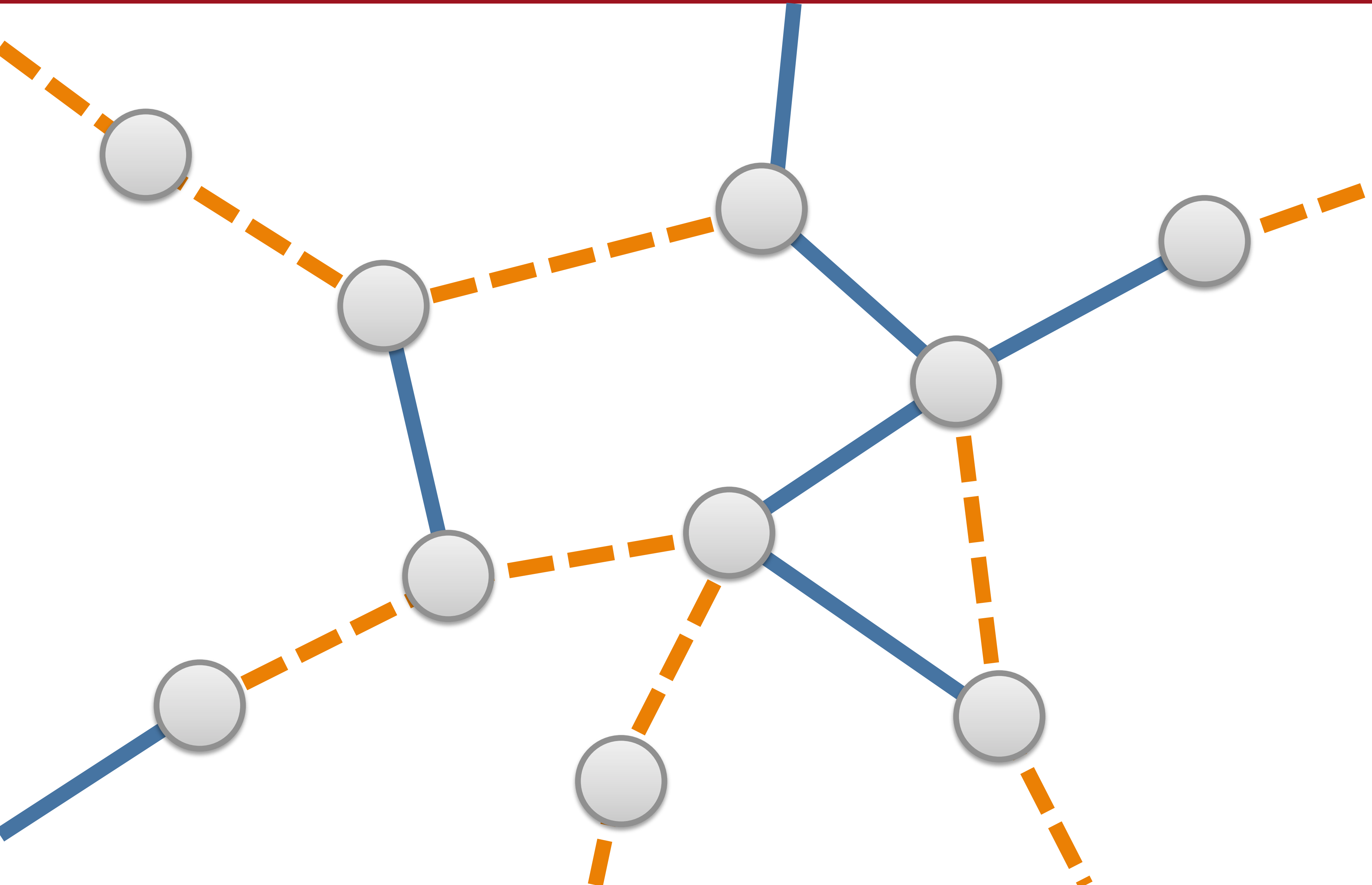
  cf. **multilayer** networks, **interdependent** networks, **interacting** networks, **coupled** networks, **network of networks**, ...



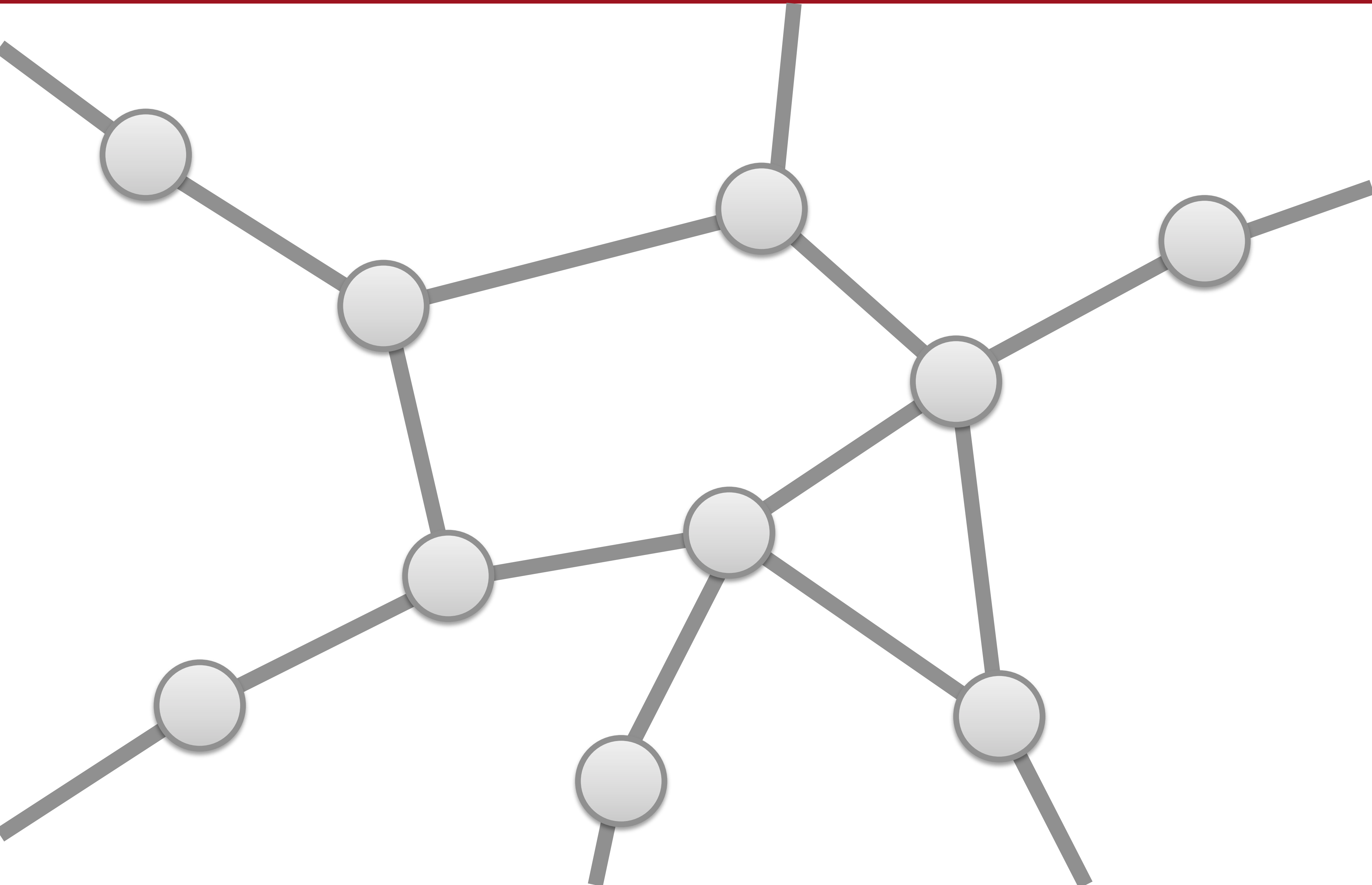
  - **Multi-relational social networks** [Padgett&Ansell (1993); Szell et al (2010)].
  - **Cellular networks** [Yeast, M. pneumoniae, etc]
  - **Interdependent critical infrastructures** [Buldyrev et al (2010)]
  - **Transportation networks** [Parshani et al (2011)].
  - **Economic networks**

- **Single/simplex-network description is incomplete.**

Multiplex networks



Simplex networks



Multiplexity on Dynamics

THRESHOLD CASCADES

- **A simple model of global cascades on random networks**

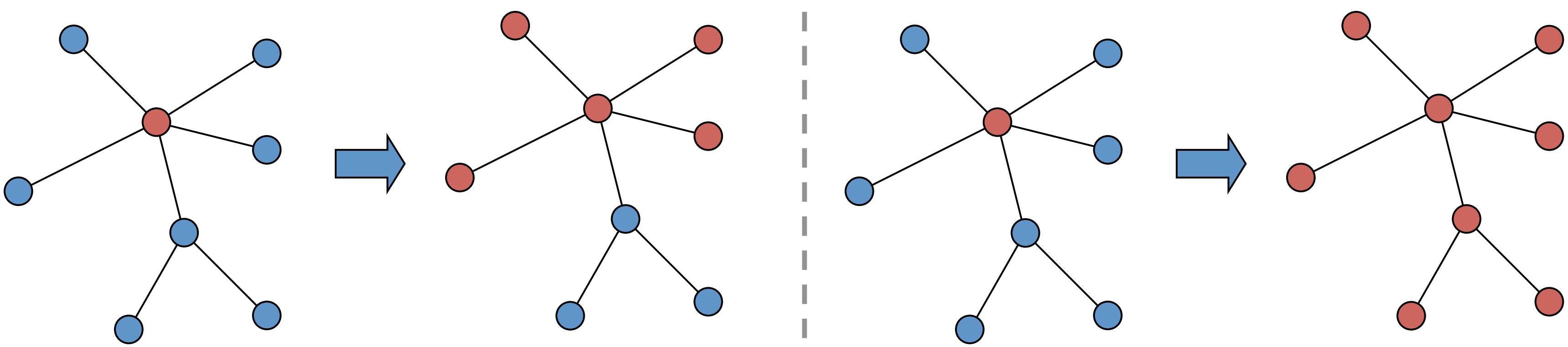
  Duncan J. Watts*

  Department of Sociology, Columbia University New York, NY 10027

  Communicated by Murray Gell-Mann, Santa Fe Institute, Santa Fe, NM, February 14, 2002 (received for review May 29, 2001)

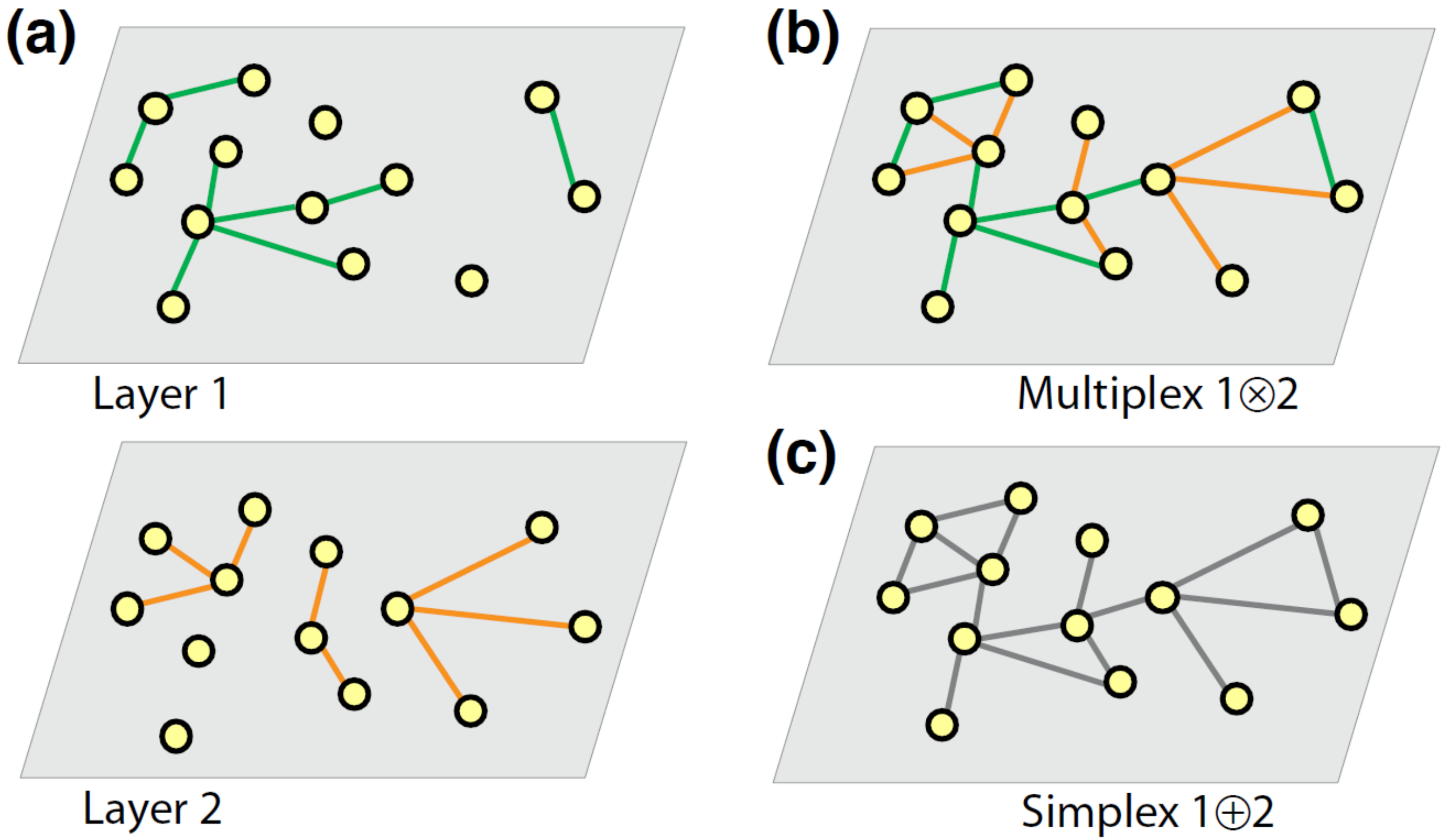
  5766–5771 | PNAS | April 30, 2002 | vol. 99 | no. 9 www.pnas.org/cgi/doi/10.1073/pnas.082090499

- Model for **behavioral adoption cascade** [Schelling, Granovetter '70s].
- E.g., using a smartphone app, or wearing a hockey helmet.
- **A node gets activated (0→1) if at least a fraction R of its neighbors are active.**
- Ex) $R = 1/2$ vs. $R = 1/4$

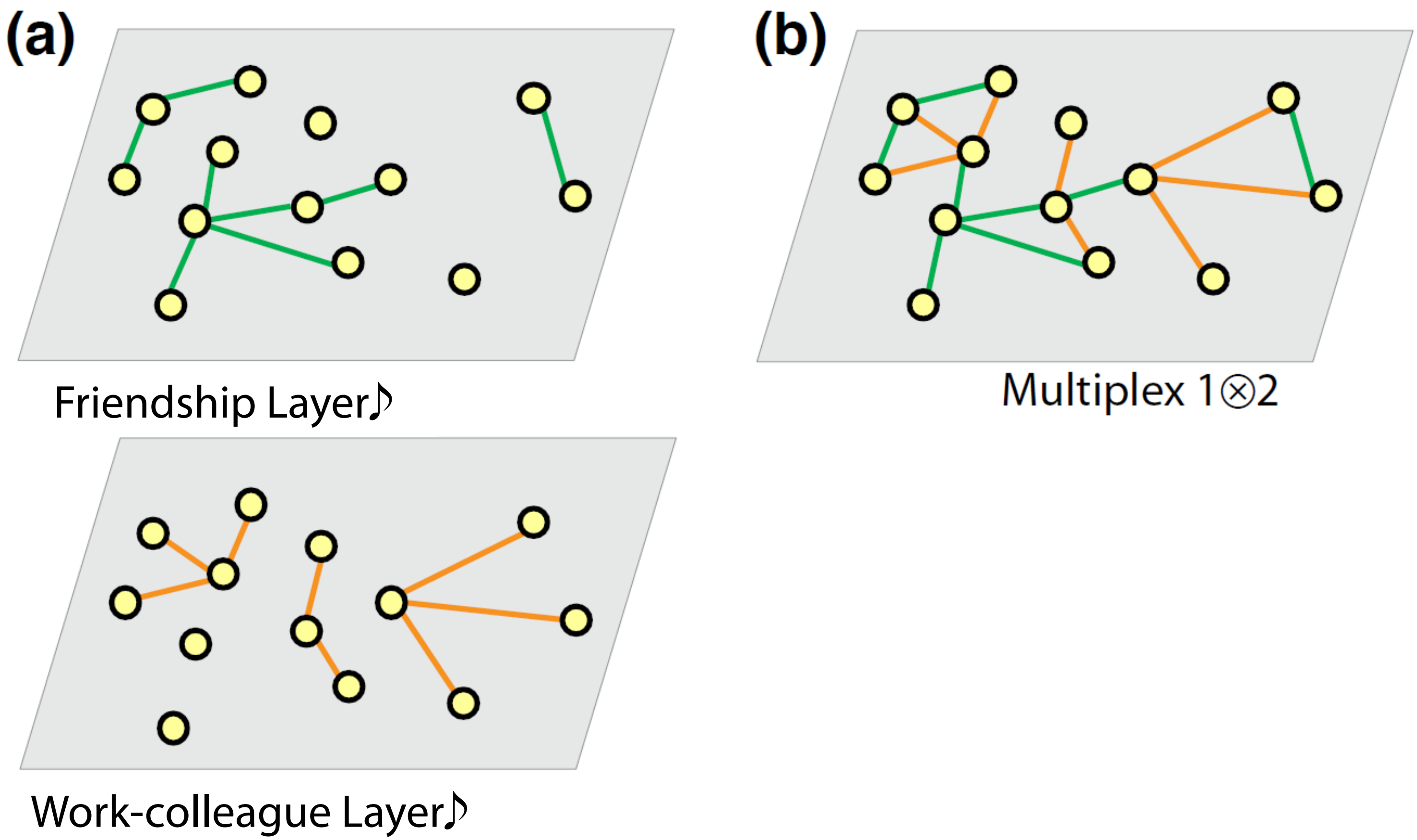


- Interested in the condition for the **global cascades** from small initial active seeds.

A duplex network system



Multiplex cascade: Multi-layer activation rules



- Nodes activate if sufficient fraction of neighbors in **ANY** layer is active (**max**).
- Nodes activate if sufficient fraction of neighbors in **ALL** layers is active (**min**).
- In-between, or mixture rule (**mix**).

Multiplex Watts model: Analysis

- Generalize Gleeson & Cahalane [PRE 2007] to multiplex networks.
- For the duplex case, we have:

$$\rho = \rho_0 + (1-\rho_0) \sum_{k_1+k_2 \geq 1} p^{(1)}_{k_1} p^{(2)}_{k_2} \sum_{m_1=0}^{k_1} \sum_{m_2=0}^{k_2} B^{k_1}_{m_1}(q^{(1)}_\infty) B^{k_2}_{m_2}(q^{(2)}_\infty) F^{k_1,k_2}_{m_1,m_2}$$

$$q^{(1)}_{n+1} = \rho_0 + (1-\rho_0) \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} \frac{k_1 p^{(1)}_{k_1}}{z^{(1)}} p^{(2)}_{k_2} \sum_{m_1=0}^{k_1-1} \sum_{m_2=0}^{k_2} B^{k_1-1}_{m_1}(q^{(1)}_n) B^{k_2}_{m_2}(q^{(2)}_n) F^{k_1,k_2}_{m_1,m_2}$$

- $\boldsymbol{F}_{\textbf{max/min/mix}}$: Response functions.

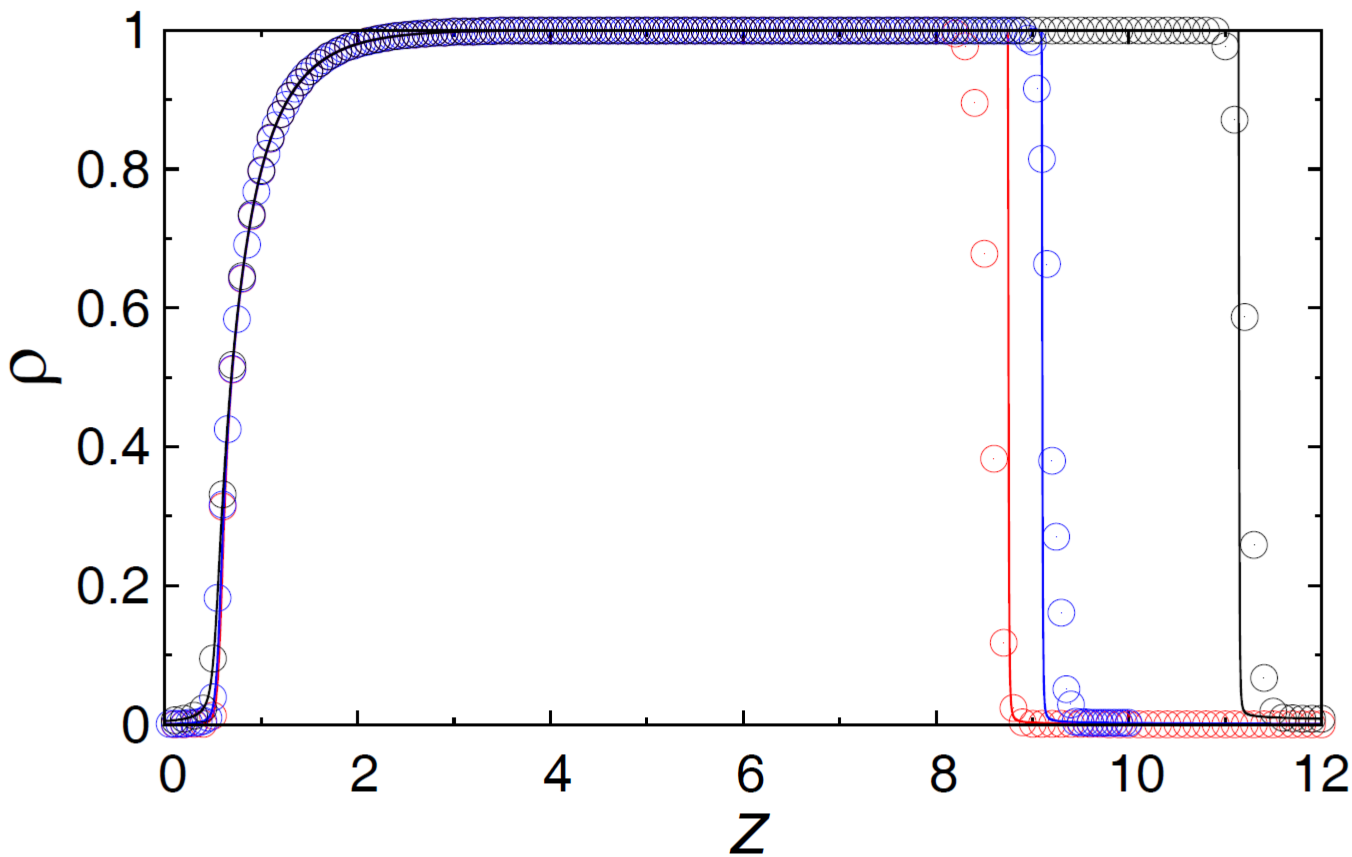
$$F^{k_1,k_2}_{m_1,m_2} = \begin{cases} 0 & \text{if } \max(m_1/k_1, m_2/k_2) \leq r, \\ 1 & \text{if } \max(m_1/k_1, m_2/k_2) > r. \end{cases}$$

$$F_{\min}((m_1,m_2),(k_1,k_2)) = \mathbb{1}_{\{\min(m_1/k_1, m_2/k_2) > r\}}.$$

$$F_{\text{mix}} = \mathcal{E} F_{\max} + (1-\mathcal{E}) F_{\min}.$$

Max-model: Theory and simulations agree well

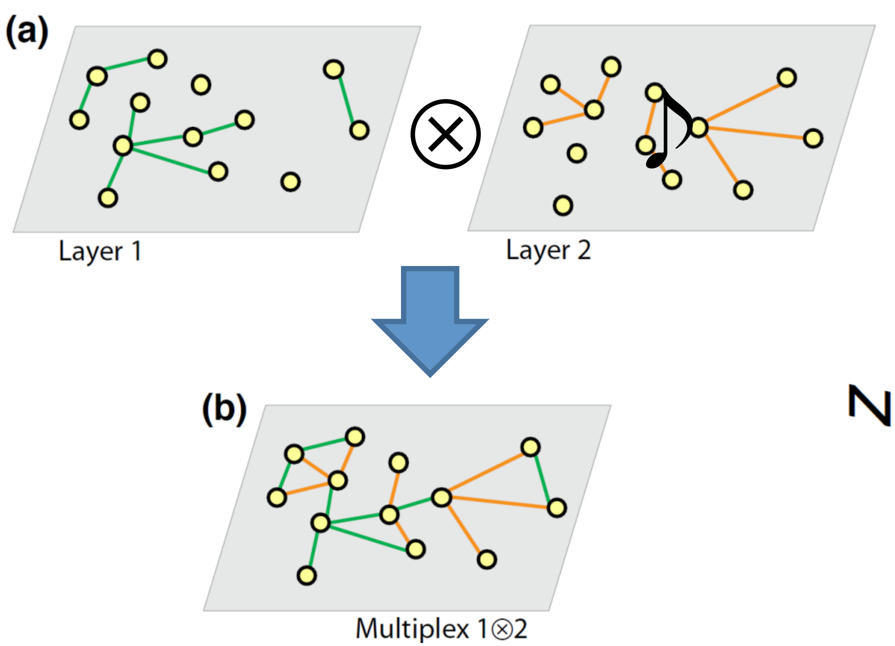
[CD Brummitt, Kyu Min Lee, KIG, *PRE* 85, 045102(R) (2012)]

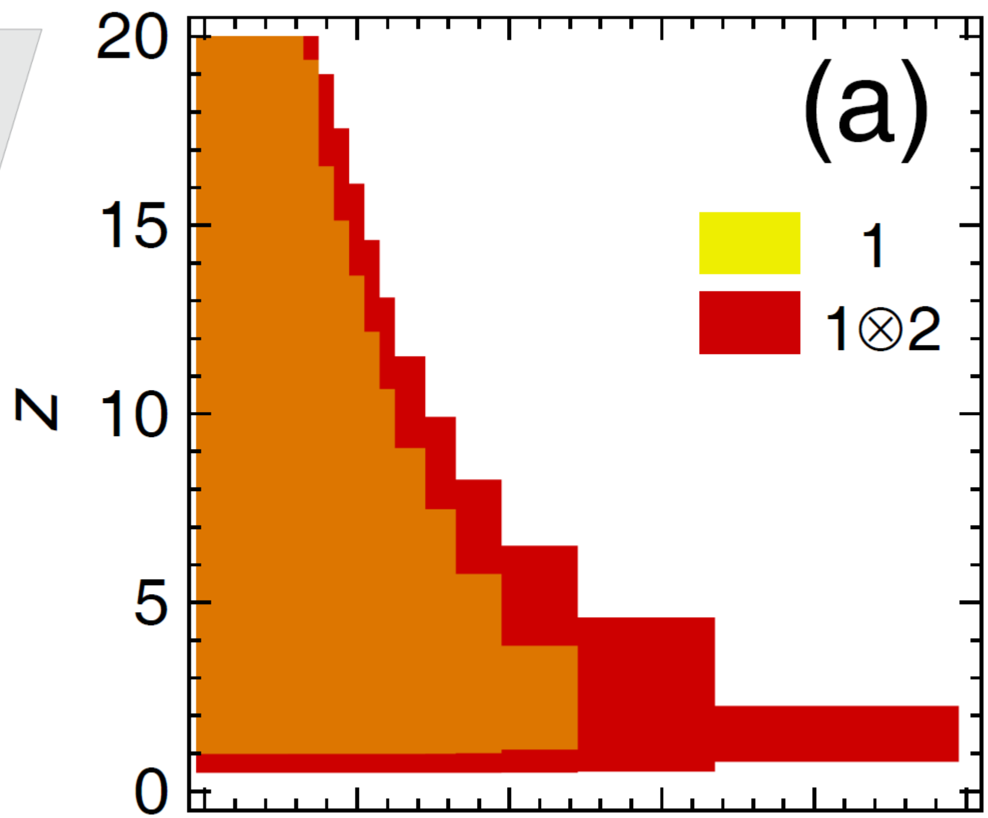


E-R network with mean-degree *z*

R=0.18 ; ρ_0=5x10^{-4} (O), 10^{-3} (O), 5x10^{-3} (O)

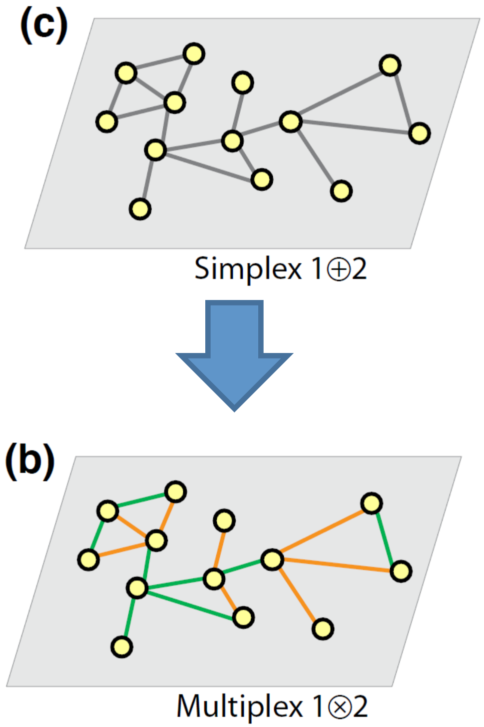
Multiplexity effect I: Adding layers

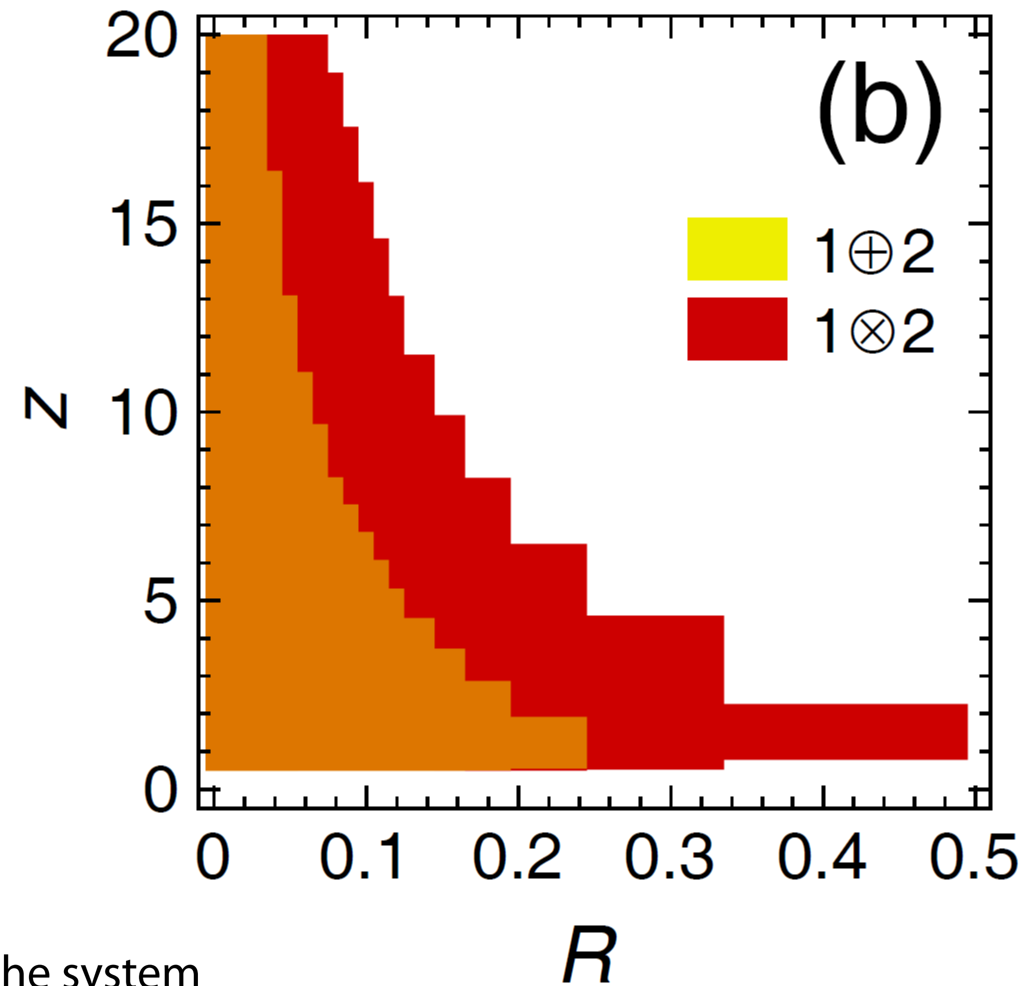




- Adding another layer (i.e. recognizing another type of interaction at play in the system) enlarges the cascade region.
- The **max-dynamics is more vulnerable to global cascades** than single-layer system.

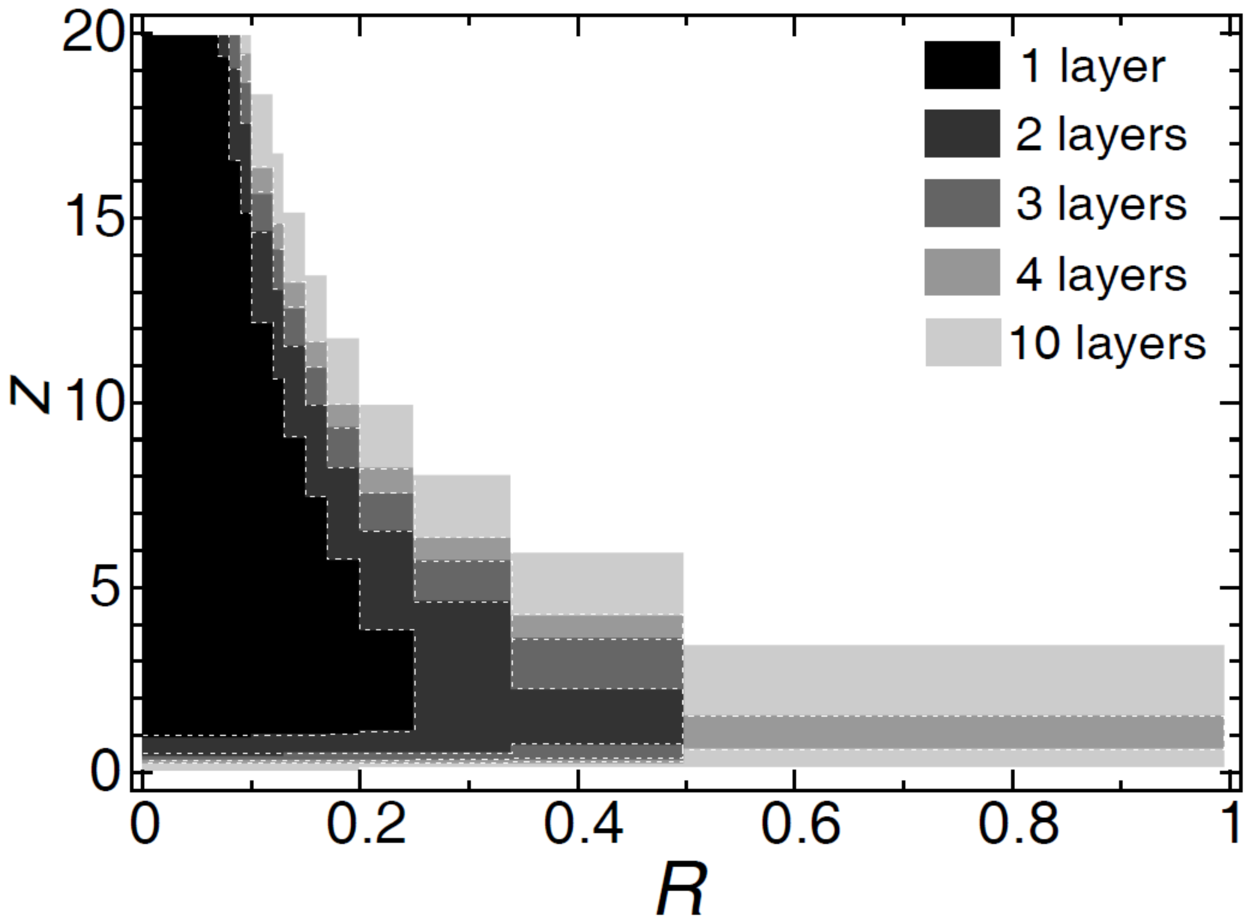
Multiplexity effect II: Splitting into layers





- Splitting into layers (i.e. recognizing the system in fact consists of multiple channels of interaction) also enlarges the cascade region.
- The **max-rule is more vulnerable to global cascades** than the simplex system.

More than two layers: ℓ-plex networks

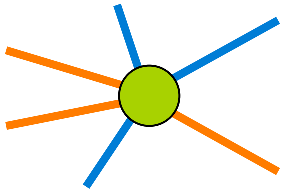


- Cascade possible even for R>1/2 with enough layers (ℓ >=4).
- Even people extremely difficult to persuade would ride on a bandwagon if she participate a little (z~1) in many social spheres (ℓ >=4).

Multiplexity on Structure

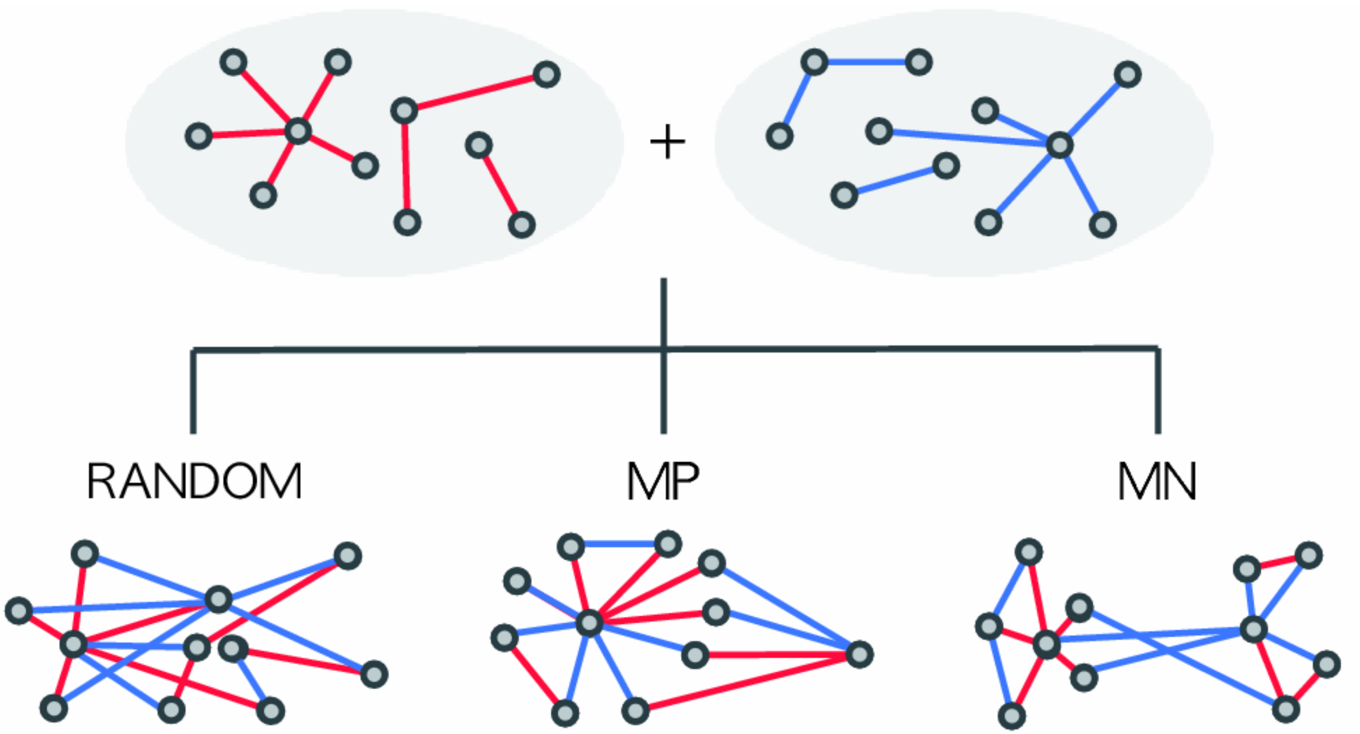
Network couplings are non-random: Correlated multiplexity

- Interlayer couplings are non-random.
  - A node's degree in one layer and those in the others are not randomly distributed.



  - A person with many friends is likely to have many work-related acquaintances.
  - Hub in one layer tends to be hub in another layer.
  - Pair of people connected in one layer is likely to be connected, or at least closer, in another layer.
- Uncorrelated (random) coupling.
- Anti-correlated coupling.

CORRELATED MULTIPLEXITY



Multiplex ER Networks with Correlated Multiplexity

[Kyu-Min Lee, Jung Yeol Kim, WKCho, KIG, IMKim, *New J Phys* 14, 033027 (2012)]

- **Multiplex network of two ER layers (duplex ER network).**
  - Same set of N nodes.
  - Generate two ER networks independently, with mean degree z_1 and z_2.
  - Interlace them in some way:
    - ¬ unc (uncorrelated): Match nodes randomly.
    - ¬ MP (maximally positive): Match nodes in perfect order of degree-ranks.
    - ¬ MN (maximally negative): " in perfect anti-order of degree-ranks.
  - Obtain the multiplex network.

- Cf. Interacting network model by Leicht & D'Souza arXiv:0907.0894
  Interdependent network model by Buldyrev et al. Nature2010.

Generating function analysis

$$\pi^{(\ell)}(k) \;\rightarrow\; \Pi(k_1, k_2) \;\rightarrow\; P(k)$$

$$S = 1 - g_0(u) \qquad \left[g_0(x) = \sum_{k=0}^{\infty} P(k) x^k \right]$$

$$u = g_1(u) \equiv g_0'(u) / g_0'(1) = \frac{1}{\langle k \rangle} \sum_{k=1}^{\infty} k P(k) u^{k-1}$$

$$\langle s \rangle = 1 + \frac{g_0'(1) u^2}{g_0(u)[1 - g_1'(u)]}$$

$$S > 0 \;\rightarrow\; u < 1 \;\rightarrow\; \sum_k k(k-2)P(k) = \langle k^2 \rangle - 2\langle k \rangle > 0$$

[Newman, Watts, Strogatz (2001); Molloy & Reed (1996)]

- A crucial step is to obtain P(k) from π(k).

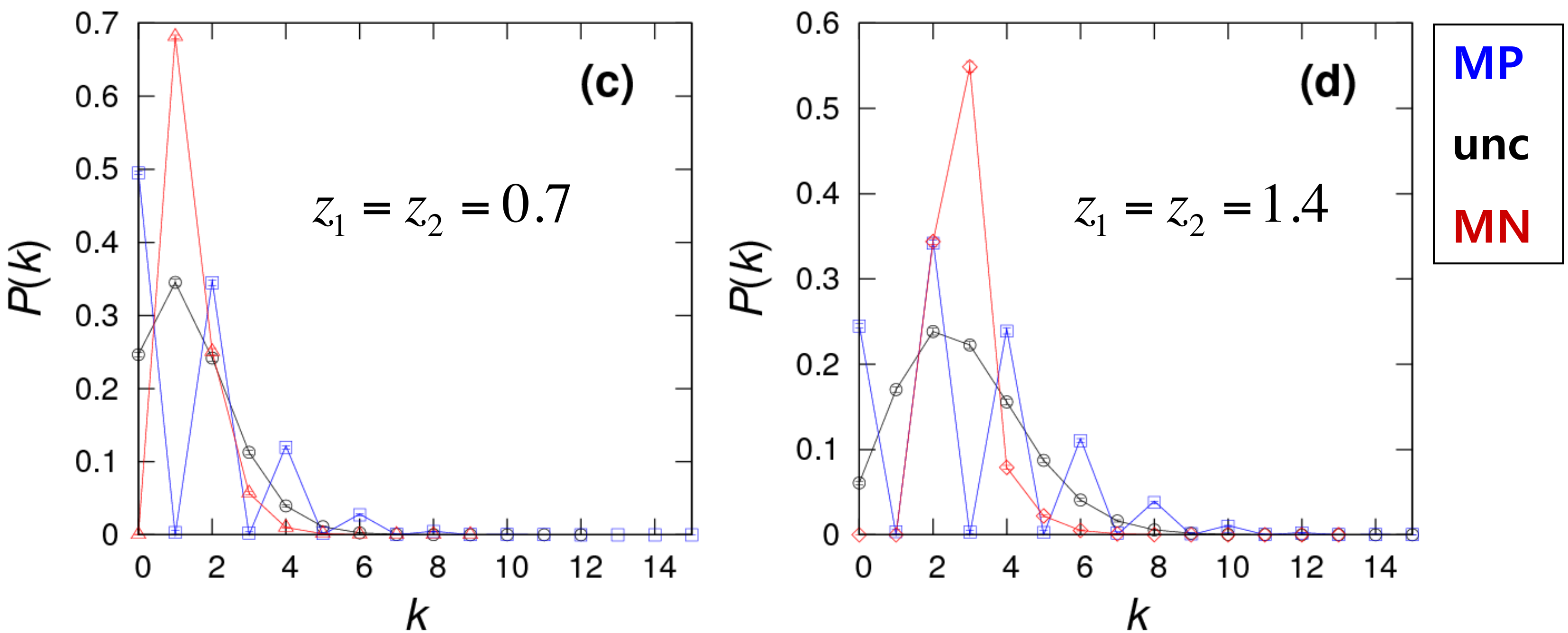
Superposed degree distributions for $z_1 = z_2$

i) MP: $P(k) = \begin{cases} 0 & (k \text{ odd}), \\ \dfrac{e^{-z_1} z_1^{k/2}}{(k/2)!} & (k \text{ even}). \end{cases}$ ii) unc: $P(k) = \dfrac{e^{-2z_1}(2z_1)^k}{k!}.$

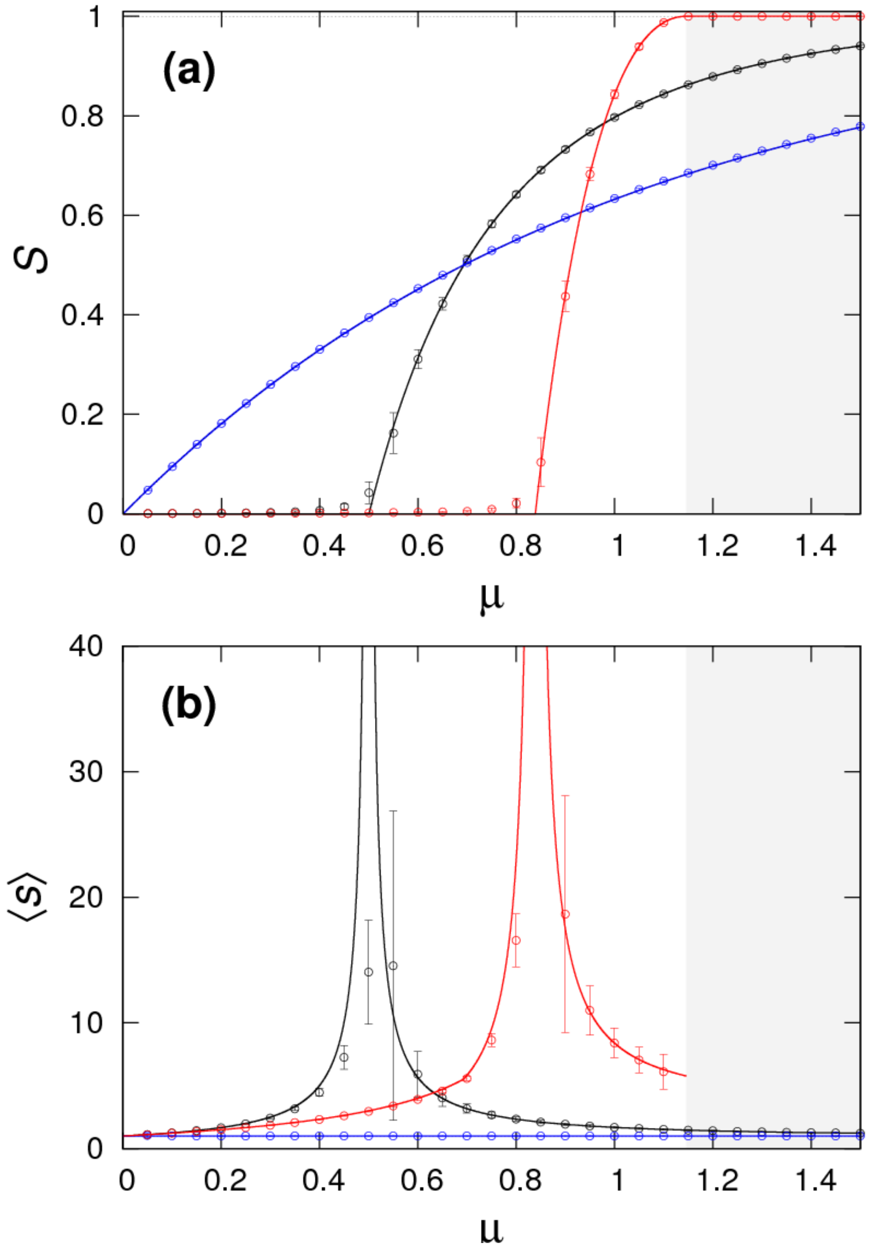
iii) MN: $z_1 < \ln 2: \quad P(0) = 2\pi(0) - 1, P(k \geq 1) = 2\pi(k).$

$\ln 2 < z_1 < z^*: \quad P(0) = 0, P(1) = 2[2\pi(0) + \pi(1) - 1],$

$P(2) = 2[\pi(2) - \pi(0)] + 1, P(k \geq 3) = 2\pi(k).$



Giant component sizes for $z_1 = z_2$



MP

unc

MN

$$z_c^{MP} = 0$$

$$z_c^{unc} = 0.5$$

$$z_c^{MN} = 0.838587497\ldots$$

For MP,

$$S = 1 - e^{-z_1}, \quad \langle s \rangle = 1 \text{ for all } z > 0.$$

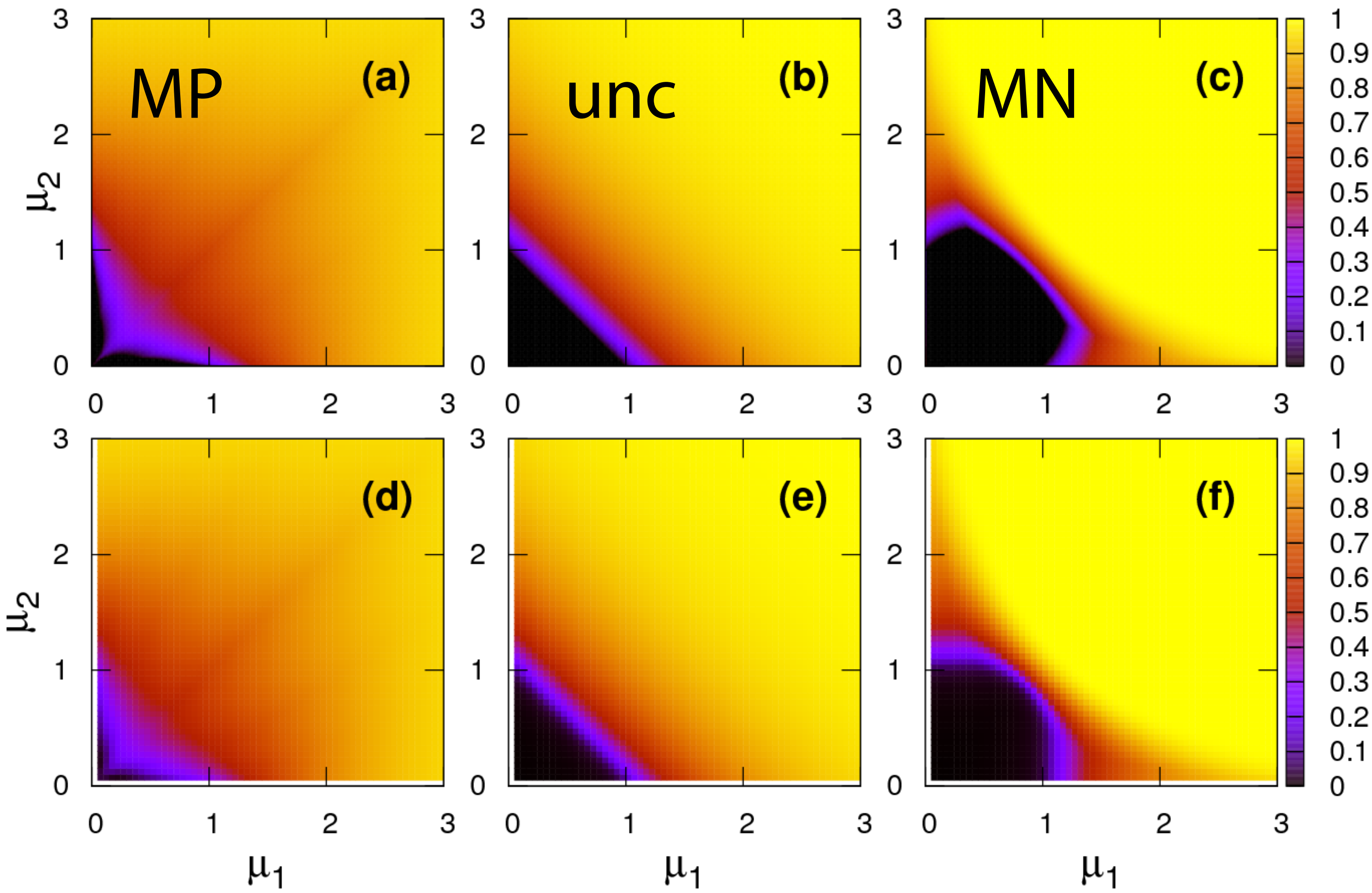
For MN,

$$S = 1 \text{ for } z_1 \geq z^* = 1.14619322\ldots$$

$$z_c^{MN} : z^2 - z - e^{-z} + 1 = 0$$

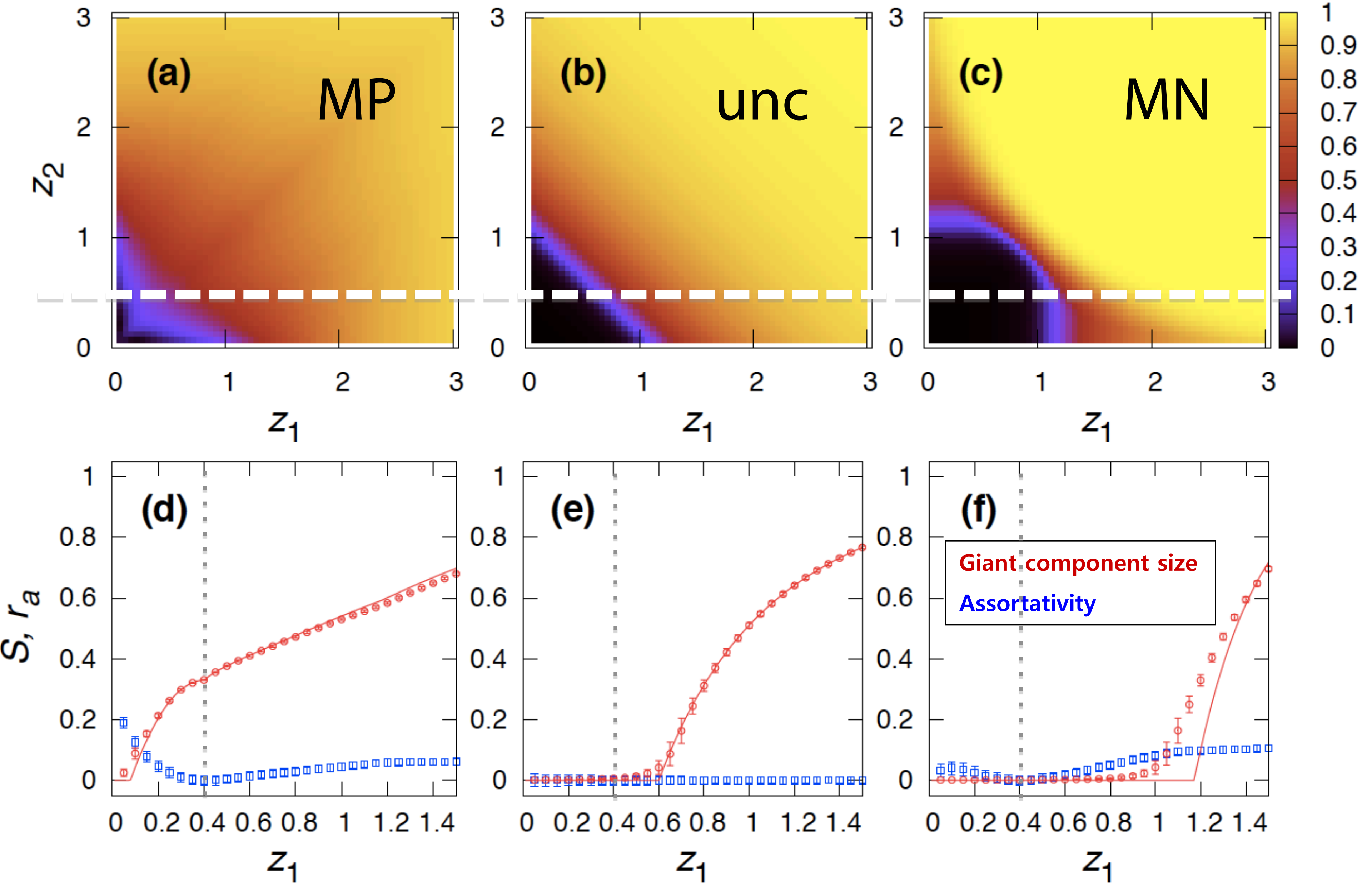
$$z^* : \quad (2 + z)e^{-z} = 1$$

Giant component sizes with $z_1 \neq z_2$



- Analytics agrees overall but not perfectly – Degree correlations!

Assortativity via correlated multiplexity



TAKE-HOME MESSAGE

- ***Think Multiplexity!***

- **Network multiplexity as a new layer of complexity in complex systems' structure and dynamics.**

Further recent related works…

- Sandpile dynamics [KM Lee, KIG, IMKim, *J Korean Phys Soc* **60**, 641 (2012)].
- Weighted threshold cascade [Yagan et al. arXiv:1204.0491].
- Boolean network [Arenas/Moreno arXiv:1205.3111].
- Diffusion dynamics [Diaz-Guilera/Moreno/Arenas arXiv:1207.2788]
- *More to come!*

STATPHYS25, July 2013 @ Korea

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