UNIVERSITY OF TWENTE.

Uncovering disassortativity in large scale-free networks

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Joint work with Remco van der Hofstad

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- Faloutsos, Faloutsos, Faloutsos (1999): power laws in Internet

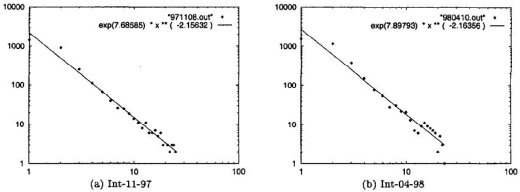

Figure 5: The outdegree plots: Log-log plot of frequency f_d versus the outdegree d.

But Power Law is not everything!

Example: Robustness of the Internet.

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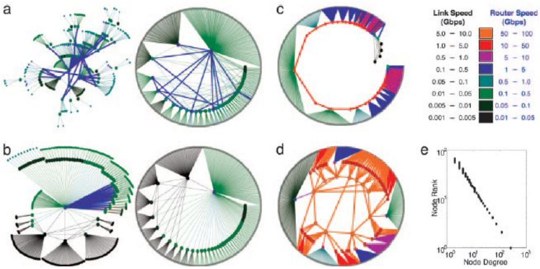
- Doyle et al. (2005): Robust yet fragile nature of Internet: Internet is not a random graph, it is designed to be robust



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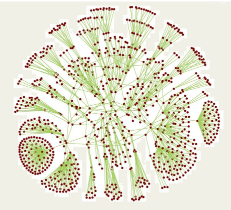
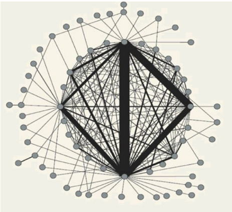
Example: Technological versus economical networks





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- To start with: do hubs connect to each other?
  YES for banks, NO for Internet
- Assortative networks: nodes with similar degree connect to each other.
- Disassortative networks: nodes with large degrees tend to connect to nodes with small degrees.

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- Newman (2002): assortativity measure ρ_n

$$\rho_n = \frac{\frac{1}{|E|}\sum_{ij\in E} d_i d_j - \left(\frac{1}{|E|}\sum_{ij\in E}\frac{1}{2}(d_i + d_j)\right)^2}{\frac{1}{|E|}\sum_{ij\in E}\frac{1}{2}(d_i^2 + d_j^2) - \left(\frac{1}{|E|}\sum_{ij\in E}\frac{1}{2}(d_i + d_j)\right)^2}$$

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- Very popular measure of assortativity!

Is there something wrong with ρ_n?

- Preferential Attachment graph appears to be assortatively neutral (Newman 2003, Dorogovtsev et al. 2010)
- Recent criticism: ρ_n depends on the size of the networks (Raschke et al. 2010; Dorogovtsev et al. 2010)

What IS assortativity measure?

- ρ_n is a statistical estimation for the coefficient of variation
$$\rho = \frac{E(XY) - [E(X)]^2}{Var(X)},$$
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- X and Y are power law r.v.'s, exponent $\alpha - 1$

$$P(X = k) = kp_k / E(\text{degree}).$$

- In real networks (WWW) we often have $2 < \alpha < 3$, so
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$$E(X) = \sum_k k \frac{kp_k}{E(\text{degree})} = \infty$$
- ρ is not defined in the power law model! Then: what are we measuring?

Assortative and disassortative graphs

- Newman(2003)

	network	type	size n	assortativity r	error σ_r	ref.
social	physics coauthorship	undirected	52 909	0.363	0.002	a
	biology coauthorship	undirected	1 520 251	0.127	0.0004	a
	mathematics coauthorship	undirected	253 339	0.120	0.002	b
	film actor collaborations	undirected	449 913	0.208	0.0002	c
	company directors	undirected	7 673	0.276	0.004	d
	student relationships	undirected	573	−0.029	0.037	e
	email address books	directed	16 881	0.092	0.004	f
technological	power grid	undirected	4 941	−0.003	0.013	g
	Internet	undirected	10 697	−0.189	0.002	h
	World-Wide Web	directed	269 504	−0.067	0.0002	i
	software dependencies	directed	3 162	−0.016	0.020	j
biological	protein interactions	undirected	2 115	−0.156	0.010	k
	metabolic network	undirected	765	−0.240	0.007	l
	neural network	directed	307	−0.226	0.016	m
	marine food web	directed	134	−0.263	0.037	n
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- Note: large networks are never strongly disassortative...

ρ_n in terms of moments of the degrees

- Write

$$\sum_{ij\in E} \tfrac{1}{2}(d_i + d_j) = \sum_{i\in V} d_i^2, \qquad \sum_{ij\in E} \tfrac{1}{2}(d_i^2 + d_j^2) = \sum_{i\in V} d_i^3$$

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- Then

$$\rho_n = \frac{\sum_{ij\in E} d_i d_j - \frac{1}{|E|}\Big(\sum_{i\in V} d_i^2\Big)^2}{\sum_{i\in V} d_i^3 - \frac{1}{|E|}\Big(\sum_{i\in V} d_i^2\Big)^2}.$$

Extreme value theory

Theorem (Extreme value theory)

$D_1, D_2, \ldots, D_n$ are i.i.d. with $1 - F(x) = P(D > x) = Cx^{-\alpha+1}$. Then

$$\lim_{n\to\infty} P\left(\frac{\max\{D_1, D_2, \ldots, D_n\} - b_n}{a_n} \leqslant x\right) = \exp(-(1+\delta x)^{-1/\delta}),$$

with $\delta = 1/(\alpha - 1)$, $a_n = \delta C^\delta n^\delta$, $b_n = C^\delta n^\delta$.
(Therefore, the maximum is 'of the order' $n^{1/(\alpha-1)}$)

CLT for heavy tails

Theorem (CLT for heavy tails)

$D_1, D_2, \ldots, D_n$ are i.i.d. with $1 - F(x) = P(D > x) = Cx^{-\alpha+1}$. If $p > \alpha - 1$ then

$$\frac{1}{a_n}\sum_{i=1}^{n} X_i^p \xrightarrow{d} Z,$$

where $a_n = [1 - F]^{-1}(1/n^p) = C^{1/(\alpha-1)} n^{p/(\alpha-1)}$ and Z has a stable distribution with parameter $(\alpha - 1)/p$. (Therefore, the sum is 'of the order' $n^{p/(\alpha-1)}$)

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- If $p > \alpha - 1$ then $\mathbb{E}(D^p) = \infty$
- CLT: for $p > \alpha - 1$ holds

$$\frac{1}{n} \sum_{i \in V} d_i^p \sim c_p n^{p/(\alpha-1)-1},$$

- But we get the same result just by adding up $k^p p_k$ from $k = 1$ to $k = n^{1/(\alpha-1)}$.

Assumptions

$$cn \leqslant |E| \leqslant Cn, (SLLN)$$
$$cn^{1/(\alpha-1)} \leqslant \max_{i\in[n]} d_i \leqslant Cn^{1/(\alpha-1)},$$
$$cn^{\max\{p/(\alpha-1),1\}} \leqslant \sum_{i\in[n]} d_i^p \leqslant Cn^{\max\{p/(\alpha-1),1\}}, \quad p = 2, 3,$$

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where $C, c > 0$.
Very natural and non-restrictive assumptions for power law graphs.

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- We have $\sum_{i\in V} d_i^3 \geqslant cn^{3/(\alpha-1)}$
- But also

$$\frac{1}{|E|}\Big(\sum_{i\in V} d_i^2\Big)^2 \leqslant (C^2/c)n^{\max\{4/(\alpha-1)-1,1\}}.$$

- When $\alpha \in (2,4)$ we have $\max\{4/(\alpha-1)-1,1\} < 3/(\alpha-1)$, so that the denominator of ρ_n^- outweighs its numerator.

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- Reason: high variability in values $\Rightarrow$ dependence on n

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- Factor n cancels, no influence of high dispersion

Classical approach!

H. Hotelling and M.R. Pabst (1936):
‘Certainly where there is complete absence of knowledge of the form of the bivariate distribution, and especially if it is believed not to be normal, the rank correlation coefficient is to be strongly recommended as a means of testing the existence of relationship.’

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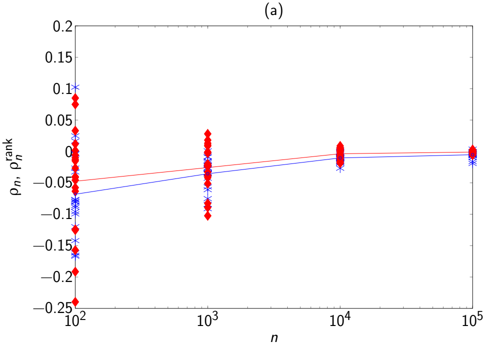
- ρ_n (blue), ρ_n^{rank} (red), and mean ρ_n^- (black) in 20 simulations for different n



Configuration model with intermediate edge (CMIE)

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Configuration model with intermediate edge: results

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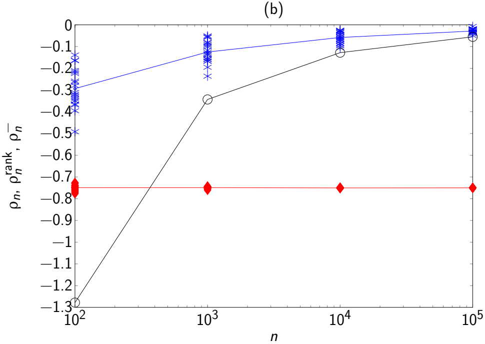
- ρ_n (blue), ρ_n^{rank} (red), and mean ρ_n^- (black) in 20 simulations for different n



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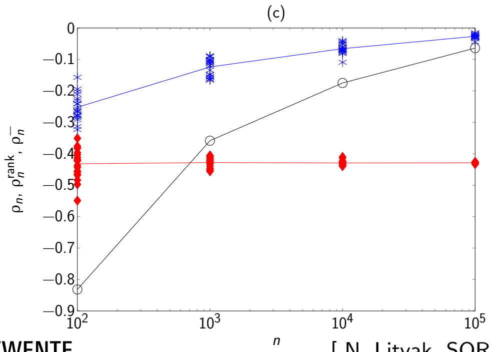

Assortative networks

$$\rho_n = \frac{\sum_{ij\in E} d_i d_j - \frac{1}{|E|}\Big(\sum_{i\in V} d_i^2\Big)^2}{\sum_{i\in V} d_i^3 - \frac{1}{|E|}\Big(\sum_{i\in V} d_i^2\Big)^2}.$$

Two possible scenarios:

- Denominator outweighs numerator, $\rho_n \to 0$
- Denominator and numerator are of the same order of magnitude. Limit?

Collection of bipartite graphs

- $((X_i, Y_i))_{i=1}^n$ i.i.d.
$$X = bU_1 + bU_2, \quad Y = bU_1 + aU_2, \quad b > 0, a > 1$$
U_1, U_2 i.i.d. random variables with power law tail, exponent α.
- For $i = 1, \ldots, n$, we create a complete bipartite graph of X_i and Y_i vertices, respectively.
- These n complete bipartite graphs are not connected to one another.
- Extreme scenario of a network consisting of highly connected clusters of different size. Such networks can serve as models for physical human contacts and are used in epidemic modelling (Eubank et al. 2004).
- Disassortative for $n = 1$ but positive dependence between X and Y prevails for larger n.

Collection of bipartite graphs: analysis

- $|V| = \sum_{i=1}^{n}(X_i + Y_i)$, $|E| = 2\sum_{i=1}^{n} X_i Y_i$,

$$\sum_{i\in V} d_i^p = \sum_{i=1}^{n}(X_i^p Y_i + Y_i^p X_i) \qquad \sum_{ij\in E} d_i d_j = 2\sum_{i=1}^{n}(X_i Y_i)^2.$$

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- Then $|E|/n \xrightarrow{p} 2E[XY] < \infty$ and $\frac{1}{n}\sum_{i \in V} d_i^2 \xrightarrow{p} E[XY(X+Y)] < \infty$.

Collection of bipartite graphs: analysis

Theorem (L& van der Hofstad, 2012)

$$n^{-4/(\alpha-1)} b^{-4} \sum_{i=1}^{n} (X_i^3 Y_i + Y_i^3 X_i) \xrightarrow{d} (a^3 + a) Z_1 + 2 Z_2,$$

$$n^{-4/(\alpha-1)} b^{-4} \sum_{i=1}^{N} (X_i Y_i)^2 \xrightarrow{d} a^2 Z_1 + Z_2,$$

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Result:

$$\rho_n \xrightarrow{d} \frac{2a^2 Z_1 + 2Z_2}{(a + a^3) Z_1 + 2Z_2}, \qquad \text{as } n \to \infty,$$

which is a random variable taking values in $(2a/(1 + a^2), 1)$, $a > 1$.

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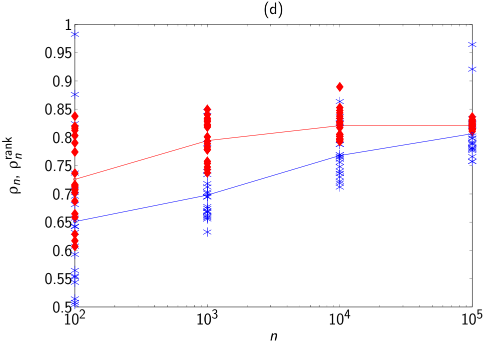
ρ_n (blue), ρ_n^{rank} (red), and mean ρ_n^- (black) in 20 simulations for different n



Web and social networks

Dataset	Description	# nodes	max d	ρ_n	ρ_n^{rank}	ρ_n^-
stanford-cs	web domain	9,914	340	-0.1656	-0.1627	-0.4648
eu-2005	.eu web crawl	862,664	68,963	-0.0562	-0.2525	-0.0670
uk@100,000	.uk web crawl	100,000	55,252	-0.6536	-0.5676	-1.117
uk@1,000,000	.uk web crawl	1,000,000	403,441	-0.0831	-0.5620	-0.0854
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  - In a graph the degrees on the ends of random edges are in general dependent. Can we analyse Spearman's rho? Work in progress.

Thank you!