Spectra and dynamics for assortative and disassortative networks

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Antonio Scala

CNR-ISC, Uos "La Sapienza" Rome Italy

with G. D'Agostino, V. Zlatic and G. Caldarelli

Effects of Topology on Networks

Can we have some general ideas on *how Topology affects the Dynamics* of the networks?

To this purpose we considered two different simple dynamical processes

- Epidemics
- Diffusion

Final Goal

Spotting systemic risks with few informations, very important for *Critical Infrastructures*

Assortativity

We focus on the role of assortativity on the dynamics.

Assortative Coefficient

r is the degree-degree Pearson correlation coefficient of two vertices connected by an edge

$$r(G) = \frac{\langle kq \rangle_e - \langle (k+q)/2 \rangle_e^2}{\langle (k^2+q^2)/2 \rangle_e - \langle (k+q)/2 \rangle_e^2}$$

where k,q are the degrees of the nodes at the vertices of the same edge and $\langle \bullet \rangle_e$ is the average over edges

Monte Carlo

Monte Carlo sampling

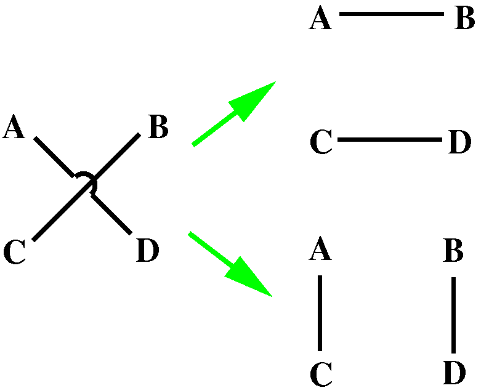
Gibbs measure $\mu[G] \propto \exp(-H_J[G])$ with coupling J

$$H_J[G] = -J \sum_{ij} A_{ij} k_i k_j$$

Assortativityis an increasing function of the coupling

Compare $H_J[G]$ with the assortativity dependent term of the assortativity coefficient r

$$\langle kq \rangle_e = \frac{1}{N_e} \sum_{ij} A_{ij} k_i k_j$$



Such swapping moves leave the degree distribution invariant

$$P\left(G \to G'\right) = \min\left[1, \exp\left(-\Delta H_J\right)\right]$$

Adjacency Matrix & Branching

SIS epidemics

SIS Model

Infected nodes can either infect their neighbors or recover. The epidemic threshold tells us if an epidemics spreads system-wide.

Adjacency Matrix

The epidemic threshold in networks scales as the inverse Λ_1^{-1} of the biggest eigenvalue of the adjacency matrix.

SIS can capture also failure propagation

SIS treshold

MF equations

$$\partial_t I_i = -I_i + (1 - I_i)\,\tau \sum_j A_{ij} I_j$$

for $\tau < \tau_C$ stable solution $\vec{I} = 0$

Small perturbation

$\vec{I} \sim \epsilon$

$$\partial_t \vec{I} = \tau A \vec{I} + \mathcal{O}(\epsilon)$$

$\vec{I} = 0$ solution is stable if

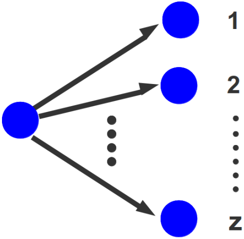
$$\|A\|\,\tau < 1$$

i.e.

$$\tau_C = 1/\Lambda_1$$

Branching processes are useful to describe percolation-like processes on trees (random networks are trees with few loops)

z branching number, p branching probability



zp average number of descendants

$(zp)^n$ at the n^{th} generation

$zp_c = 1$ critical probability

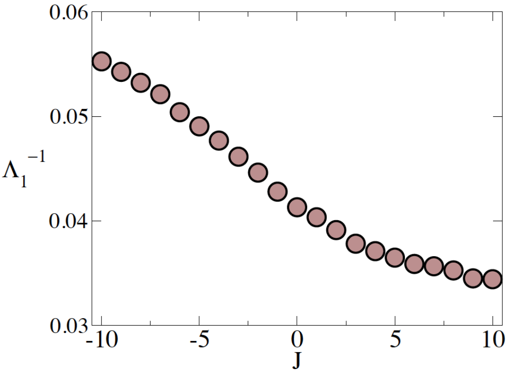
Percolation on a network

$z_i = \sum_j A_{ij}$ branching number

$\sum_j pA_{ij}$ descendants

$\sum_j p^n (A^n)_{ij}$ at the n^{th} generation

$p_c \|A\| \sim 1$ critical probability

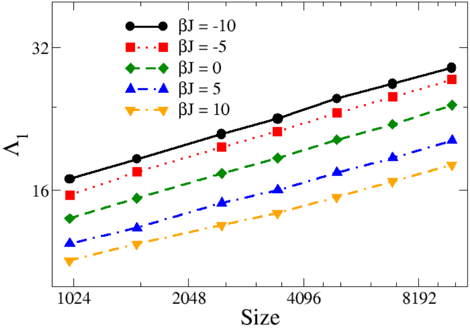


Data averaged on 10^2 networks of 10^4 nodes.

Result

Assortative networks are more prone to epidemic spreading.

Scaling

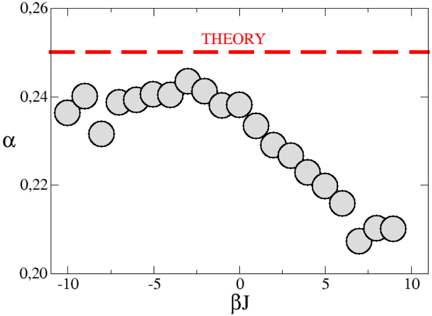


Data averaged on 10^2 networks for each size.

Result

Assortative networks have bigger scaling amplitudes.

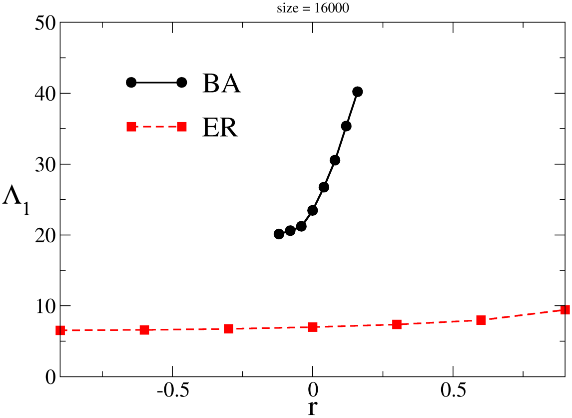
Exponents



Result

Scaling deviations at "small" sizes (THEORY is exact for $N \to \infty$)

Power Law vs Poisson



Laplacian Matrix & Diffusion

Laplacian

A adjacency matrix

sparse matrix with $A_{ij} = 1$ iff nodes i and j are linked

K degree matrix

diagonal matrix with $K_{ii} = \sum_j A_{ij}$ degree k_i of node i

$\mathcal{L}$ Laplacian matrix

$$\mathcal{L} = K - A$$

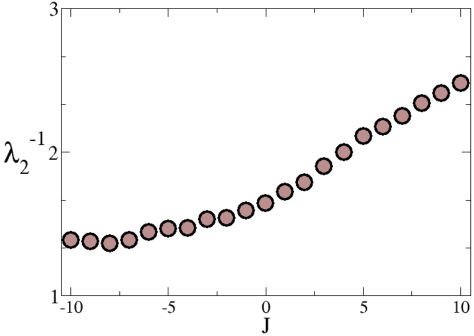
- Diffusion in the network is dictated by the Laplacian matrix

$$\partial_t \rho = -\mathcal{L}\rho$$

- The eigenvalues of $\mathcal{L}$ are $\lambda_1 = 0 \leq \lambda_2 \leq \ldots \leq \lambda_N$

The first non-zero eigenvalue λ_2 is the inverse timescale of slowest mode of diffusion (the most extended mode). In general, we can think of λ_2^{-1} as the timescale after which a perturbation (like the infection of a site) that spreads diffusively will settle a new state (like an epidemics) *in the whole network*.

Numerical Results



- Data are averaged on 10^2 networks of 10^4 nodes.
- No relevant size depenence

Result

Assortative networks allow for a longer intervention time.

Laplacian again

- Network vibrations are dictated by the Laplacian matrix

$$\partial_t^2 \rho = -\mathcal{L}\rho$$

- Synchronizability is linked to the spectrum of $\mathcal{L}$
- Controllability is linked to the spectrum of $\mathcal{L}$

Minimal Cut

A partition of the nodes into two sets can be represented by a vector $\vec{x}$ with $x_i \in \{-1, 1\}$

- Ω_+ sites with $x_i > 0$
- Σ_+ links between nodes in Ω_+
- $\partial\Omega$ sites at the border of the partitions
- $\partial\Sigma$ links among Ω_+ and Ω_-

Min-Cut

find $\min \mathcal{H}[\vec{x}]$ s.t. $x_i \in \{-1, 1\}$

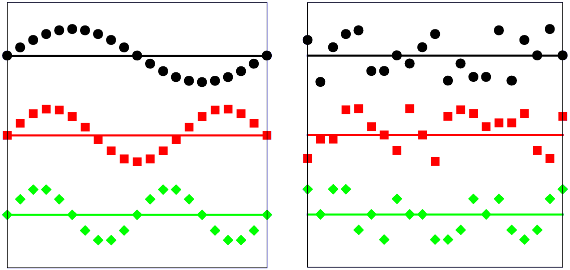
$$\mathcal{H}[\vec{x}] = \sum_{ij}^{n.n.} \left(\frac{x_i - x_j}{2} \right)^2 = \frac{\vec{x}\mathcal{L}\vec{x}}{4}$$

Relax the *min* $-$ *cut* conditions and let $\vec{x} \in \mathbb{R}^N$ s.t $\|\vec{x}\| = 1.$; then I can look at the *eigenvectors* $\vec{u}_\alpha$ *of* $\mathcal{L}$

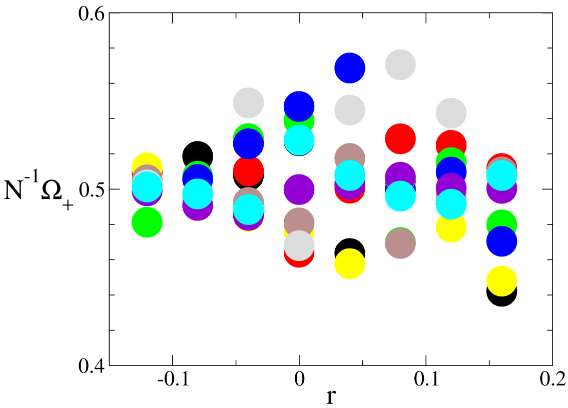
by expressing $\vec{x} = \sum_\alpha a_\alpha \vec{u}_\alpha$ we get the relation

$$\vec{x}\mathcal{L}\vec{x} = \sum_\alpha a_\alpha^2 \lambda_i \geq \lambda_2 \|\vec{x}\|^2 = \lambda_2$$

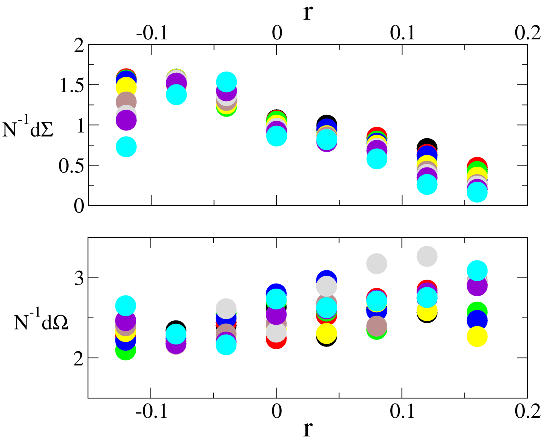
therefore the minimal solution is $\vec{u}_2$



Partitions can be identified by a sequence $(1, -1, \ldots)$

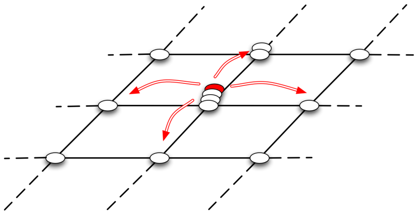

0.6
0.5
0.4
$N^{-1}\Omega_+$
-0.1
0
0.1
0.2
r

Scaling of the Min-Cut Boundaries



Sandpiles & Finance

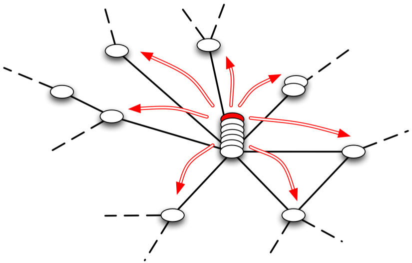
Sandpile model has been the prototype of Self-Organised Criticality



- Defined on a lattice
- Sand accumulates on vertices
- Until threshold
- Then topples
- Until reaches lattice boundaries

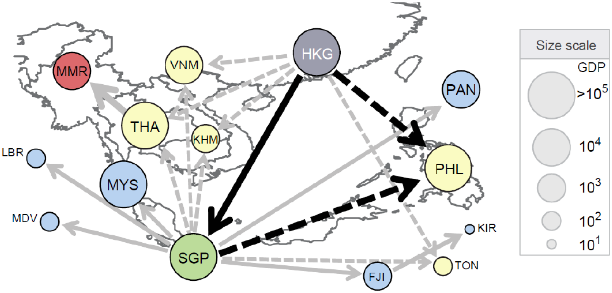
Network Sandpiles

Substitute sand with distress/energy/stress/...



- Defined on a Network
- Threshold is the degree
- Boundaries ?

Avalanches can be seen as propagation of distress



K.-M. Lee, J.-S. Yang, G. Kim, J. Lee, K.-I. Goh, I-M. Kim, PLOS1 6, e18443 (2011)

Pinning of Sandpiles

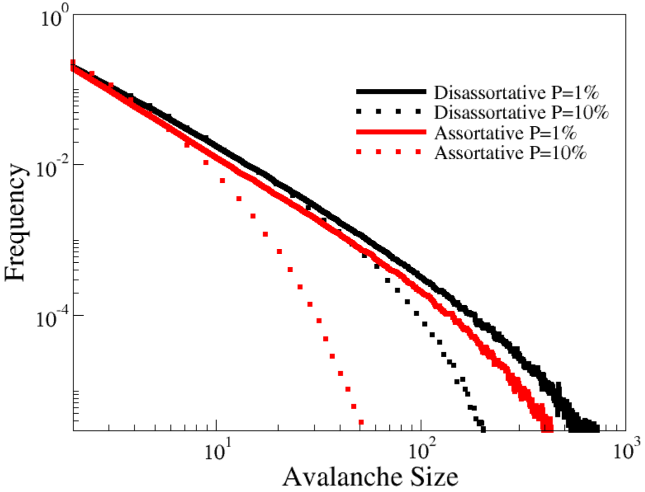
A bank "*too big to fail*" is a site that does not topple.
Pinning Bailing out in the language of sandpile corresponds to pinning. I.e. stopping topples on sites.

- You can pin randomly (standard sandpile)
- You can pin the hubs (too connected to fail)

The pinned sites are the boundary ∂G of the sandpile

Results of Pinning

On assortative networks domino-effect have a larger cut-off.



The toppling of a site is described by

$$s_i\,(t+1) = s_i\,(t) + T_{ij}$$

Toppling Matrix

$$T_{ij} = \begin{cases} k_i & if \quad i=j \\ -1 & if \quad i\; n.n.\; j \\ 0 & \quad otherwise \end{cases}$$

Toppling & Laplacian

$$T_{ij} = \begin{cases} 0 & if \quad i \in \partial G \vee j \in \partial G \\ -\mathcal{L}_{ij} & \quad otherwhise \end{cases}$$

Conclusions

Spectra

Adjacency Matrix dictates irreversible propagation
Laplacian Matrix dictates diffusive propagation
Toppling dynamics are linked to Diffusive dynamics

Eigenvectors

Min-Cut
Community Finding

To Do

Signed Graphs
Directed Graphs

Open problems

- Assortativity ***vs*** Size: do non-neutral configurations disappear or is assortativity ill defined?
- MC Sampling with a non-extensive Hamiltonian
- Fixed Assortativity simulations introduce bias in the assortativity structure

Advertising

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CRISIS LAB

IMT Lucca & CNR-ISC - Italian government funded

Networks of Networks

6 June Chicago, NetSci 2012 sites.google.com/site/netonets2012

Complex Interacting Networks

8 Sept Bruxelles, ECCS 2012 sites.google.com/site/coinets2012