# Anomalous statistics of dynamical systems on networks

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# Why are networks cool?

- Tell you who interacts with whom
- Same statistical system on different networks can behave totally different

# How ?

- Simple example: Ising spins on constant-connectency networks
- Show: this is not of Boltzmann Gibbs type – give exact statistics

## Why Statistics ?

• Central concept: understanding macroscopic system behavior on the basis of microscopic elements and interactions $\rightarrow$ *entropy*

• Functional form of entropy: must encode information on interactions too!

• Entropy relates number of states to an extensive quantity, plays fundamental role in the thermodynamical description

• Hope: 'thermodynamical' relations $\rightarrow$ phase diagrams, etc.

# 3 Ingredients

• Entropy has scaling properties → what are entropies for non-ergodic systems?

• How does entropy grow with system size? → what n.e. system is realized?

• Symmetry in thermodynamic systems → if broken: entropy has no thermodynamic meaning → forget dream about handling system with TD

What is the entropy of strongly interacting systems?

# Appendix 2, Theorem 2

C.E. Shannon, The Bell System Technical Journal **27**, 379-423, 623-656, 1948.

# Entropy

$$S[p] = \sum_{i=1}^{W} g(p_i)$$

$p_i$ ... probability for a particular (micro) state of the system, $\sum_i p_i = 1$

$W$ ... number of states

$g$ ... some function. What does it look like?

# The Shannon-Khinchin axioms

- SK1: $S$ depends continuously on $p$ $\rightarrow$ $g$ is continuous
- SK2: entropy maximal for equi-distribution $p_i = 1/W$ $\rightarrow$ $g$ is concave
- SK3: $S(p_1, p_2, \cdots, p_W) = S(p_1, p_2, \cdots, p_W, 0) \rightarrow g(0) = 0$
- SK4: $S(A + B) = S(A) + S(B|A)$

**Theorem:**
If SK1-SK4 hold, the only possibility is Boltzmann-Gibbs-Shannon entropy

$$S[p] = \sum_{i=1}^{W} g(p_i) \quad \text{with} \quad g(x) = -x \ln x$$

# Shannon-Khinchin axiom 4 is non-sense for NWs

$\rightarrow$ SK4 violated for strongly interacting systems

$\rightarrow$ nuke SK4

SK4 corresponds to weak interactions or Markovian processes

## The Complex Systems axioms

- SK1 holds
- SK2 holds
- SK3 holds
- $S_g = \sum_i^W g(p_i)$ , $W \gg 1$

**Theorem:** All systems for which these axioms hold

(1) can be uniquely classified by 2 numbers, $c$ and $d$

(2) have the unique entropy

$$S_{c,d} = \frac{e}{1 - c + cd} \left[ \sum_{i=1}^{W} \Gamma \left( 1 + d \, , \, 1 - c \ln p_i \right) - \frac{c}{e} \right] \qquad e \cdots \text{Euler const}$$

## The argument: generic mathematical properties of $g$

- Scaling transformation $W \to \lambda W$: how does entropy change ?

# Mathematical property I: an unexpected scaling law !

$$\lim_{W\to\infty} \frac{S_g(W\lambda)}{S_g(W)} = ... = \lambda^{1-c}$$

**Theorem 1:** Define $f(z) \equiv \lim_{x\to 0} \frac{g(zx)}{g(x)}$ with $(0 < z < 1)$. Then for systems satisfying SK1, SK2, SK3: $f(z) = z^c$, $0 < c \leq 1$

## Theorem 1

Let $g$ be a continuous, concave function on $[0, 1]$ with $g(0) = 0$ and let $f(z) = \lim_{x \to 0^+} g(zx)/g(x)$ be continuous, then $f$ is of the form $f(z) = z^c$ with $c \in (0, 1]$.

*Proof.* Note that $f(ab) = \lim_{x \to 0} g(abx)/g(x) = \lim_{x \to 0}(g(abx)/g(bx))(g(bx)/g(x)) = f(a)f(b)$. All pathological solutions are excluded by the requirement that $f$ is continuous. So $f(ab) = f(a)f(b)$ implies that $f(z) = z^c$ is the only possible solution of this equation. Further, since $g(0) = 0$, also $\lim_{x \to 0} g(0x)/g(x) = 0$, and it follows that $f(0) = 0$. This necessarily implies that $c > 0$. $f(z) = z^c$ also has to be concave since $g(zx)/g(x)$ is concave in $z$ for arbitrarily small, fixed $x > 0$. Therefore $c \leq 1$. $\square$

# Mathematical properties II: yet another one !!

$$\lim_{W\to\infty} \frac{S(W^{1+a})}{S(W)W^{a(1-c)}} = ... = (1+a)^{d}$$

**Theorem 2:** Define $h_c(a) \equiv ...$

## Theorem 2

Let $g$ be like in Theorem 1 and let $f(z) = z^c$ then $h_c$ given in Eq. (8) is a constant of the form $h_c(a) = (1+a)^d$ for some constant $d$.

*Proof.* We determine $h_c(a)$ again by a similar trick as we have used for $f$.

$$\begin{aligned} h_c(a) &= \lim_{x\to 0} \frac{g(x^{a+1})}{x^{ac}g(x)} = \frac{g\left((x^b)^{\left(\frac{a+1}{b}-1\right)+1}\right)}{(x^b)^{\left(\frac{a+1}{b}-1\right)c} g(x^b)} \frac{g(x^b)}{x^{(b-1)c}g(x)} \\ &= h_c\left(\frac{a+1}{b}-1\right) h_c(b-1) \quad , \end{aligned}$$

for some constant $b$. By a simple transformation of variables, $a = bb' - 1$, one gets $h_c(bb'-1) = h_c(b-1)h_c(b'-1)$. Setting $H(x) = h_c(x-1)$ one again gets $H(bb') = H(b)H(b')$. So $H(x) = x^d$ for some constant $d$ and consequently $h_c(a)$ is of the form $(1+a)^d$. □

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# Summary

Strongly interacting systems $\rightarrow$ SK1-SK3 hold

$$\rightarrow \lim_{W\to\infty} \frac{S_g(W\lambda)}{S_g(W)} = \lambda^{1-c} \qquad 0 \le c < 1$$

$$\rightarrow \lim_{W\to\infty} \frac{S(W^{1+a})}{S(W)W^{a(1-c)}} = (1+a)^{d} \qquad d \text{ real}$$

## Remarkable:

- all systems are characterized by 2 exponents: $(c, d)$ – universality class
- Which $S$ fulfills above? $\rightarrow$ $S_{c,d} = \sum_{i=1}^{W} re\,\Gamma\left(1+d\,,\,1-c\ln p_i\right) - rc$
- Which distribution maximizes $S_{c,d} \rightarrow p_{c,d}(x) = e^{-\frac{d}{1-c}\left[W_k\left(B(1+\frac{x}{r})^{\frac{1}{d}}\right)-W_k(B)\right]}$

$r = \frac{1}{1-c+cd}$, $B = \frac{1-c}{cd}\exp\left(\frac{1-c}{cd}\right)$, $\Gamma(a,b) = \int_b^\infty dt\, t^{a-1}\exp(-t)$; Lambert-$W$: solution to $x = W(x)e^{W(x)}$

# Holds very generically

- for all non-ergodic systems
- for all non-Markovian systems

(complex systems)

# Examples

- $S_{1,1} = \sum_i g_{1,1}(p_i) = -\sum_i p_i \ln p_i + 1$ (BG entropy)
- $S_{q,0} = \sum_i g_{q,0}(p_i) = \frac{1-\sum_i p_i^q}{q-1} + 1$ (Tsallis entropy)
- $S_{1,d>0} = \sum_i g_{1,d}(p_i) = \frac{e}{d} \sum_i \Gamma\left(1+d\,,\, 1-\ln p_i\right) - \frac{1}{d}$ (AP entropy)
- ...

# Classification of entropies: order in the zoo

| entropy | | $c$ | $d$ |
|---|---|---|---|
| $S_{BG} = \sum_i p_i \ln(1/p_i)$ | | 1 | 1 |
| • $S_{q<1} = \frac{1-\sum p_i^q}{q-1}$ | $(q < 1)$ | $c = q < 1$ | 0 |
| • $S_\kappa = \sum_i p_i (p_i^\kappa - p_i^{-\kappa})/(-2\kappa)$ | $(0 < \kappa \leq 1)$ | $c = 1 - \kappa$ | 0 |
| • $S_{q>1} = \frac{1-\sum p_i^q}{q-1}$ | $(q > 1)$ | 1 | 0 |
| • $S_b = \sum_i (1 - e^{-bp_i}) + e^{-}b - 1$ | $(b > 0)$ | 1 | 0 |
| • $S_E = \sum_i p_i (1 - e^{\frac{p_i - 1}{p_i}})$ | | 1 | 0 |
| • $S_\eta = \sum_i \Gamma(\frac{\eta+1}{\eta}, -\ln p_i) - p_i \Gamma(\frac{\eta+1}{\eta})$ | $(\eta > 0)$ | 1 | $d = 1/\eta$ |
| • $S_\gamma = \sum_i p_i \ln^{1/\gamma}(1/p_i)$ | | 1 | $d = 1/\gamma$ |
| • $S_\beta = \sum_i p_i^\beta \ln(1/p_i)$ | | $c = \beta$ | 1 |

| | | |
|---|---|---|
| $S_{c,d} = \sum_i er\Gamma(d+1, 1 - c \ln p_i) - cr$ | $c$ | $d$ |

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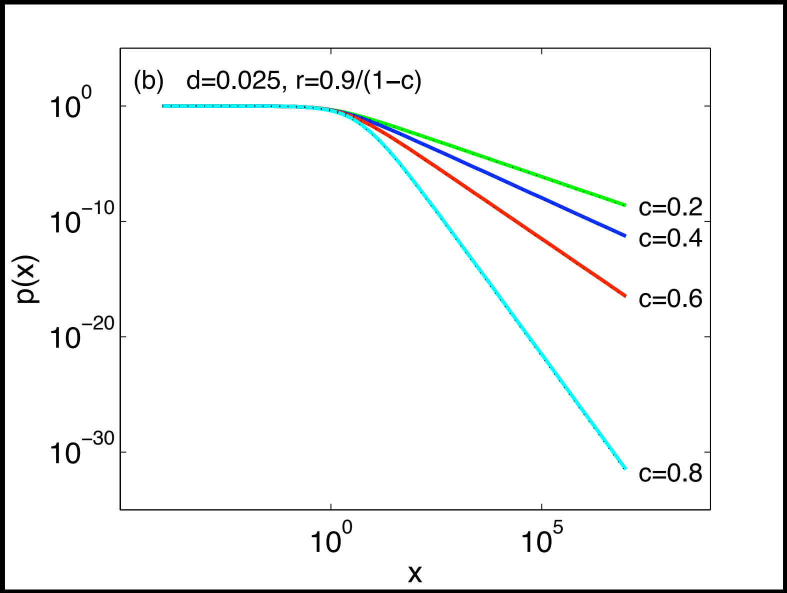
# Distribution functions of CS

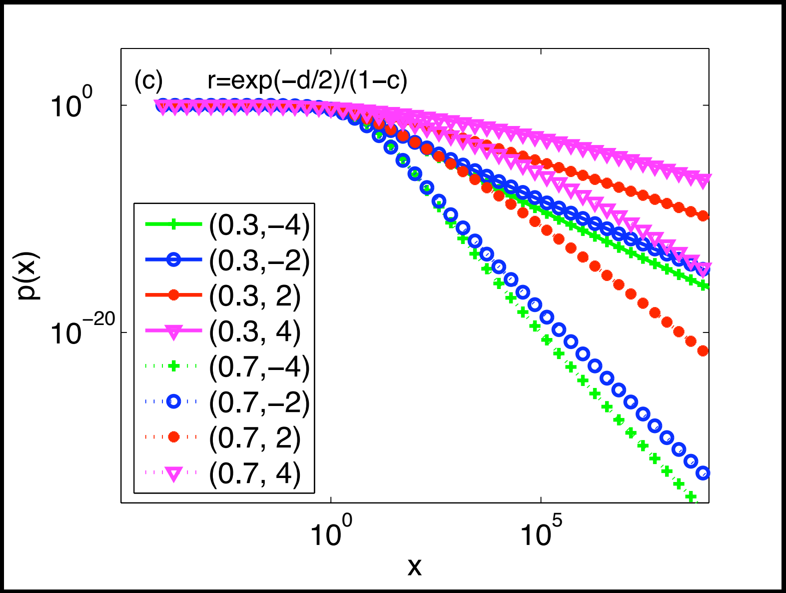
- $p_{(1,1)} \to$ exponentials (Boltzmann distribution)
- $p_{(q,0)} \to$ power-laws ($q$-exponentials)
- $p_{(1,d>0)} \to$ stretched exponentials
- $p_{(c,d)}$ all others $\to$ Lambert-$W$ exponentials

NO OTHER POSSIBILITIES

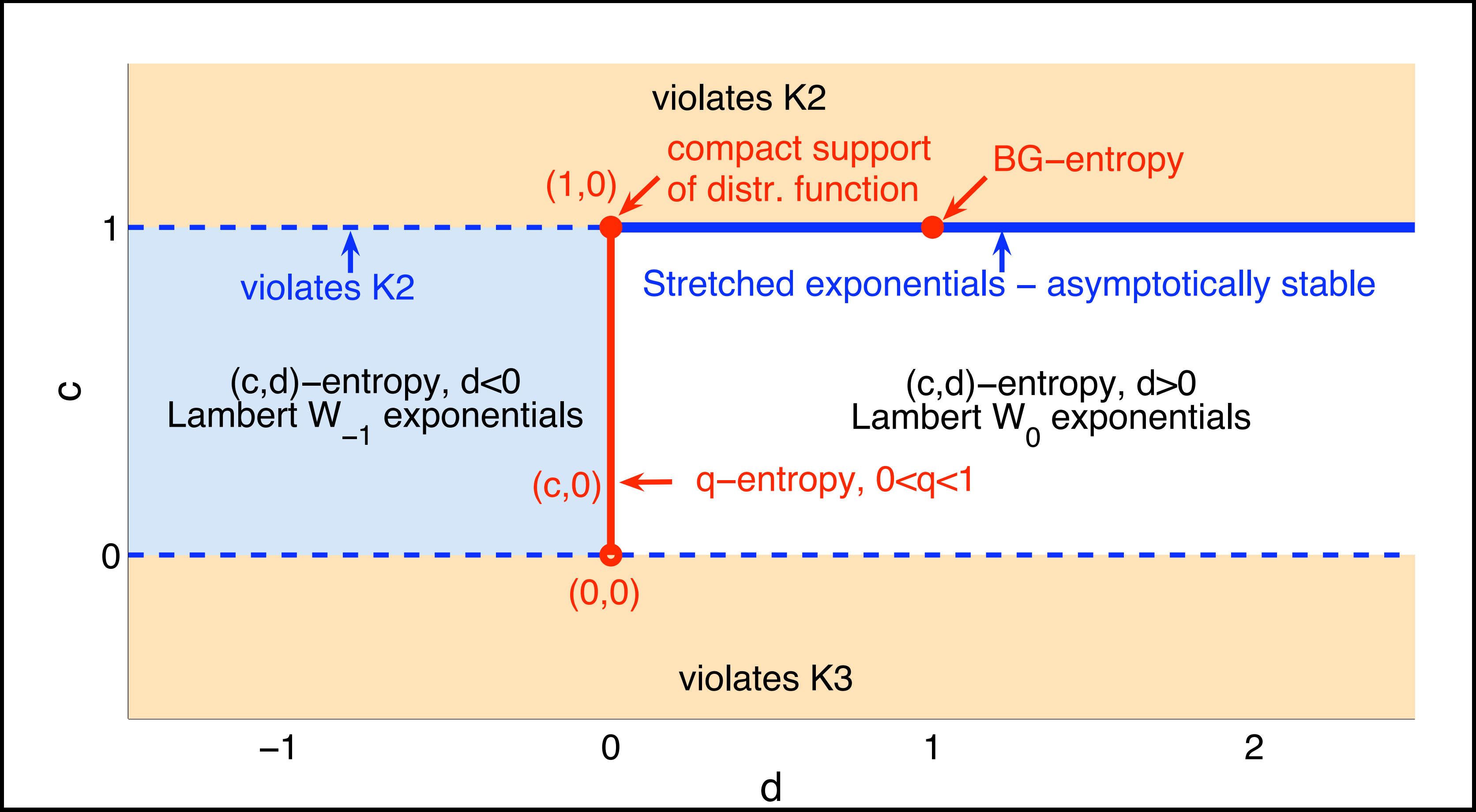
## $q$-exponentials

## Lambert-exponentials



# The world beyond Shannon



# Scaling property opens door to ...

- ...bring order in the zoo of entropies through universality classes
- ...understand ubiquity of power laws (and extremely similar functions)
- ...understand where Tsallis entropy comes from
- ...understand statistical systems on networks

# The requirement of extensivity

# Needed for TD program to work: extensive entropies

System has $N$ elements $\rightarrow W(N)$... phasespace volume (system property)

Extensive: $S(W_{A+B}) = S(W_A) + S(W_B) = \cdots$ [use scaling property I] $\rightarrow$

**Can proof:** extensive is equivalent to $W(N) = \exp\left[\frac{d}{1-c} W_k\left(\mu(1-c)N^{\frac{1}{d}}\right)\right]$

$$\begin{aligned} c &= \lim_{N\to\infty} 1 - \frac{1}{N}\frac{W'(N)}{W(N)} \\ d &= \lim_{N\to\infty} \log W \left(\frac{1}{N}\frac{W}{W'} + c - 1\right) \end{aligned}$$

**Message:** Growth of phasespace volume determines entropy and *vice versa*

## Examples

- $W(N) = 2^N \rightarrow (c,d) = (1,1)$ and system is BG
- $W(N) = N^b \rightarrow (c,d) = (1-\frac{1}{b}, 0)$ and system is Tsallis
- $W(N) = \exp(\lambda N^\gamma) \rightarrow (c,d) = (1, \frac{1}{\gamma})$
- ...

Can explicitly verify statements in theory of binary processes and spin-systems on networks

## What does this imply further ?

- almost all systems are Boltzmann Gibbs type
- to be non-BG: phasespace has to collapse to a set of measure zero
- this means: bulk of statistically relevant degrees of freedom is frozen
- only systems where dynamics is confined its surface can be non-BG

**Hypothesize applications in:**

- Self Organized Critical systems, sandpiles ...
- Spin systems with dense meta-structures, such as spin-domains, vortices, instantons, caging, etc.
- Anomalous diffusion (porous media)

# 2 Examples

## Spin system on networks

- each node $i$ has 2 states: $s_i = \pm 1$ ; YES / NO (e.g. opinion)
- each node $i$ has initial ('kinetic') energy $\epsilon$ (e.g. free will)
- (anti) parallel spins add $J^{+(-)}$ to energy $E$; $\Delta J = J^- - J^+$
- total energy in the system: $E = \epsilon N$
- spin-flip of node can occur if node has enough energy for it (microcanonic)

$\rightarrow$ **Can show** entropy depends on network !!!

# Phasespace volume

- $N$ nodes, $L$ links, $k = N/L$, $\phi = N/N(N-1)$

$n^+$ ... spins pointing up, $\mu$ cost for link

- phase space volume: $\Omega = \binom{N}{n^+}$ (MC partition function)

- derive $n^+$

$E$ can be estimated by

$$E = \frac{L\left[\left(n^+(n^+-1) + n^-(n^- - 1)\right) J^+ + 2n^+n^-J^-\right]}{N(N-1)} + \mu L \sim 2\phi n^+(N - n^+)\Delta J$$

and

$$n^+ = \frac{N}{2}\left(1 - \sqrt{1 - \frac{2\epsilon}{k\Delta J}}\right) \sim \frac{\epsilon}{2\phi\Delta J}$$

# Phasespace volume and NW growth

- Example 1: NW grows such that connectivity $k$ is constant as it grows

$k = \text{const.} \rightarrow n^+ = aN$ with $0 < a < 1$ constant

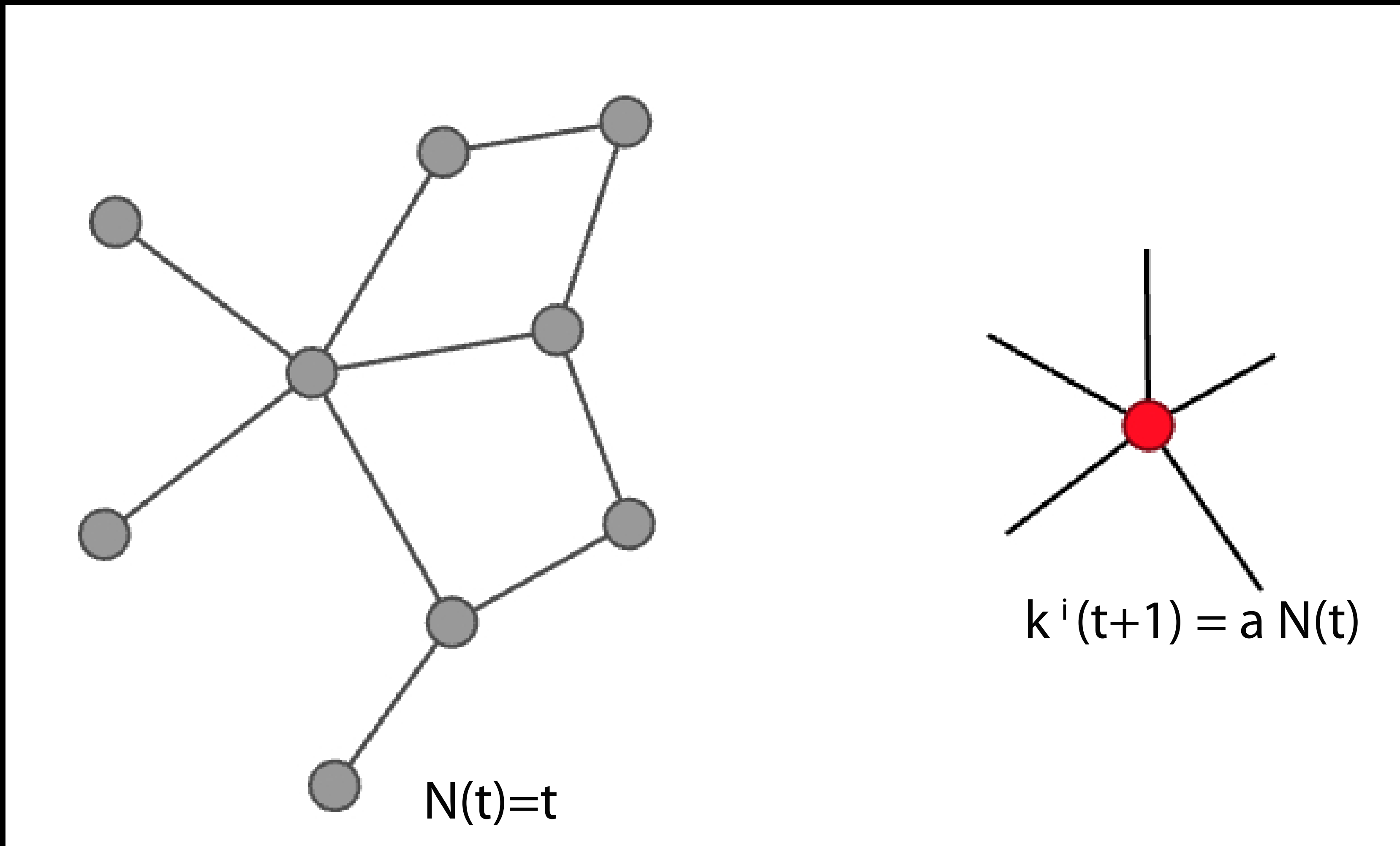
Sterling's approximation $W = \binom{N}{aN} \sim b^N$ with $b = a^{-a}(1-a)^{a-1} > 1$

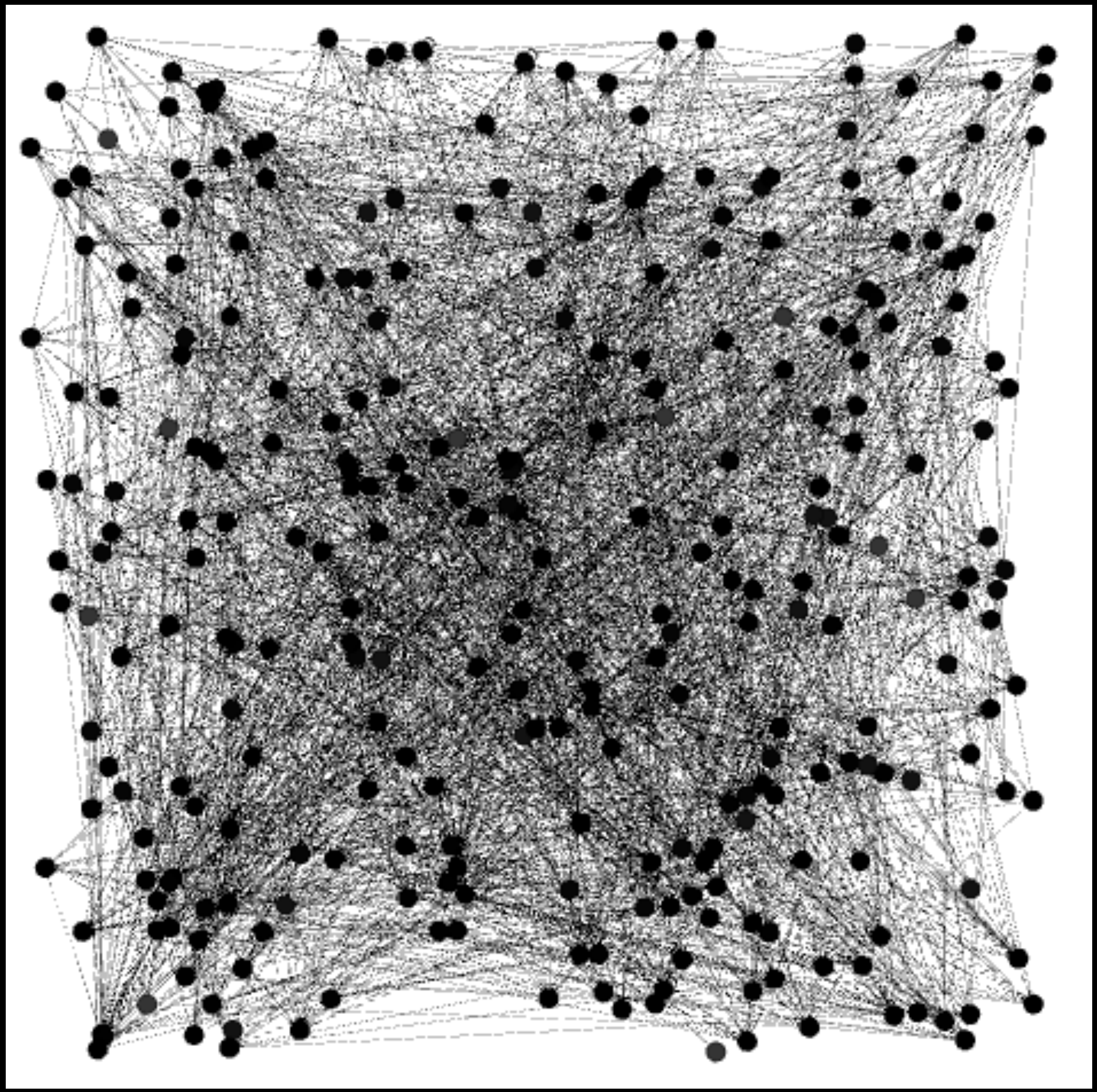
From before: $c = 1$ and $d = 1 \rightarrow$ entropy of the system is BG

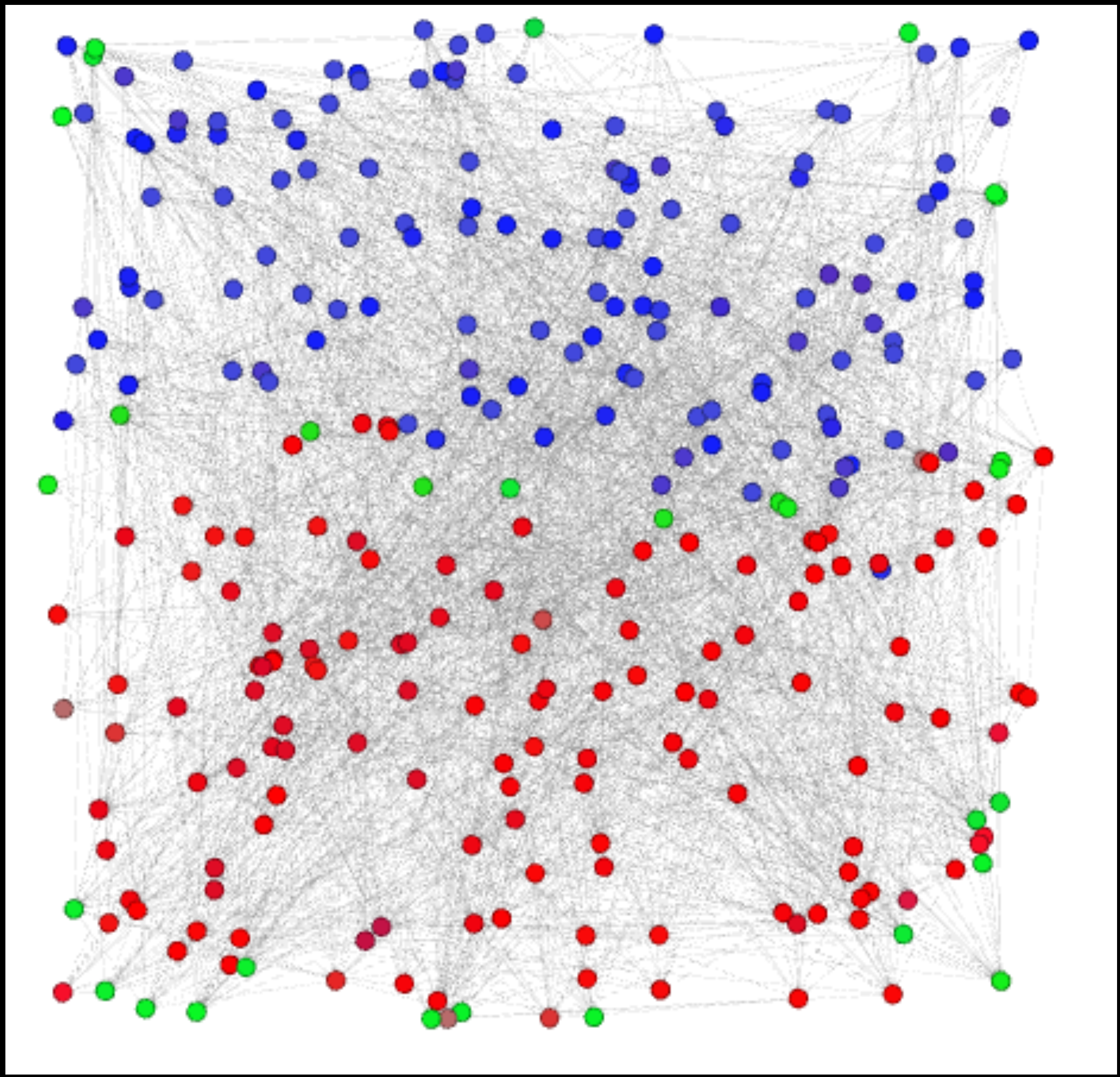
- Example 2: NW growth: join-a-club network

new node links to $\alpha N(t)$ random neighbors, $\alpha < 1$

**What is this ?**


N(t)=t
k i (t+1) = a N(t)





## Phasespace volume and NW growth

- Example 2: NW growth: join-a-club network

new node links to $\alpha N(t)$ random neighbors, $\alpha < 1$

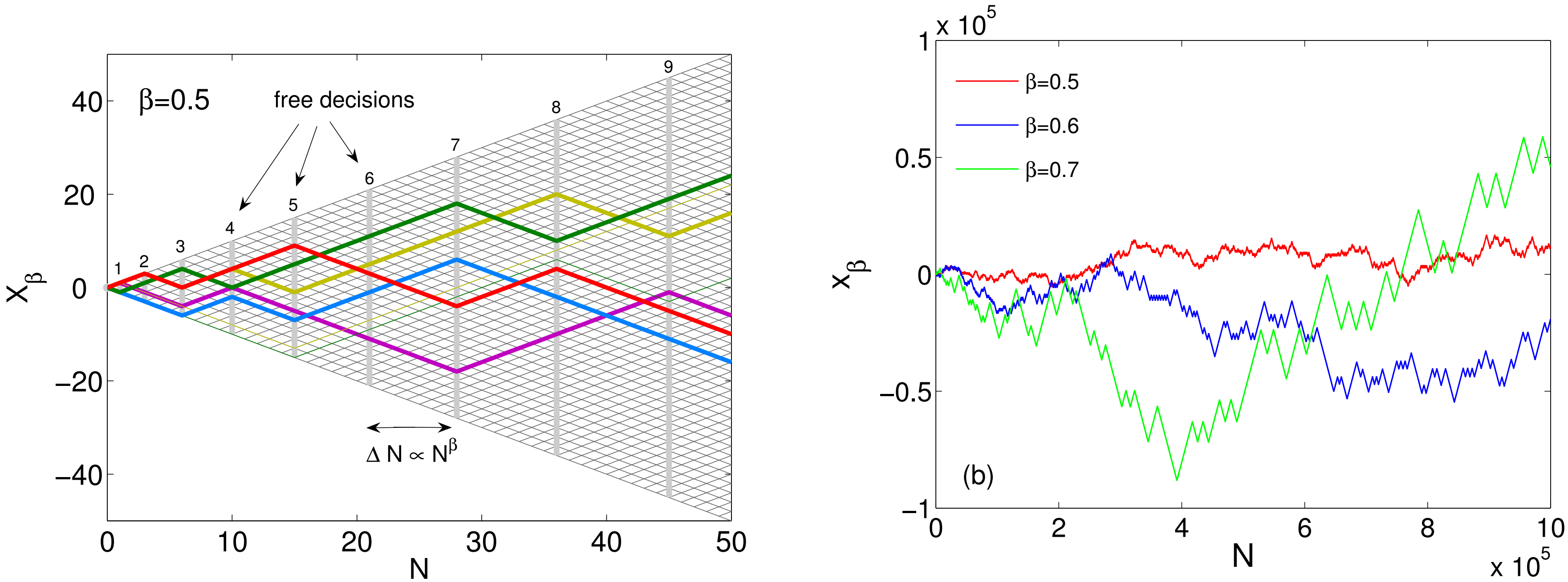
constant connectancy, $\phi = \mathrm{const.} \to k = \phi N$ and $n^+ \sim \epsilon / 2\phi \Delta J = \mathrm{const.}$

$W = \binom{N}{n^+} \sim (N/n^+)^{n^+} exp(-n^+) \propto N^{n^+}$

From before $(c, d) = (1 - \frac{1}{n^+}, 0)$, meaning Tsallis $q$-entropy with $q = c$

- Note that intermediate cases with $k \propto N^\gamma$ with $0 < \gamma < 1$, require generalized entropies with $c = 1$ and $d = 1/\gamma$.

# Bonus track: Super-diffusion: Accelerating random walks




- up-down decision of walker is followed by $[N^\beta]_+$ steps in same direction
- $k(N)$ number of random decisions up to step $N \to k(N) \sim N^{1-\beta}$
- number of all possible sequences $W(N) \sim 2^{N^{1-\beta}} \to (c,d) = (1, \frac{1}{1-\beta})$
- note continuum limit of such processes is well defined !

## Conclusions

- Interactions on networks may violate Shannon-Khinchin axiom 4
- Keep Shannon-Khinchin axioms 1-3, and $S = \sum g$ (CS in general)
- Showed: macroscopic statistical systems can be uniquely classified in terms of 2 scaling exponents $(c, d)$ – analogy to critical exponents
- Single entropy covers all systems: $S_{c,d} = re \sum_i \Gamma\left(1 + d\,,\, 1 - c \ln p_i\right) - rc$
- All known entropies of SK1-SK3 systems are special cases
- Distribution functions of *all* systems are Lambert-$W$ exponentials. There are no other options
- Phasespace growth uniquely determines entropy
- Statistical systems on networks: examples

  constant connectivity, $k \rightarrow$ Boltzmann-Gibbs

  constant connectancy $\phi \rightarrow$ Tsallis entropy

## A note on Rényi entropy

It is it not sooo relevant for CS. Why?

- Relax Khinchin axiom 4:

$S(A+B) = S(A) + S(B|A) \to S(A+B) = S(A) + S(B) \to$ Rényi entropy

- $S_R = \frac{1}{\alpha-1} \ln \sum_i p_i^\alpha$ violates our $S = \sum_i g(p_i)$

But: our above argument also holds for Rényi-type entropies !!!

$$S = G\left(\sum_{i=1}^{W} g(p_i)\right)$$

$$\lim_{W\to\infty} \frac{S(\lambda W)}{S(W)} = \lim_{R\to\infty} \frac{G\left(\frac{f_g(z)}{z} G^{-1}(R)\right)}{R} = [\text{for } G \equiv \ln] = 1$$

# The Lambert-W: a reminder

- solves $x = W(x)e^{W(x)}$
- inverse of $p \ln p = [W(p)]^{-1}$
- delayed differential equations $\dot{x}(t) = \alpha x(t-\tau) \rightarrow x(t) = e^{\frac{1}{\tau}W(\alpha\tau)t}$

# Example: a physical system

equation of motion for particle $i$ in system of $N$ overdamped particles

$$\mu \vec{v}_i = \sum_{j \neq i} \vec{J}(\vec{r}_i - r_j) + \vec{F}(\vec{r}_i) + \eta(\vec{r}_i, t)$$

$v_i$ ... velocity of $i$ th particle   $\mu$ ... viscosity of medium   $F$ ... external force
$\vec{J}(\vec{r}) = G\left(\frac{|\vec{r}|}{\lambda}\right)\hat{r}$ ... repulsive particle-particle interaction
$\eta$ ... uncorrelated thermal noise $\langle \eta \rangle = 0$ and $\langle \eta^2 \rangle = \frac{kT}{\mu}$
$\lambda$ ... characteristic length of short range pairwise interaction

Shown with FP approach and simulation (Curado, Nobre, et al. PRL 2011)

- low temperature: Tsallis system $(c, d) = (q, 0)$
- high temperature limit $\rightarrow$ BG system $(c, d) = (1, 1)$