# Interplay of nonlinearity, interactions and Anderson localization

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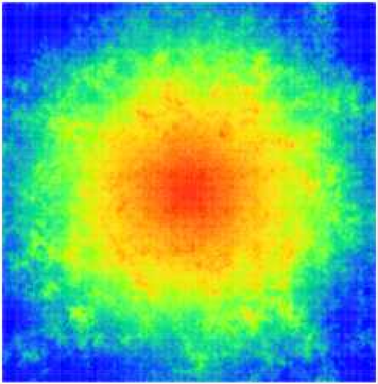
with I.García-Mata (Toulouse), K.Ahnert, M.Mulansky, A.Pikovsky (Potsdam)



I.García-Mata, DLS arXiv:0805.0539 (2008)

- Discrete Anderson nonlinear Schrödinger equation (DANSE) ($d = 1, 2$)
- Nonlinear delocalization on disordered Stark ladder
- Dynamical thermalization of nonlinear disordered lattices: anti-FPU (poster N2 K.Ahnert *et al.*)
- Two interacting particles effect
- Åberg criterion for dynamical thermalization in many-body quantum systems

# Nonlinearity and Anderson localization: estimates

$$i\hbar \frac{\partial \psi_n}{\partial t} = E_n \psi_n + \beta |\psi_n|^2 \psi_n + V(\psi_{n+1} + \psi_{n-1}) \; ; [-W/2 < E_n < W/2]$$

localization length $l \approx 96(V/W)^2$ (1D); $\ln l \sim (V/W)^2$ (2D) Amplitudes $C$ in the linear eigenbasis are described by the equation

$$i\frac{\partial C_m}{\partial t} = \epsilon_m C_m + \beta \sum_{m_1 m_2 m_3} U_{m m_1 m_2 m_3} C_{m_1} C^*_{m_2} C_{m_3}$$

the transition matrix elements are $U_{m m_1 m_2 m_3} = \sum_n Q^{-1}_{nm} Q_{nm_1} Q^*_{nm_2} Q_{nm_3} \sim 1/l^{3d/2}$. There are about $l^{3d}$ random terms in the sum with $U \sim l^{-3d/2}$ so that we have $idC/dt \sim \beta C^3$. We assume that the probability is distributed over $\Delta n > l^d$ states of the lattice basis. Then from the normalization condition we have $C_m \sim 1/(\Delta n)^{1/2}$ and the transition rate to new non-populated states in the basis $\boldsymbol{m}$ is $\Gamma \sim \beta^2 |C|^6 \sim \beta^2/(\Delta n)^3$. Due to localization these transitions take place on a size $l$ and hence the diffusion rate in the distance $\Delta R \sim (\Delta n)^{1/d}$ of $d-$ dimensional $\boldsymbol{m}-$ space is $d(\Delta R)^2/dt \sim l^2 \Gamma \sim \beta^2 l^2/(\Delta n)^3 \sim \beta^2 l^2/(\Delta R)^{3d}$. At large time scales $\Delta R \sim R$ and we obtain

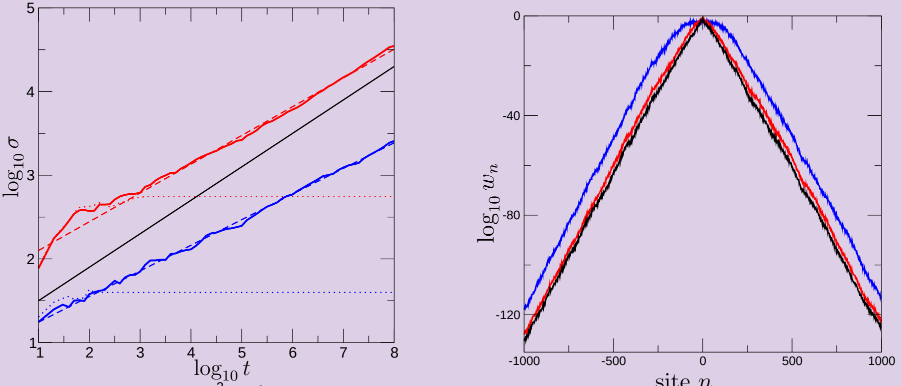
$$\Delta n \sim R^d \sim (\beta l)^{2d/(3d+2)} t^{d/(3d+2)}; \; (\Delta n)^2 \propto t^\alpha; \; \alpha = 2/(3d+2)$$

Chaos criterion:

$$S = \delta\omega/\Delta\omega \sim \beta > \beta_c \sim 1$$

there $\delta\omega \sim \beta |\psi_n|^2 \sim \beta/\Delta n$ is nonlinear frequency shift
and $\Delta\omega \sim 1/\Delta n$ is spacing between exites eigenmodes
DLS PRL **70**, 1787 (1993) ($d = 1$); I.García-Mata, DLS arXiv:0805.0539 (2008) ($d \geq 1$)

# Nonlinearity and Anderson localization (1D)




$W/V = 2, 4$, $\beta = 0, 1$; $\sigma = (\Delta n)^2 \propto t^{\alpha}$;

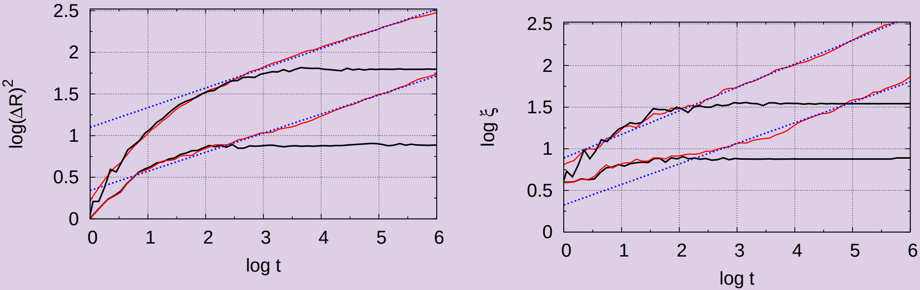
$\alpha = 2/5$ (theory) 0.34, 0.31 numerics

$W/V = 4$, $\beta = 1$, $t = 10^8$, $\beta = 0$

$$i\hbar \frac{\partial \psi_n}{\partial t} = E_n \psi_n + \beta | \psi_n |^2 \psi_n + V(\psi_{n+1} + \psi_{n-1}) \; ; [-W/2 < E_n < W/2]$$

A.S.Pikovsky, DLS PRL **100**, 094101 (2008)

# Nonlinearity and Anderson localization (2D)



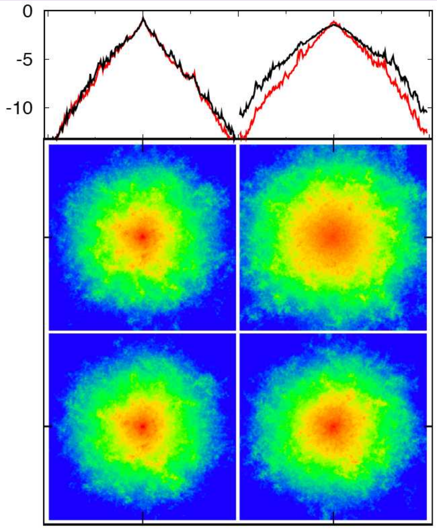

$W/V = 10, 15$, $\beta = 0, 1$; $\alpha_2 = 0.236, 0.229 \pm 0.003$ (theory 0.25)

$\nu = 0.282, 0.247 \pm 0.005$ (theory 0.25); $\xi$ is participation ratio

$$i\hbar \frac{\partial \psi_n}{\partial t} = E_n \psi_n + \beta |\psi_n|^2 \psi_n + V(\psi_{n+1} + \psi_{n-1})$$

I.García-Mata, DLS arXiv:0805.0539 (2008)

# Nonlinearity and Anderson localization (2D)

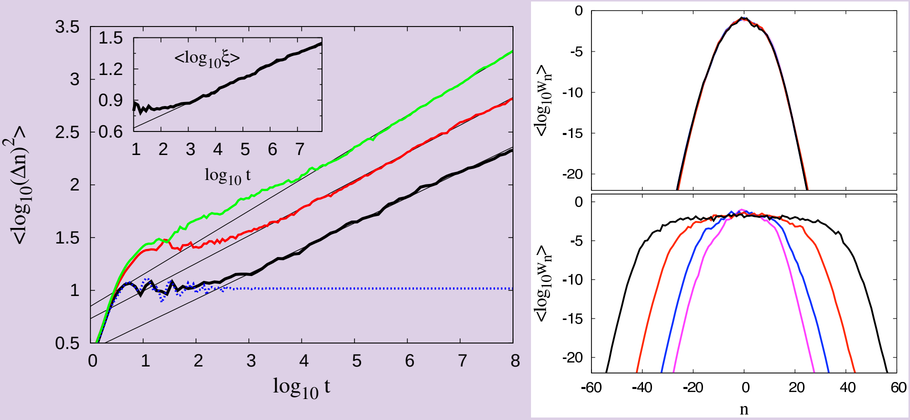



$W = 10$; $\beta = 0$(left), 1(right);
$t = 10^4$ (bottom), $10^6$ (middle),
projecton on $x-$axis (top);
$256 \times 256$ lattice

[also: kicked nonlinear rotator model (1d)]

I.García-Mata, DLS arXiv:0805.0539 (2008)

# Delocalization on disordered Stark ladder




Static field $f$ along Stark ladder ($W = 4$): *statistical entanglement*

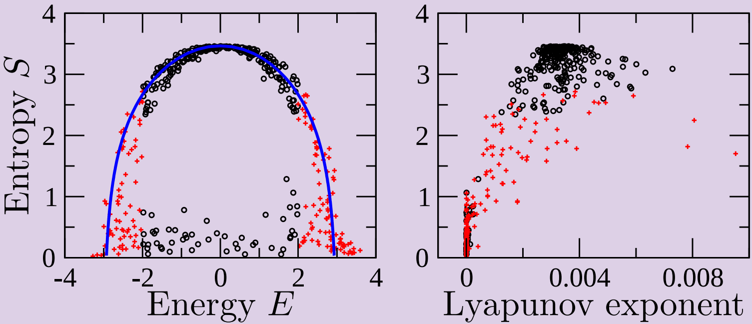
Left: $f = 0, 0.25, 0.5$, $\alpha = 0.30, 0.26, 0.24$, $\beta = 1; 0$ top to bottom; inset IPR at $f = 0.5$;

Right: probabability distribution at $f = 0.5$, $t = 10^2, 10^4, 10^6, 10^8$, $\beta = 0; 1$ (top/bottom)

I.García-Mata, DLS arXiv:0903.2103 (2009)

# Dynamical thermalization in DANSE (1D)

starting from Fermi-Pasta-Ulam problem (1955):
regular lattice, delocalized linear modes → disorder localized modes

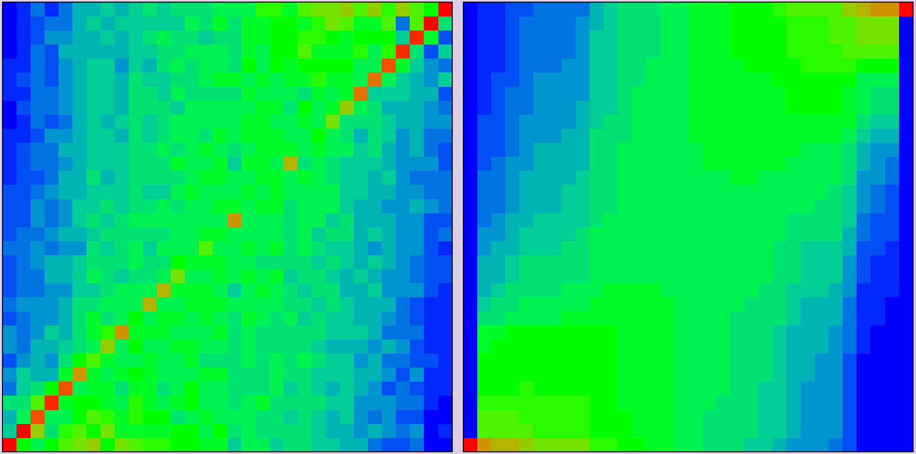



$N = 32$, $W = 4$, $\beta = 1$, $t = 10^7 + 10^6$, initial state: linear eigenmode

Gibbs distribution with temperature $T$ for localized linear modes, $\rho_m = |C_m|^2$:
entropy $S = -\sum_m \rho_m \ln \rho_m$, $\rho_m = Z^{-1} \exp(-\epsilon_m/T)$, $Z = \sum_m \exp(-\epsilon_m/T)$,
$E = T^2 \partial \ln Z/\partial T$, $S = E/T + \ln Z$ . $\langle \ln Z \rangle \approx \ln N + \ln \sinh(\Delta/T) - \ln(\Delta/T)$, $\Delta \approx 3$

M.Mulansky, K.Ahnert, A.Pikovsky, DLS arxiv:0903.2191 (2009)

# Dynamical thermalization in DANSE (1D)

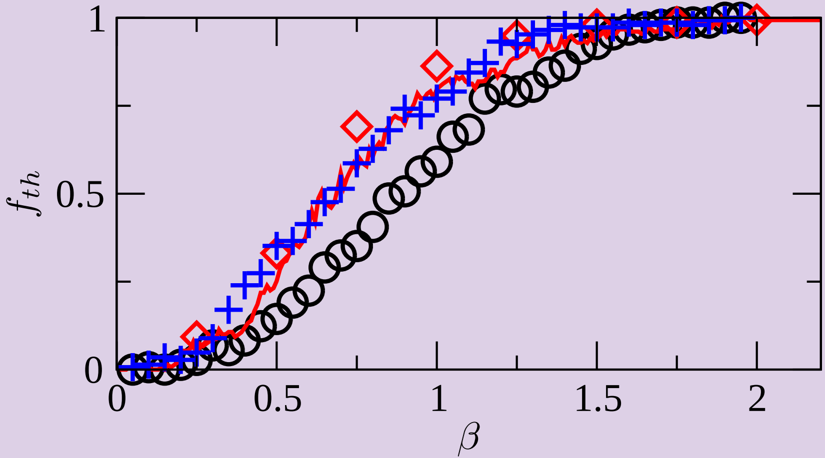


$N = 32$, $W = 4$, $\beta = 1$, $t = 10^6$, initial state: linear eigenmode $m'$, averaged over 8 disoder realisations

Gibbs distribution: time, disorder averaged $\rho_m$ in mode $m$ ($y$ - axis) for initial eigenmode $m'$ ($x$ -axis); left: numerics, right: Gibbs theory

M.Mulansky, K.Ahnert, A.Pikovsky, DLS arxiv:0903.2191 (2009)

# Dynamical thermalization in DANSE (1D)




Fraction of thermalized states: $N = 16$ (circles), 32 (curve), 64($+$) ; $W = 4, t = 10^6$, (diamonds $N = 32, t = 10^7$)

M.Mulansky, K.Ahnert, A.Pikovsky, DLS arxiv:0903.2191 (2009)

# Possible experimental tests & applications

- BEC in disordered potential (Aspect, Inguscio)
- kicked rotator with BEC (Phillips)
- nonlinear wave propagation in disordered media (Segev, Silberberg)
- lasing in random media (Cao)
- energy propagation in complex molecular chains (proteins, Fermi-Pasta-Ulam problem)
- NONLINEAR SPIN-GLASS ?
- OTHER GROUPS:
  S.Aubry *et al.* PRL **100**, 084103 (2008)
  A.Dhar *et al.* PRL **100**, 134301 (2008)
  S.Fishman *et al.* J. Stat. Phys. bf 131, 843 (2008)
  S.Flach *et al.* arXiv:0805.4693[cond-mat] (2008)
  W.-M.Wang *et al.* arXiv:0805.4632[math.DS] (2008)
  see also the participant list of the NLSE Workshop
  at the Lewiner Institute, Technion, June 2008

# Quntum systems:
# Two Interacting Particles (TIP) effect

Anderson model in $d$-space + onsite Hubbard interaction $U$, $V \sim E_F$ is one-particle hopping; exited states $\psi_n \sim \exp(-|n-m|/l)/\sqrt{l}$; $l \gg 1$.
Equation in the basis of noninteracting eigenstates $\chi_{m_1 m_2}$:

$$i\partial\chi_{m_1 m_2}/\partial t = \epsilon_{m_1 m_2}\chi_{m_1 m_2} + \sum_{m'_1 m'_2} U_{m_1 m_2 m'_1 m'_2}\chi_{m'_1 m'_2}$$

Sum runs over $M \sim l^d$ coupled states; interaction induced matrix elements $U_s \sim U_{m_1 m_2 m'_1 m'_2} \sim (U/(l^{2d}) \times \sqrt{M}$, density of coupled states is $\rho_2 \sim l^{2d}/V$, TIP transition rate $\Gamma_s \sim {U_s}^2 \rho_2 \sim U^2/(l^d V)$. Enhancement factor

$$\kappa = \Gamma_s \rho_2 \sim (U/V)^2 l^d > 1$$

TIP localization:
$l_2/l \sim (U/V)^2 l$ (1d);
$\ln(l_2/l) \sim (U/V)^2 l^2$ (2d);
delocalization for $\kappa \sim (U/V)^2 l^3 > 1$ (3d)

DLS PRL **73**, 2607 (1994); Y.Imry EPL **30**, 405 (1995)

# Many electrons near the Fermi level (Coulomb interaction, no spin)

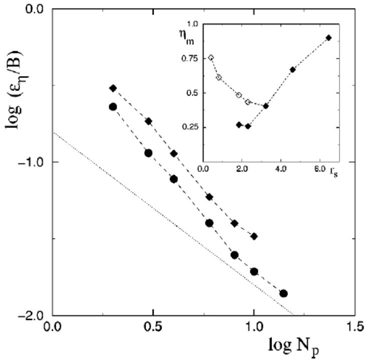



FIG. 4. Dependence of $\epsilon_\eta/B$ on the number of particles $N_p$, obtained from Fig. 2: $W/V=10$ with $\eta(E_\eta)=0.4$ (full diamond) and $W/V=7$ with $\eta(E_\eta)=0.2$ (●), where $\epsilon_\eta=E_\eta/N_p$. The straight line shows the slope when $E_\eta=$const. The inset gives the dependence of maximal $\eta$ on $r_s$ for $W/V=7$ and $N_p=6$: $U/V=2$, $8\leqslant L\leqslant 28$ (full diamond), and $L=14$, $0.25\leqslant U/V\leqslant 2$ (◇).

Level-spacing statistics $P(s)$:
$\eta = 1$ Poisson distribution,
$\eta = 0$ Wigner-Dyson distribution
$\epsilon_\eta$ - exitation energy per particle
at a given $\eta = const$ ($B = 4V$)
$r_s = U/(2V\sqrt{\pi\nu})$, $\nu = N_p/L^2 \approx 1/32$
usually $U = 2V$, $r_s \approx 3.2$,
$2 \leq N_p \leq 20$, $8 \leq L \leq 25$

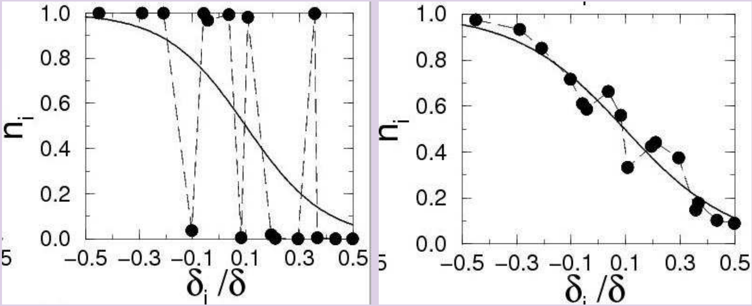
Result: chaotic, ergodic states at temperature going to zero

Problems: transport properties ?

DLS PRB **61**, 4588 (2000); P.H.Song, DLS PRB **61**, 15546 (2000)

# Dyn-thermalization in many-body Q-systems

Åberg criterion $J > J_c \approx \Delta_c$: two-body matrix element $J$ should be larger than energy spacing between directly coupled states $\Delta_c$

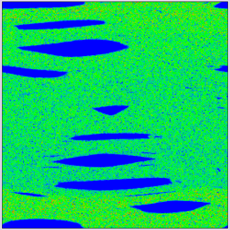



Example: Quantum computer with $n_q = 16$ qubits. One quantum eigenstate: Occupation numbers $n_i$ vs. rescaled exitation energies $\epsilon_i = \delta_i$. Left: $J/J_c \approx 0.15$, $T = 0.15\delta$, $\delta E = 0.97\delta$, $S = 0.49$. Right: $J/J_c \approx 1.5$, $T = 0.20\delta$, $\delta E = 1.19\delta$, $S = 8.41$. Full curves: Fermi-Dirac thermal distribution with given temperature $T$. G.Benenti *et al.* EPJD **17**, 265 (2001)

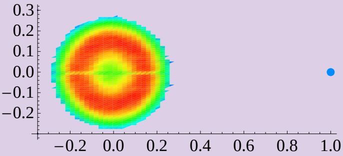
QC Hamiltonian: $H = \sum_i \Gamma_i \sigma_i^z + \sum_{i<j} J_{ij} \sigma_i^x \sigma_j^x$;
$\Gamma_i = \Delta_0 + \delta_i$, $-\delta < 2\delta_i < \delta$, $-J < J_{ij} < J$; $\rightarrow$ $J_c \approx 4\delta/n_q$

S.Åberg PRL **64**, 3119 (1990); DLS, O.Sushkov EPL **37**, 121 (1997);
P.Jacquod, DLS PRL **79**, 1837 (1997); B.Georgeot, DLS PRE **62**, 3504 (2000)

# Quantware posters from Toulouse



- Klaus Frahm
  "Diffusion and localization in the Chirikov typical map"




- Bertrand Georgeot
  "Delocalization transition in Google Matrix"