Quantum chaos applications: from simple models to quantum computers and Google matrix

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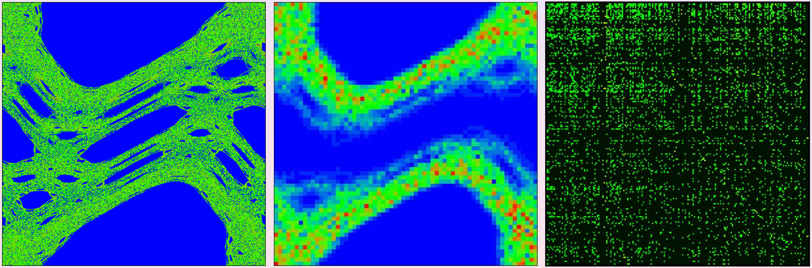


L1: Simple models of classical and quantum chaos

L2: Anderson localization in presence of nonlinearity and interactions

L3: Quantum chaos in many-body systems and quantum computers

L4: Google matrix and directed networks

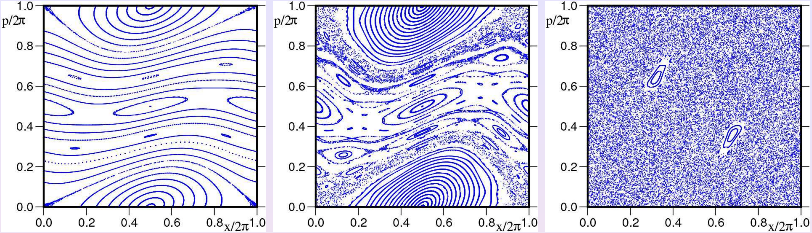
L1: Simple models of classical and quantum chaos

Poincaré (1890) - Einstein (1917)

Chirikov standard map: classical and quantum

- Classical map (Chirikov 1969, 1979):
  $\bar{p} = p + K \sin x, \quad \bar{x} = x + \bar{p}$
- Quantum map (kicked rotator) (1979):
  $\bar{\psi} = e^{-i\hat{p}^2/2\hbar} e^{-iK/\hbar \cos \hat{x}} \psi = e^{-iTn^2/2} e^{-ik \cos x} \psi;$
  $\hat{p} = -i\hbar\partial/\partial x, (T = \hbar \sim period, k = K/\hbar, K = kT)$
- Hamiltonian classical/quantum chaos:
  $H(\hat{p},\hat{x}) = \hat{p}^2/2 + K \cos \hat{x} \sum_m \delta(t - m); [\hat{p},\hat{x}] = -i\hbar$
  periodic conditions $\psi(x + 2\pi) = \psi(x)$ => rotator
  free space => cold atoms in kicked optical lattice
- Chaos border: $K < K_c = K_G = 0.97163540631...$
  Kolmogorov-Arnold-Moser (KAM) invariant curves with irrational rotation numbers $\nu =< (x(t) - x(0)) > /(2\pi t)$;
  most robust golden curve $\nu = (\sqrt{5} - 1)/2$ (golden mean curve)
  Global diffusion $K > K_c$ with the diffusion rate $D =< (p(t) - p(0))^2 > /t$

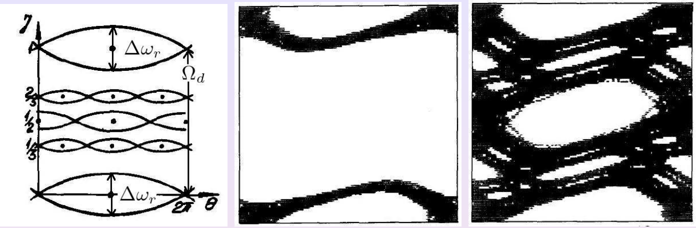
Poincaré section: KAM curves and chaos



$K = 0.5$; $K = K_G = 0.97163540631...$; $K = 5$

- $K < K_G$ exponentially narrow chaotic layers around separatrix of width $\delta w_s \sim \lambda^3 \exp(-\pi\lambda/2)$, $\lambda = 2\pi/\sqrt{K}$ (adiabatic parameter)
- $K = K_G$ => universal self-similar structure on small scales
  Greene (1979), MacKay (1983)
- Chaos: exponential divergence of nearby trajectories: $\Delta x(t) \sim e^{ht} \Delta x(0)$;
  Kolmogorov-Sinai entropy $h \approx \ln(K/2)$ *for* $(K > 4)$ (Lyapunov exponent)
- $K > K_G$ diffusion: $D \approx (K - K_G)^3/3$ for $K_G < K < 4.5$;
  $D \approx K^2[1 - 2J_2(K) + 2J_2^2(K)]/2 \approx K^2[1 - (8/\pi K)^{1/2} \cos(K - 5\pi/4)]/2$
  for $K > 4.5$
- Statistical description by the Fokker-Planck equation
  $\partial w(p,t)/\partial t = D/2 \;\; \partial^2 w(p,t)/\partial^2 p$

Chirikov criterion (1959)



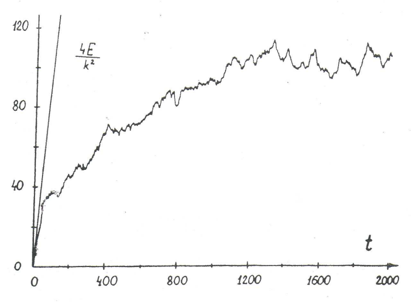
two resonances; $K = 0.96$; $K = 1.13$

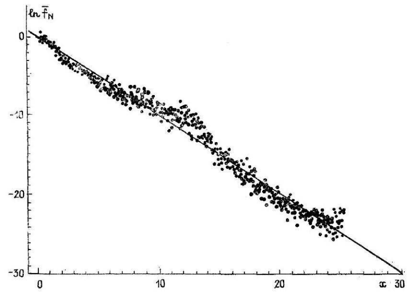
- resonance overlap (separatrix of unperturbed resonances):
  $K \approx 2.5S^2 > 1; S = \Delta\omega_r/\Omega_d$
  here the sum of two unperturbed resonance half-width is $\Delta\omega_r$; the frequency distance between resonance frequencies is $\Delta\Omega_d$
- example of the standard map: $H = I^2/ + K\sum_m \cos(\theta - 2\pi mt)$ (action-angle variables $p \to I$, $x \to \theta$)
- pendulum Hamiltonian $H = I^2/2 + K\cos\theta$, separatrix at energy $H(I,\theta) = E = K$, separatrix half width in frequency $\Delta I = 2\sqrt{K}$, hence $\Delta\omega_r = 2\Delta I = 4\sqrt{K}$, the frequency distance is $\Delta\Omega_d = 2\pi$
- $S = \Delta\omega_r/\Omega_d = 2\sqrt{K}/\pi > 1$ gives $K > \pi^2/4 \approx 2.5$.
- Counter-example: integrable nonlinear systems, e.g. Toda lattice

Quantum chaos models

- Conservative systems: discrete spectrum → no local exponential instability, regular time evolution of $\psi-$function
  Manifestations of classical chaos in quantum systems
- Ergodicity of eigenfunctions: Shnirelman theorem (1974)
  $\int \psi_n^* \hat{A} \psi_n dx = \int_\mu A d\mu$ (for chaotic billiards, e.g. Sinai billiard)
- Bohigas-Giannoni-Schmit conjecture (1984)
  for chaotic billiards the level spacing statistics is like for the Random Matrix Theory,
  Wigner-Dyson distribution $P_W(s) = (\pi s/2)\exp(-\pi s^2/4)$
  (Wigner surmise, time reversible systems)
- Integrable billiards → Poisson distribution: $P_P(s) = \exp(-s)$
  random independent levels (Berry, Tabor (1977))
- Ehrenfest time scale:
  exponentially rapid spreading of minimal coherent wave packet
  $t_E \sim |\ln N|/h \sim \ln(1/\hbar_{eff})/h$
  (Berman-Zaslavsky (1978); Chirikov, Izrailev, DS (1981), (1988))
- Exponentially many terms in a semicalssical expansion of wave function
  $\psi(x,t) = \sum_s \exp(-iS_s/\hbar)/\sqrt{D} + O(\hbar)$
- Gutzwiller quantization via unstable periodic orbits

Quantum standard map (kicked rotator) (1979)

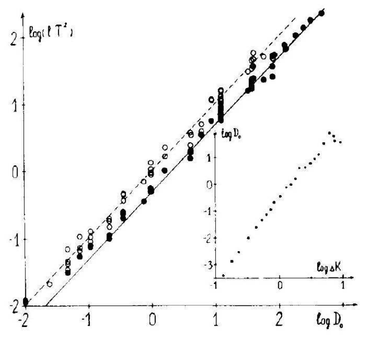


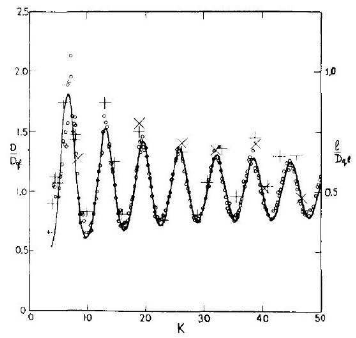


$E = < n^2/2 >, K = 5, k = 20; K = 5, k = 10, x = 2n/l_s, \bar{f}_N = < w_n > 2l_s/(1+x)$

- Time evolution: $\bar{\psi}_n = e^{-Tn^2/2} \sum_m J_{n-m}(k)(-i)^{n-m} \psi_n$
- Numerical simulations: via Bessel functions or fast Fourier transform 10^6 kicks in 1 CPU min for vector size 1024
- quantum suppression of classical chaotic diffusion (Casati, Chirikov, Ford, Izrailev (1979));
- estimate of number of excited states $\ell \sim \Delta n \sim D/\hbar^2$ (Chirikov, Izrailev, DS (1981); Chirikov, DS (1986));
- analogy with Anderson localization (Fishman, Grempel, Prange (1982))

Chirikov localization (dynamical localization)





diffusion and localization: $k = 30$, $K_q = 2k \sin T/2$; theory $\ell = D/2$ (DS (1987))

- Estimate of localization length: $\Delta n \sim (Dt)^{d/2}$, all corresponding frequencies are *homogeneously* distributed in the interval $(0, 2\pi)$, frequency spacing $\Delta\omega \sim 1/\Delta n$; discrete frequencies start to be resolved when $t > t_D \sim 1/\Delta\omega \sim \Delta n \sim (Dt_D)^{d/2} \sim D \sim \ell$ (for $d = 1$); here $D \approx k^2/2$ is diffusion rate measured in number of levels squared per time period; exponential localization of quasi-energy eigistates;
- $d = 2$ is critical dimension; $d = 3$ delocalization for $D > 1$

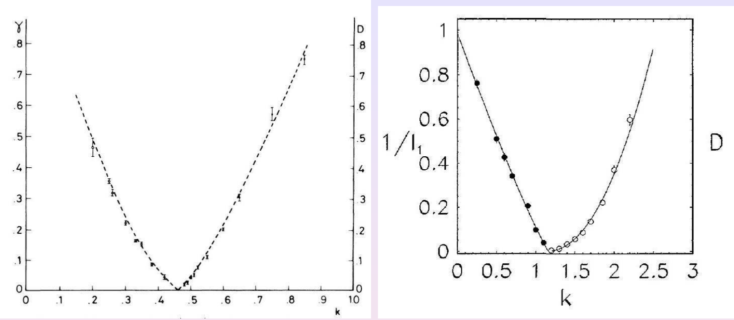
Anderson localization (1956)

- Anderson model:

  $E_n \psi_n + V(\psi_{n+1} + \psi_{n-1}) = E\psi_n$

  $-W/2 \le E_n \le W/2$ random on site energies;
- $d = 1, 2$ exponentially localized eigenstates, no spreading; localization length $\ell \approx 96(V/W)^2$ ($d = 1$ center of energy band); $\ln \ell \approx C(V/W)^2$ with a constant $C \sim 1$;
- physical origin: enhanced scattering due to quantum interference of forward and back paths in time
- $d = 3$ delocalized eigenstates for $W < W_c = 16.5V$
- numerics: direct diagonalization, transfer matrix technique
- critical exponents in a vicinity of transition $s = (d-2)\nu$;
  ($D \sim |W - W_c|^s$, $\ell \sim 1/|W - W_c|^\nu$; $r^d \sim t$ at $W = W_c$)
  (see more details e.g. Evers, Mirlin Rev. Mod. Phys. (2008))

Chirikov localization: various cases



frequency modulated kicked rotator (DS (1983), ... Borgonovi, DS (1996))

- Quantum resonance: $T = 4\pi m/q$ periodic quantum rotational phases $e^{-iT(n+\alpha)^2/2}$ (kicked rotator/particle)
  ballistic propagation like is perfect crystal
- Frequency modulated kicked rotator:
  $H = H_0(n) + K(t)V(\theta)\sum_m \delta(t-m)$; $H_0(n) = n^2/2$ or random;
  $V(\theta) = \cos\theta$, $K(t) = K[1 + \epsilon\sum_{m=1}^{d-1}\cos(\omega_m t)]$, $\hbar_{eff}$
- Lloyd model $V = -2tan^{-1}[2k(\cos\theta + \cos(\omega_1 t) + \cos(\omega_2 t)]$
  (effective 3d Lloyd model)

Cold atoms experiments for quantum chaos

the atomic state as $\psi_g(x,t)|g\rangle + \psi_e(x,t)e^{-i\omega_L t}|e\rangle$ with equations of motion

$$i\hbar\frac{\partial\psi_g}{\partial t} = -\frac{\hbar^2}{2M}\frac{\partial^2\psi_g}{\partial x^2} - \frac{\hbar\Omega}{2}\cos[k_L(x-\Delta L\sin\omega t)]\psi_e ,$$

$$i\hbar\frac{\partial\psi_e}{\partial t} = -\frac{\hbar^2}{2M}\frac{\partial^2\psi_e}{\partial x^2} + \hbar\delta_L\psi_e - \frac{\hbar\Omega}{2}\cos[k_L(x-\Delta L\sin\omega t)]\psi_g .$$

(see e.g. Graham, Schlautman, Zoller (1992))

Cold atoms experiments for quantum chaos

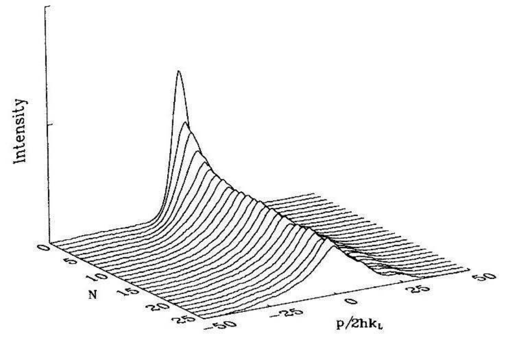
Here we neglect spontaneous emission from the upper atomic level, which is justified for sufficiently high detuning $\delta_L = \omega_0 - \omega_L$; $\Omega/2 = d\mathcal{E}_0/\hbar$ is the Rabi frequency. Adiabatic elimination of the excited-state amplitude with the assumption that the detuning δ_L is large compared to the Rabi frequency Ω and the excited-state kinetic energy term leads to

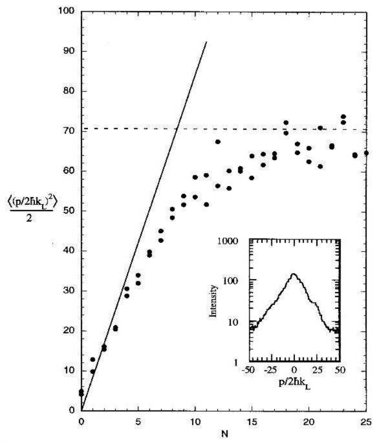
$$i\hbar \frac{\partial \psi_g}{\partial t} = -\frac{\hbar^2}{2M} \frac{\partial^2 \psi_g}{\partial x^2} - \frac{\hbar \Omega_{\rm eff}}{4} \cos^2[k_L(x - \Delta L \sin\omega t)]\psi_g ,$$

where $\Omega_{\rm eff} = \Omega^2/\delta_L$ is the effective Rabi frequency. (Note

(see e.g. Graham, Schlautman, Zoller (1992))

Experimental realization of kicked rotator by Raizen group (1995)

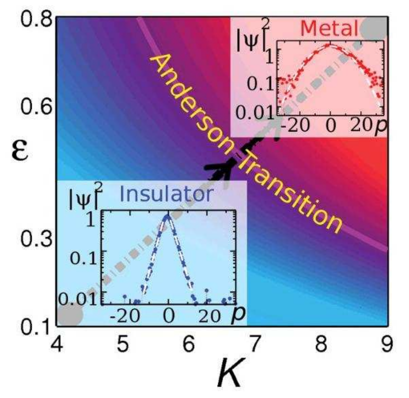


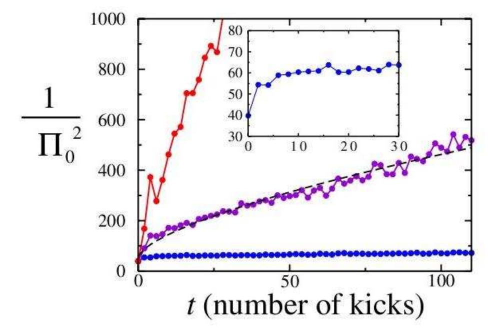


$K = 11.6 = kT$, $\hbar_{eff} = T = 2 => 1.5\mu s$, $\ell = D/2 = k^2/4 \approx 8.3$ (Raizen *et al.* (1995))

- Cold sodium atoms in a kicked optical lattice: 10^5 atoms ($\Delta x \approx 0.17mm$, $\Delta p \approx 4.6\hbar k_L$ at lattice recoil $k_L(589nm)$, $\hbar_{eff} = 8\omega_r \tilde{T}$, $\omega_r = \hbar k_L^2/2M$)

Anderson transition with frequency modulated kicked rotator by Garreau group (2008-2011)





$\hbar_{eff} = T = 2,89$, $\tilde{T} = 27.7\mu s$, $\omega_{1,2}/2\pi = \sqrt{5}, \sqrt{13}$ (Garreau *et al.* (2008))

- Cold sodium atoms in a kicked optical lattice: 10^7 atoms, $T_{cool} = 3.2\mu K$; $\Pi_0 \approx 1/\Delta p$ is probability at zero momentum

Anderson transition with frequency modulated kicked rotator by Garreau group (2008-2011)

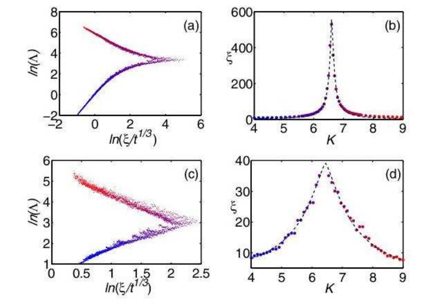


FIG. 3 (color online). Finite-size scaling applied to the results of numerical simulations (top) and to the experimental results (bottom). Graphs (a) and (c) emphasize that all data points corresponding to the quantity $\Lambda(k, t) = \Pi_0^{-2}(K, t)t^{-2/3}$ obtained for various values of K and t can actually be described by a scaling function $f(X)$ depending only on the variable $X = \xi(K)t^{-1/3}$. The finite-size technique makes it possible to determine both $f(X)$, shown in (a) and (c), and the scaling parameter $\xi(K)$, shown in (b) and (d). Close to K_c, the behavior of $\xi(K)$ is well fitted by Eq. (2) (dashed lines), giving $K_c = 6.6 \pm 0.1$ (simulation) and $K_c = 6.4 \pm 0.2$ (experiment). The critical exponent is $\nu = 1.60 \pm 0.05$ (simulation) and $\nu = 1.4 \pm 0.3$ (experiment).

scaling and critical exponent of Anderson transition (Garreau *et al.* (2008))

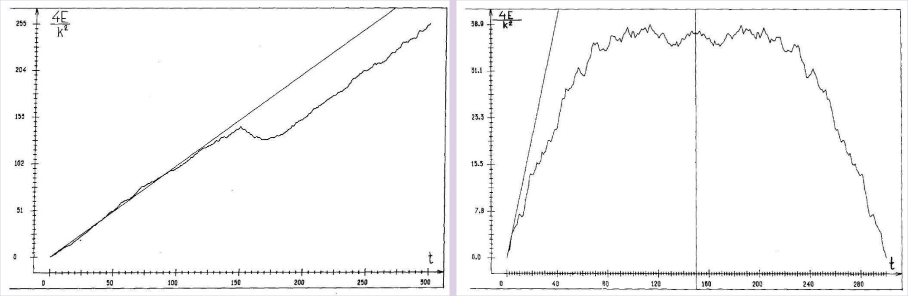
- Anderson transition for atomic matter waves

Boltzmann - Loschmidt dispute on time reversibility (1876)

* irreversible kinetic theory from reversible equations

Sitzungsberichte der Akademie der Wissebschaften, Wien,
II **73**, 128 (1876); **75**, 67 (1877)

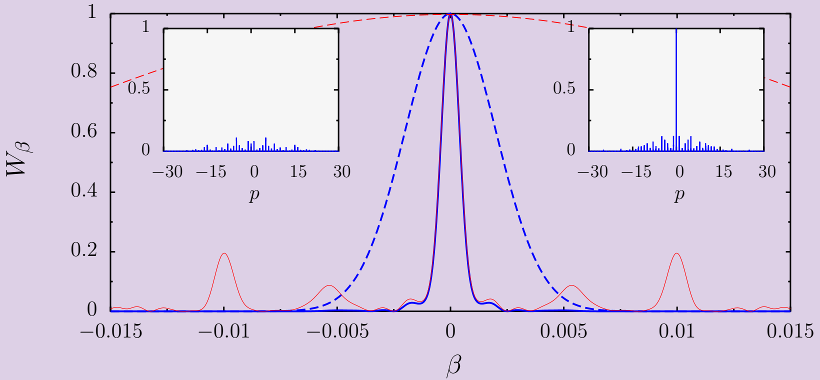
Time reversal for the Chirikov standard map



BESM-6 computation, rescaled energy or squared momentum vs. time t:
$K = 5$, $\hbar = 0$ (left), $\hbar = 1/4$ (right)
(DS (1983))

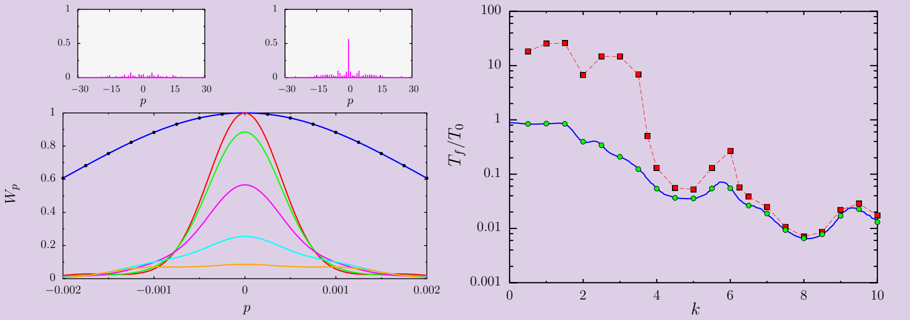
* Experimental realization of time reversal:
spin echo (E.L.Hahn (1950)); acoustic waves (M.Fink (1995));
electromagnetic waves (M.Fink (2004))

* Loschmidt cooling by time reversal of atomic matter waves



proposal of time reversal in kicked optical lattices:
$k = K/\hbar$, $\hbar = 4\pi + \epsilon$ (forward), $\hbar = 4\pi - \epsilon$ (back) and $k \to -k$;
Fig: $k = 4.5, \epsilon = 2, t_r = 10$, $k_B T_o/E_r = 2 \times 10^{-4}$ (red), $k_B T_o/E_r = 2 \times 10^{-6}$ (blue);
momentum β and energy E_r are give in recoil units
(Martin, Georgeot, DS (2008))

* Time reversal of Bose-Einstein condensates

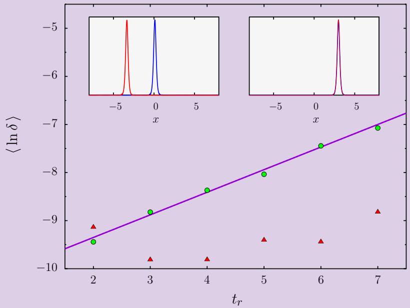


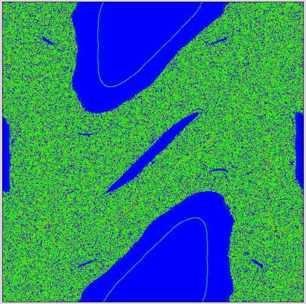
The Gross-Pitaevskii equation with kicks:

$$i\hbar\frac{\partial}{\partial t}\psi = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - g|\psi|^2 + k\cos x\ \delta_T(t)\right)\psi$$

Left: same as in previous Fig. for $g = 0, 5, 10(insets), 15, 20$ (top to bottom), ($t = 0$);
Right: cooling ratio T_f/T_0 for $g = 0$ (blue curve), $g = 0.5$ (green), $g = 10$ (red)
(Martin, Georgeot, DS (2008))

* Loschmidt paradox for Bose-Einstein condensates





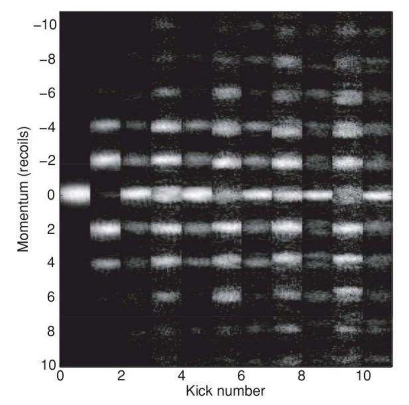
Soliton initial condition (Zakharov, Shabat (1971)):

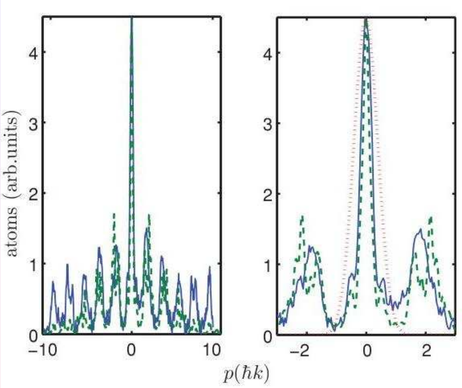
$$\psi(x,t) = \frac{\sqrt{g}}{2} \frac{\exp\left(ip_0(x-x_0-p_0t/2)+ig^2t/8\right)}{\cosh\left(\frac{g}{2}(x-x_0-p_0t)\right)}$$

Left: time resersal of soliton at $g = 10, k = 1, T = \hbar = 2, K = kT = 2, t_r = 40$ inside chaotic (left inset) and regular (right inset) domains; line shows divergence given by the Kolmogorov-Sinai entropy $h = 0.45$. Right: Poincaré section at $K = 2$

But the real BEC is quantum and should return back since the Ehrenfest time $t_E \sim |\ln \hbar_{eff}|/h \sim \ln N/2h \sim 13$ for BEC with $N = 10^5$ atoms
(Martin, Georgeot, DS (2008))

Time reversal of atomic matter waves by Hoogerland group (2011)

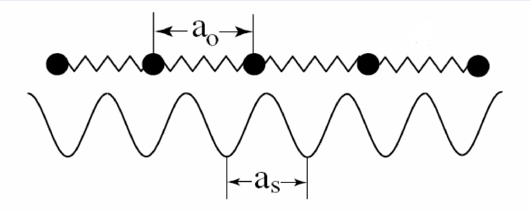




$k = 2 - 3$, $T = 4\pi \pm \epsilon$ (Ullah, Hoogerlan (2011))

- Ultracold ^{67}Rb atoms BEC: 10^4 atoms, $T_{cool} = 50nK$, $\lambda = 2\pi/k_L = 760nm$, $\epsilon = 1$; $5 + 5$ kicks; right panel shows zoom near initial distribution shown by red dotted curve (initial/final width is $0.43/0.21$ recoils; full/dashed curve for experiment/numerics).

Frenkel-Kontorova model (1938)



Schematic presentation of the Frenkel-Kontorova model: A chain of particles with their mean separation a_0 interacting via harmonic forces is subjected to the action of an external periodic potential with period a_s and amplitude K

$$H = \sum_i \left(\frac{dx}{dt}\right)^2 + \underbrace{\frac{(x_i - x_{i-1} - a_0)^2}{2} + K\cos(x)}_{U_{pot}}$$

The ratio $\nu = a_0/a_s$ defines the *rotation number*.

see details in O.M.Braun abd Yu.S.Kivshar, *The Frenkel-Kontorova model: concepts, methods, adn applications*, Spriner, Berlin (2004)

FK model: Aubry sliding-pinning transition

Static minimal energy equilibrium configurations are given by conditions

$$\frac{dU_{pot}}{dx_i} = 0, \quad i = -\infty, \ldots, 0, \ldots, \infty \tag{1}$$

This conditions can be written in the form of the Chirikov standard map

$$\begin{aligned} p_{i+1} &= p_i + K \sin(x_i) \\ x_{i+1} &= x_i + p_{i+1}, \end{aligned}$$

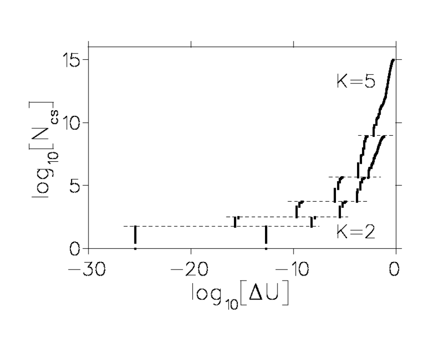
[*S.Aubry (1978-1983), I.C.Persival (1979)*].

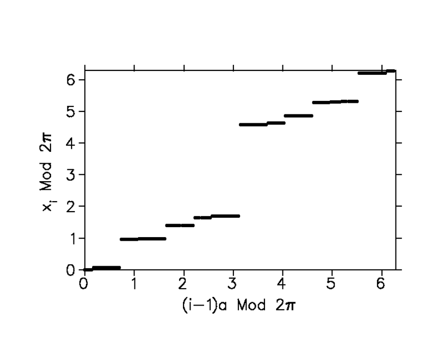
For irrational winding number ν the chain can freely slide if $K < K_c$, and is pinned for $K > K_c$, that corresponds for the standard map to the regimes of regular and chaotic dynamics.

This critical behavior is known as *Breaking of analitycity or Aubry transition.*

For $\nu = (\sqrt{5} - 1)/2$ (the golden mean) the critical point is at $K_c = K_G = 0.971635\ldots$
Phonon spectrum of oscillations $\nu = ck$ at $K < K_c$,
gaped optical phonons $\nu^2 = \Delta^2 + (ck)^2$ at $K > K_c$

FK model: dynamical spin glass





Left: Integrated number of equilibrium configuration states N_{cs} as a function of the energy difference ΔU between the energy of configuration U and the ground state energy U_G, counted per one particle. Here the number of particles is $s = 89$ and the number of wells $r = 55$ for $K = 5 > K_c$ and for $K = 2 > K_c$. bands. All equilibrium configurations are counted. Right: devil's staircase hull function at $K = 2$ (Zhirov, Casati, DS (2002))

Quantum Frenkel-Kontorova model

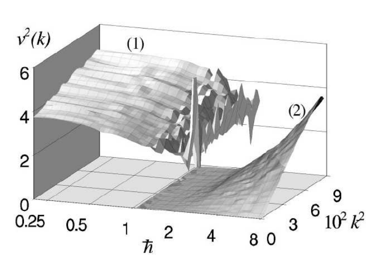
Quantum effects:

- Quantum **instanton** tunneling between degenerate classical configurations, forming the true quantum ground state, the vacuum state.
- The energy spectrum is given by quantum excitations of this vacuum state.
- The kinetic energy of particles is nonzero even in the ground state.

Zero temperature Quantum Phase Transition:

melting of pinned state at $\hbar = \hbar_c$

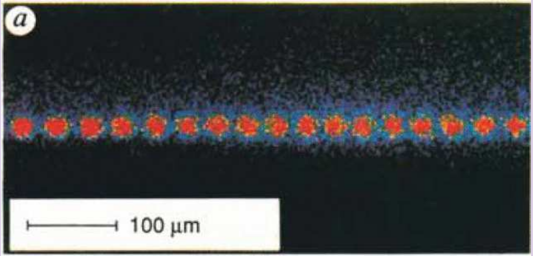
- Rearrangement of spectrum of elementary quantum excitations.
- Vanishing phonon gap.
- Melting of pinned phase and emergence of sliding regime.



$\nu(k)$ and k are the frequency and the wave number of elementary excitations.

(Zhirov, Casati, DS (2003))

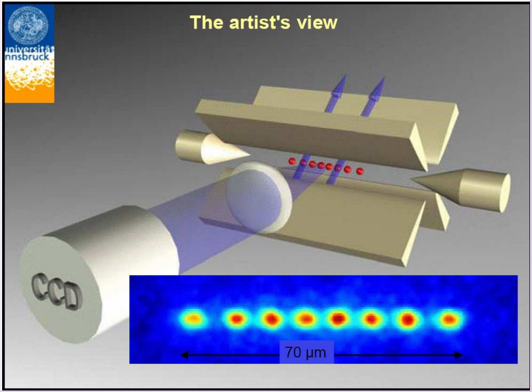
Wigner crystal in a periodic potential: ions



Experiments with cold $^{24}M^{+}$ ions in a quadrupole storage ring:
G.Bikl, S.Kassner and H.Walther, Nature **357**, 310 (1992))

Observed structures include the one-dimensional Wigner crystal,
zig-zag and helical structures in three dimensions with up to thousands of ions.

Cold ions for quantum computing



Proposal Cirac-Zoller (1995); Experiments of the Blatt group at Innsbruck (J.Eschner lecture at Varenna school 2005)

Ions in a global oscillator potential, at to 8 qubits has been realized.

Wigner crystal in a periodic potential

The dimensionless Hamiltonian has the form:

$$H = \sum_{i=1}^{N} \left(\frac{P_i^2}{2} - K \cos x_i \right) + \sum_{i>j} \frac{1}{|x_i - x_j|} \quad (2)$$

where P_i, x_i are ion momentum and position, K gives the strength of optical lattice potential and all N ions are placed in a harmonic potential with frequency ω. To make a transfer from (2) to dimensional physical units one should note that the lattice constant d in $K \cos(x_i/d)$ is taken to be unity, the energy $E = H$ is measured in units of ion charge energy e^2/d. In the quantum case $P_i = -i\hbar\partial/\partial x_i$ with dimensionless $\hbar$ measured in units $\hbar \to \hbar/(e\sqrt{md})$, m is charge mass.

Related map:

$$p_{i+1} = p_i + Kg(x_i)\ , \quad x_{i+1} = x_i + 1/\sqrt{p_{i+1}}\ , \quad (3)$$

where the effective momentum conjugated to x_i is $p_i = 1/(x_i - x_{i-1})^2$ and the kick function is $Kg(x) = -K \sin x$.

For the Frenkel-Kontorova model the equilibrium positions are described by the Chirikov standard map (1969-1979): $p_{i+1} = p_i + K \sin x_i,\ x_{i+1} = x_i + p_{i+1}$ with $K_c = 0.971635...$ for the golden mean density $\nu = (\sqrt{5} - 1)/2$.

(Garcia-Mata, Zhirov, D.S. (2007))

Wigner crystal in a periodic potential

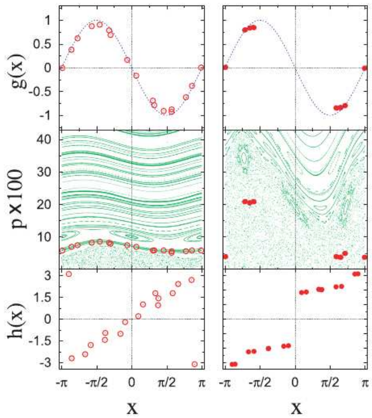
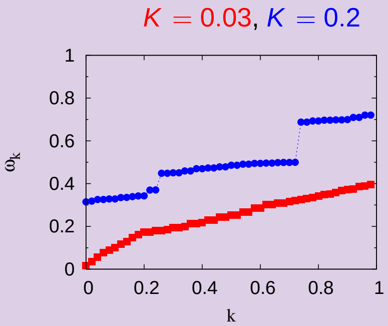
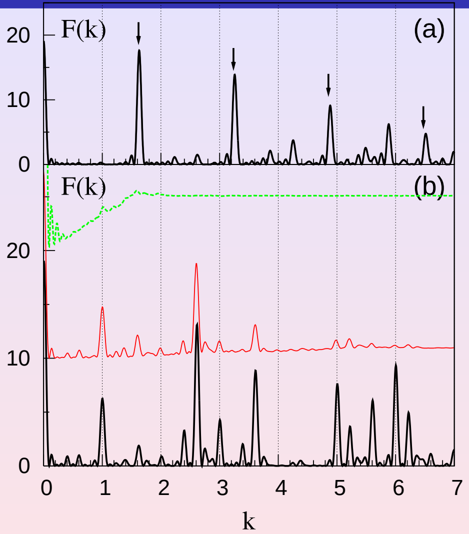


Fig. 1. (Color online) Functions related to the dynamical map (2) obtained from the ground state equilibrium positions x_i of $N = 50$ ions for $\omega = 0.014$ at $K = 0.03$ (open circles, left column) and $K = 0.2$ (full circles, right column). Panels show: the kick $g(x)$ function (top); the phase space (p, x) of the map (2) with $g(x) = -\sin x$ (green/gray points) and actual ion positions (red/black circles) (middle); the hull function $h(x)$ (bottom). The ion positions are shown as $x = x_i$ (mod 2π) for the central 1/3 part of the chain.



Quantum melting of Wigner crystal



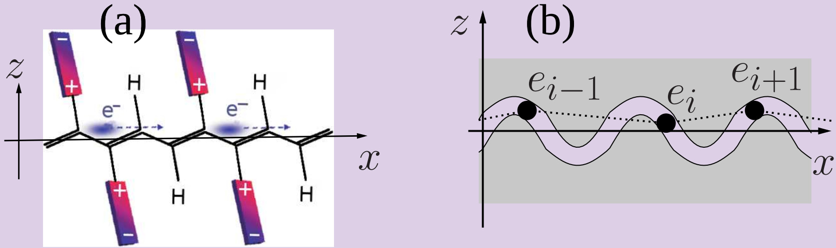
Formfactor of charge density

$F(k) = < |\sum_j \exp(ikx_j(\tau))|^2 >$

(a) The classical incommensurate phase at $K = 0.03$, $\hbar = 0$, arrows mark the peaks at integer multiples of golden mean density ν_g. (b) The pinned phase case at $K = 0.2$ for $\hbar = 0$ (bottom black curve), $\hbar = 0.1$ (red curve shifted 10 units upward), $\hbar = 2$ (green curve shifted 20 units upward, for clarity $F(k)$ is multiplied by factor 5); temperature is $T = \hbar/400 \ll K$. The quantum phase transition takes place at $\hbar_c \approx 1$.

$\rightarrow$ **SLIDING at $\nu_{c1} < \nu$, PINNING at $\nu < \nu_{c1}$**

Wigner crystal in snaked nanochannels



Left: R.Little (1964) organic conductors/superconductors;
Right: Zhirov, DS (2011) Wigner snake sliding

Wigner snake sliding

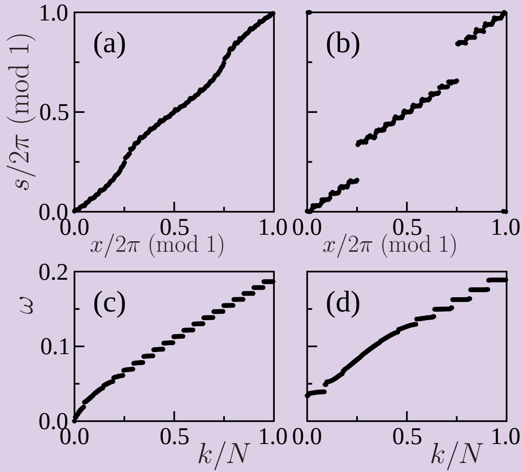
We take a finite number of electrons N for L periods of a channel of finite length. In numerical simulations we put the channel on a cylindrical surface in 3D with electron coordinates being $x_i = L\sin(s_i/L)$, $y_i = L\cos(s_i/L)$, $z = a\sin(s_i)$ where s_i is coordinate along channel for electron i. Thus the channel, filled by N electrons, wiggles in the $z-$direction making L periodic oscillations along cylinder of radius L with periodic boundary conditions. The Coulomb energy of the system is

$$E = \sum_{j>i} 1/R(s_i, s_j) \tag{4}$$

where $R(s_i, s_j)$ is the distance between two electrons. We find from geometry $R^2(s_i, s_j) = 4L^2\sin^2[(s_i - s_j)/2L] + a^2(\sin s_i - \sin s_j)^2$. Here we choose dimensionless units for charge e and length, so that the channel period length is $\ell = 2\pi$ and dimensionless amplitude of channel oscillations is a. The equilibrium static configurations are defined by the condition $\partial E/\partial s_i = 0$ with a minimal ground state energy configuration determined numerically.

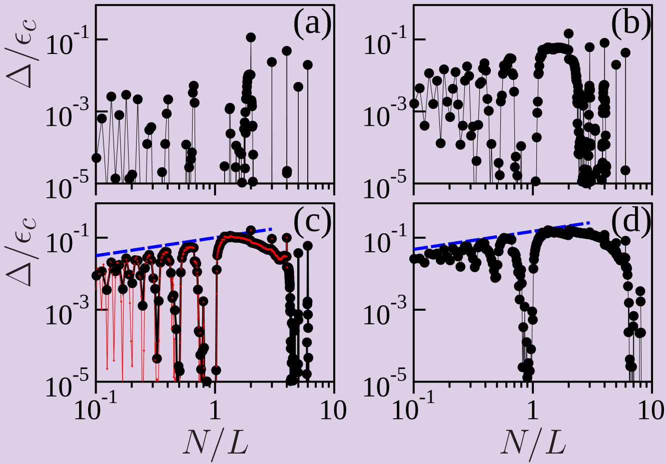
The total energy E is invariant for a homogeneous shift of all electrons by δs when the distance between nearby electrons is $s_{i+1} - s_i = 2\pi m$ that corresponds to electron density $\nu = N/L$ with resonant values $\nu_m = 1/m$.

Wigner snake sliding



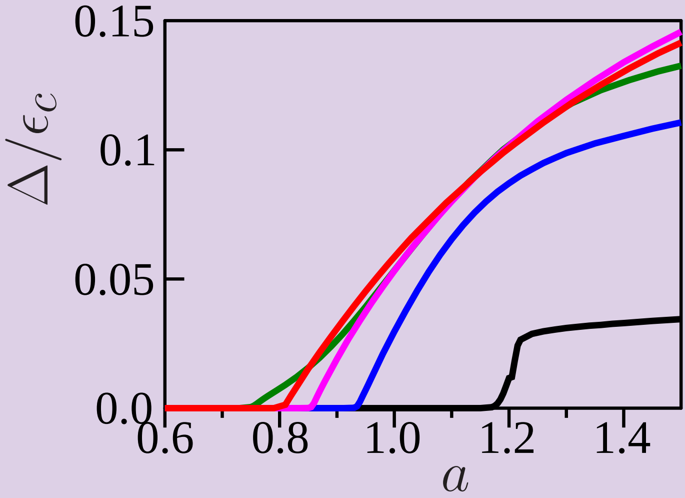
Hull function $s = h(x)$ (a,b) and phonon spectrum $\omega(k/N)$ (c,d) for incommensurate electron densities $\nu = N/L = 239/233$ (a,c) and $\nu = N/L = 244/233$ (b,d). Here $a = 1.2$ and x gives the positions s_i of electrons at $a = 0$.

Phonon gap: density dependence



Dependence of the dimensionless phonon gap Δ/ϵ_c on the electron density $\nu = N/L$ for $a = 0.7(a)$, $1(b)$, $1.2(c)$, $1.5(d)$. Here $L = 89$ (black), 233 (red). The straight line shows empirical dependence $\Delta/\epsilon_c \propto (N/L)^{1/2}$ for (c, d), where $\epsilon_c = 2\pi e^2\nu/\ell = \nu$ is the Coulomb energy.

Phonon gap: deformation dependence



Dependence of rescaled phonon gap Δ/ϵ_c on channel deformation amplitude a at various values of electron density ν with the number of electrons $N = 241$ (black), 269(blue), 337 (magenta), 377 (red), 307 (green) at $L = 233$.

Dynamical map description

An approximate dynamical map determines recursively the electron positions along the channel. The recursion is given by equilibrium conditions $\partial E/\partial s_i = 0$. Assuming that $a \ll 1$ we can expand R in a that, after keeping only nearest electron interactions, gives recursive relations between s_{i-1}, s_i, s_{i+1}. They can be presented in a form of dynamical map

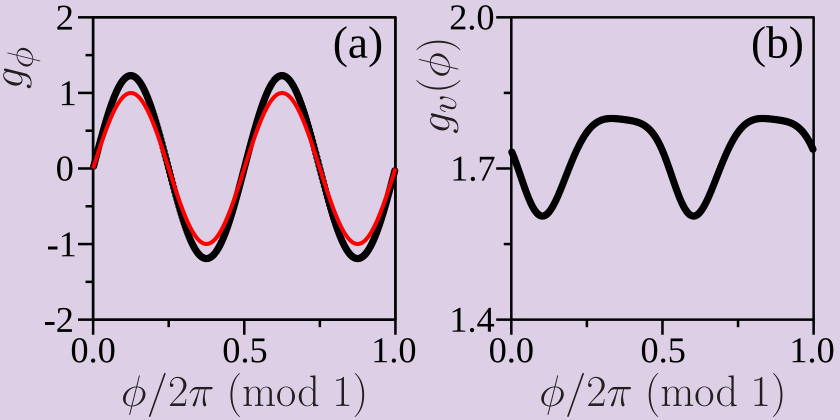
$$\begin{aligned} \bar{v} &= v + 2a^2(1 - \cos\bar{v})\sin 2\phi \,, \\ \bar{\phi} &= \phi + \bar{v} + a^2 \sin\bar{v}\cos 2\phi \,, \end{aligned} \qquad (5)$$

where $v = s_i - s_{i-1}$, $\phi = s_i$ are conjugated action-phase variables, bar marks their values after iteration. The map is implicit but symplectic. To check its validity we use the values s_i obtained for the groundstate configuration and extract from them the kick function $g_\phi = \sin 2\phi$ from the values $\bar{v} - v = 2a^2 g_v(v) g_\phi(\phi)$ with $g_v(v) = 1 - \cos v$. Such a check shows that the map indeed gives a good description of actual electron positions s_i up to moderate values of a.

Kolmogorov-Arnold-Moser (KAM) invariant curves $\rightarrow$ sliding phase

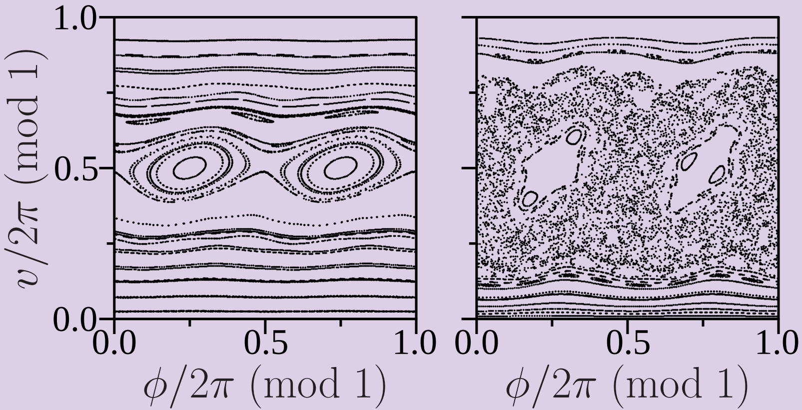
Aubry (cantori) phase $\rightarrow$ pinned phase

Dynamical map desciption



Map kick functions $g_\phi(\phi)$ (a) and $g_v(v)$ (b) obtained from the groundstate electron positions s_i in nanochannel (points), full red curve in (a) shows the theoretical dependence from the map. Here N=377, L=233, $a = 0.5$.

Poincaré section and Aubry transition



Poincare section for the dynamical map at $a = 0.25$ (left panel), 0.5 (right panel).

Dynamics is approximately described by the Chirikov standard map with the chaos parameter $K \approx 4a^2(1 - \cos v)$; the KAM curves are destroyed at $K > 1$. At small charge density ν the parameter K is small $K \approx 2a^2\nu^2 \ll 1$ that corresponds to the KAM regime and a conducting phase of Wigner crystal.

$\rightarrow$ **SLIDING at $\nu < \nu_{c2}$, PINNING at $\nu > \nu_{c2}$**

Discussion:

Kolmogorov-Arnold-Moser concept of superconductivity in organic conductors

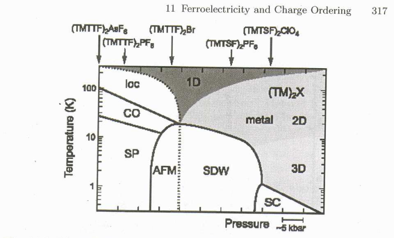
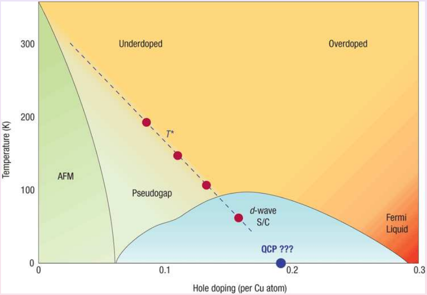


Fig. 11.1. Schematic phase diagram of the $(TMTCF)_2X$ family. SC – superconductivity; AFM – antiferromagnet – i.e., commensurate SDW; SP – spin-Peierls; CO – charge ordering, loc – charge localization (CO pretransitional effect); 1D, 2D, 3D – dimensional regimes. After Dressel, Dumm et al.



KAM CONCEPT: KAM curves and free sliding correspond to a superconducting phase induced by Coulomb repulsion in molecular wires, this phase appears at filling factors

$\nu_{c1} < \nu < \nu_{c2}$.

Microwave ionization of hydrogen atoms

Experiment Bayfield, Koch (1974), Koch *et al.* (1988)
Hamiltonian: $H(p,r) = \mathbf{p}^2/2 - 1/r + \epsilon r \cos \omega t$
(atomic units, $n_0 \approx 66$, $\omega \approx 0.43/n_0^3 (10GHz)$, $\epsilon_0 = \epsilon n_0^4 \approx 0.13$)

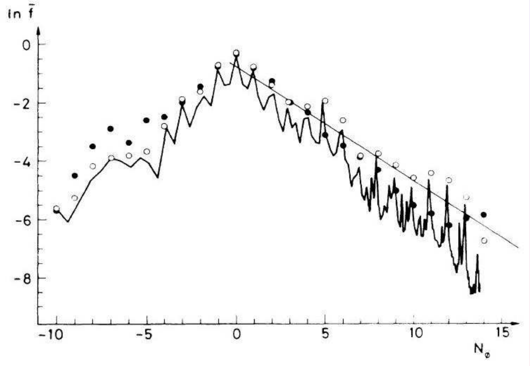
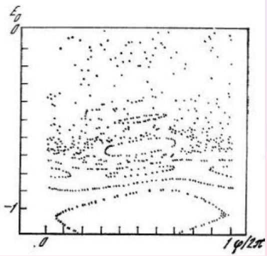


FIG. 1. The probability distribution, averaged from 80 to 120 periods vs the number of photons $N_\phi = N_I - 1/(2n^2\omega)$. Here $n_0 = 100$, $\varepsilon_0 = \varepsilon n_0^4 = 0.04$, $\omega_0 = \omega n_0^3 = 3$. For each integer value of N_ϕ, open circles indicate the probability in the interval $N_\phi - \frac{1}{2}$,

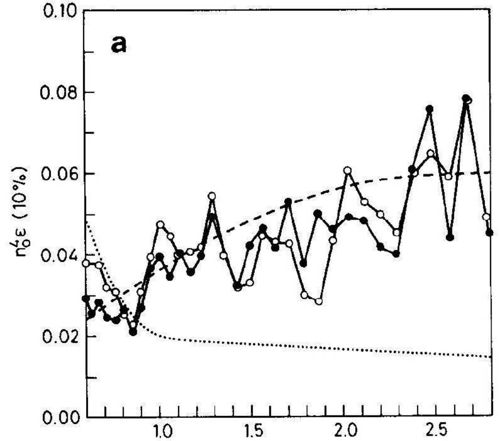
Kepler map ($E = \omega N$)
$\bar{N} = N + k \sin \phi$,
$\bar{\phi} = \phi + 2\pi\omega(-2\omega\bar{N})^{-3/2}$;
$k = 2.6\epsilon/\omega^{5/3}$, $\epsilon_0 > 1/(50\omega_0^{1/3})$



photonic localization:
$\ell_\phi = 2\pi(\epsilon\mu\rho)^2 = 3.3\epsilon^2/\omega^{10/3}$
DS, Chirikov *et al.* (1983-91)

Theory vs. microwave ionization experiments

Koch *et al.* (1988): 36GHz, $n_0 = 45 - 80$, points



theory Casati, Guarneri, DS (1988-90): quantum Kepler map (open circles);
$\bar{\psi}_N = \exp(-i\hat{H})_0 \exp(-ik\cos\phi)\psi_N$, $H_0 = 2\pi[-2\omega(N_0 + \hat{N}_\phi)]^{-1/2}$,
$\hat{N}_\phi = -i\partial/\partial\phi$; delocalization border: $\ell_\phi = N_l = n_0/(2\omega_0) => \epsilon n_0^4 = \omega^{1/6}\omega_0/8$;
dashed/dotted quantum/classical curves of analytical theory
(no fit parameters)

Halley comet map

Chirikov, Vecheslavov (1988-89): 46 appearances from history/numerics
map description of comet dynamics

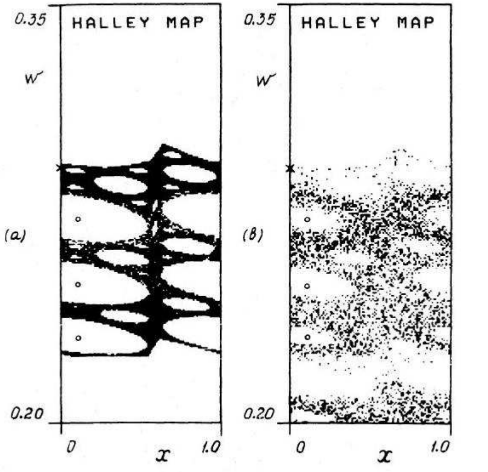


Fig. 3a and b. Phase trajectory of map (3) in the STA (6). Initial conditions (crosses) $w_1 = 0.29164$; $x_1 = 0$ (in 1986, see Table 1): **a** Jupiter's perturbation only, $N = 1.5\ 10^5$ iterations; **b** perturbation by both Jupiter and Saturn, $N = 4000$

Halley map ($w = -2E$)
$\bar{w} = w + F(x)$,
$\bar{x} = x + \bar{w}^{-3/2}$;
$x = t/T_J$ Jupiter phase at perihelion,
$F_{max} \sim 5M_J/M_S$

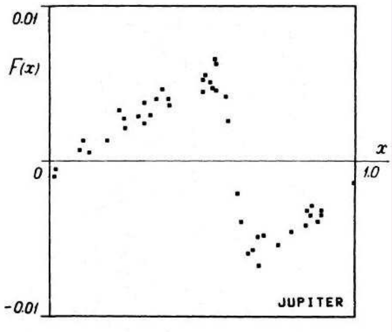


Fig. 1. The full perturbation of comet Halley vs. Jupiter's phase

diffusive life time: 10^7 years

Capture of dark matter by the Solar system

capture => inverse process to ionization

Let us now estimate the capture cross-section σ assuming that for all DMPs the dynamics is described by the Kepler map with fixed $\beta \sim 1$. Then only DMPs with energies $|w| = v^2 r_p / k m_p M = v^2/v_p^2 < \beta m_p/M$ are captured under the condition that $q < r_p$ (here v_p is the velocity of the planet). The value of q can be expressed via the DMP parameters at infinity, where its velocity is v and its impact parameter is r_d, and hence $q = (v r_d)^2/2kM$.[13] Since $q \sim r_p$ we obtain the cross-section

$$\sigma \sim \pi r_d^2 \sim \frac{2\pi k M r_p}{v^2} \sim 2\pi r_p^2 \left(\frac{v_p}{v}\right)^2 \sim \frac{2\pi r_p^2 M}{\beta m_p}, \qquad (9)$$

where the last relation is taken for those typical velocities, $v^2 \sim \beta v_p^2 m_p/M$, at which the capture of DMPs takes place (for $q \approx 1.4 r_p$ we have $\beta \approx 5$). Then Eqs. (4) and (9) give the captured mass Δm_p of (7) with an additional numerical factor $\beta \sim 1$.

According to the above estimates, DMPs captured by Jupiter have typical velocities at infinity $v \sim (\beta m_p/M)^{1/2} v_p \sim 1\,\mathrm{km/s}$ for typical $\beta \sim 5$ and $m_p/M \approx 10^{-3}$, $v_p \approx 13\,\mathrm{km/s}$. This value of v is in good agreement with the numerical simulations of Ref. 5, which give typical captured DMP velocities for Jupiter of $1\,\mathrm{km/s}$.

Khriplovich, DS (2009)

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