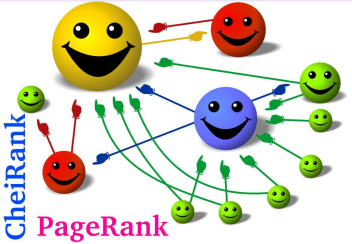
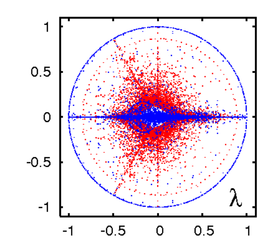
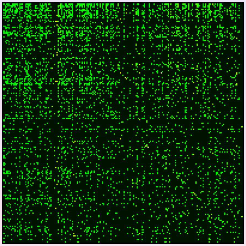
# Google matrix of social networks

Dima Shepelyansky (CNRS, Toulouse)
www.quantware.ups-tlse.fr/dima



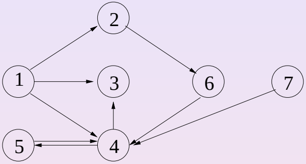



1945: Nuclear physics → Wigner (1955)→ Random Matrix Theory
1991: WWW, small world social networks → Markov (1906) → Google matrix

S.Brin and L.Page, Comp. Networks ISDN Systems **30**, 107 (1998)

## Markov chains (1906) and Directed networks




Weighted adjacency matrix

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For a directed network with $N$ nodes the adjacency matrix $\mathbf{A}$ is defined as $A_{ij} = 1$ if there is a link from node $j$ to node $i$ and $A_{ij} = 0$ otherwise. The weighted adjacency matrix is

$$S_{ij} = A_{ij} / \sum_k A_{kj}$$

In addition the elements of columns with only zeros elements are replaced by $1/N$.

## Google Matrix and Computation of PageRank

$\mathbf{P} = \mathbf{SP} \Rightarrow \mathbf{P}$= stationary vector of $\mathbf{S}$; can be computed by iteration of $\mathbf{S}$.

To remove convergence problems:

- Replace columns of 0 (dangling nodes) by $\frac{1}{N}$:

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & \frac{1}{7} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{7} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{7} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{7} & 0 & 1 & 1 & 1 \\ 0 & 0 & \frac{1}{7} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{7} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{7} & 0 & 0 & 0 & 0 \end{pmatrix} ; \mathbf{S}^* = \begin{pmatrix} \frac{1}{7} & 1 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{1}{7} \\ \frac{1}{7} & 0 & 0 & 0 & 0 & 1 & \frac{1}{7} \\ \frac{1}{7} & 0 & 0 & 0 & 0 & 0 & \frac{1}{7} \\ \frac{1}{7} & 0 & \frac{1}{2} & 0 & 1 & 0 & \frac{1}{7} \\ \frac{1}{7} & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{7} \\ \frac{1}{7} & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{7} \\ \frac{1}{7} & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{7} \end{pmatrix}.$$

- To remove degeneracies of $\lambda = 1$, replace $\mathbf{S}$ by **Google matrix**

  $\mathbf{G} = \alpha \mathbf{S} + (1-\alpha)\frac{\mathbf{E}}{N}$ ;  $GP = \lambda P$  => Perron-Frobenius operator

- $\alpha$ models a random surfer with a random jump after approximately 6 clicks (usually $\alpha = 0.85$); PageRank vector => $P$ at $\lambda = 1$ ($\sum_j P_j = 1$).
- CheiRank vector $P^*$: $G^* = \alpha \mathbf{S}^* + (1-\alpha)\frac{\mathbf{E}}{N}$, $G^* P^* = P^*$
  ($\mathbf{S}^*$ with inverted link directions)
  Fogaras (2003) ... Chepelianskii arXiv:1003.5455 (2010) ...

# Real directed networks

Real networks are characterized by:

- **small world property**: average distance between 2 nodes $\sim \log N$
- **scale-free property**: distribution of the number of ingoing or outgoing links $\rho(k) \sim k^{-\nu}$

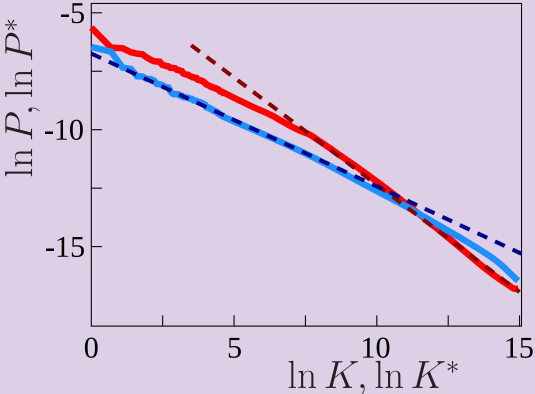
PageRank vector for large WWW:

- $P(K) \sim 1/K^{\beta}$, where $K$ is the ordered rank index
- number of nodes $N_n$ with PageRank $P$ scales as $N_n \sim 1/P^{\nu}$ with numerical values $\nu = 1 + 1/\beta \approx 2.1$ and $\beta \approx 0.9$.
- PageRank $P(K)$ on average is proportional to the number of ingoing links
- CheiRank $P^*(K^*) \sim 1/K^{*\beta}$ on average is proportional to the number of outgoing links ($\nu \approx 2.7; \beta = 1/(\nu - 1) \approx 0.6$)
- WWW at present: $\sim 10^{11}$ web pages

Donato *et al.* EPJB **38**, 239 (2004)

# Wikipedia ranking of human knowledge

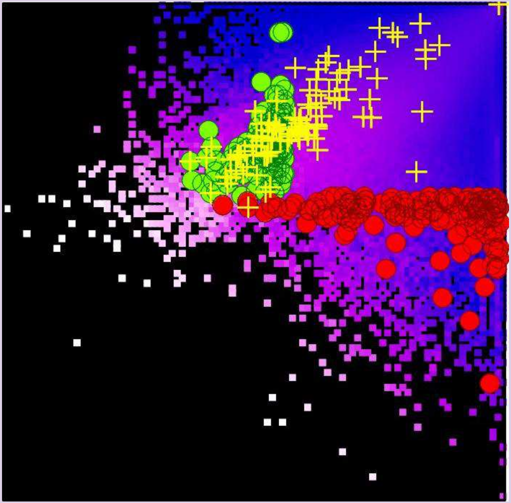
Wikipedia English articles $N = 3282257$ dated Aug 18, 2009




Dependence of probability of PagRank $P$ (red) and CheiRank $P^*$ (blue) on corresponding rank indexes $K, K^*$; lines show slopes $\beta = 1/(\nu - 1)$ with $\beta = 0.92; 0.57$ respectively for $\nu = 2.09; 2.76$.

[Zhirov, Zhirov, DS EPJB **77**, 523 (2010)]

# Two-dimensional ranking of Wikipedia articles



Density distribution in plane of PageRank and CheiRank indexes $(\ln K, \ln K^*)$: 100 top personalities from PageRank (green), CheiRank (red) and Hart book (yellow)

# Wikipedia ranking of universities, personalities

Universities:

PageRank: 1. Harvard, 2. Oxford, 3. Cambridge, 4. Columbia, 5. Yale, 6. MIT, 7. Stanford, 8. Berkeley, 9. Princeton, 10. Cornell.

2DRank: 1. Columbia, 2. U. of Florida, 3. Florida State U., 4. Berkeley, 5. Northwestern U., 6. Brown, 7. U. Southern CA, 8. Carnegie Mellon, 9. MIT, 10. U. Michigan.

CheiRank: 1. Columbia, 2. U. of Florida, 3. Florida State U., 4. Brooklyn College, 5. Amherst College, 6. U. of Western Ontario, 7. U. Sheffield, 8. Berkeley, 9. Northwestern U., 10. Northeastern U.

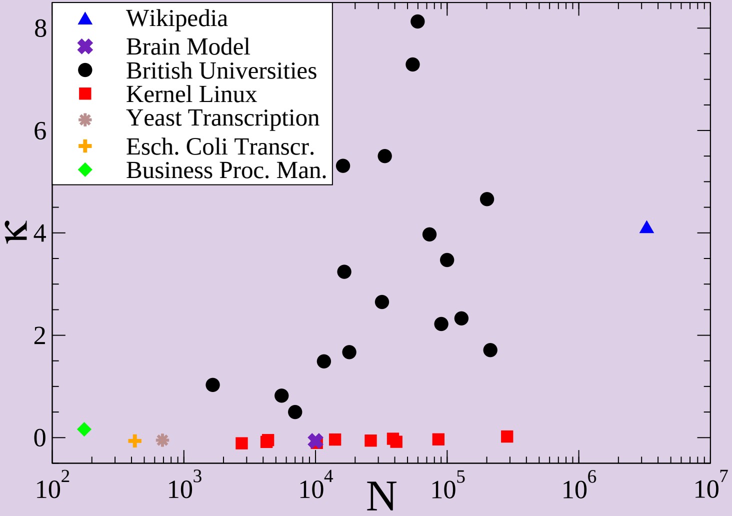
Personalities:

PageRank: 1. Napoleon I of France, 2. George W. Bush, 3. Elizabeth II of the United Kingdom, 4. William Shakespeare, 5. Carl Linnaeus, 6. Adolf Hitler, 7. Aristotle, 8. Bill Clinton, 9. Franklin D. Roosevelt, 10. Ronald Reagan.

2DRank: 1. Michael Jackson, 2. Frank Lloyd Wright, 3. David Bowie, 4. Hillary Rodham Clinton, 5. Charles Darwin, 6. Stephen King, 7. Richard Nixon, 8. Isaac Asimov, 9. Frank Sinatra, 10. Elvis Presley.

CheiRank: 1. Kasey S. Pipes, 2. Roger Calmel, 3. Yury G. Chernavsky, 4. Josh Billings (pitcher), 5. George Lyell, 6. Landon Donovan, 7. Marilyn C. Solvay, 8. Matt Kelley, 9. Johann Georg Hagen, 10. Chikage Oogi.

# Correlator of PageRank and CheiRank



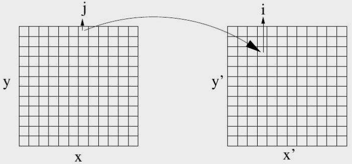

$\kappa = N \sum_i P(K(i)) P^*(K^*(i)) - 1$

# Ulam networks

## Ulam conjecture (method) for discrete approximant of Perron-Frobenius operator of dynamical systems

Discretized phase-space:
Adjacency matrix $\mathbf{A} = P(j \to i)$

$N = N_x \times N_y$ cells.



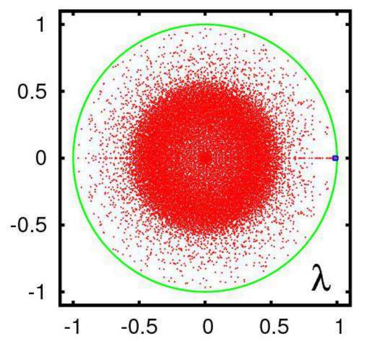

$N_c$: traj. from cell $j$
$N_i$: traj. to call $i$

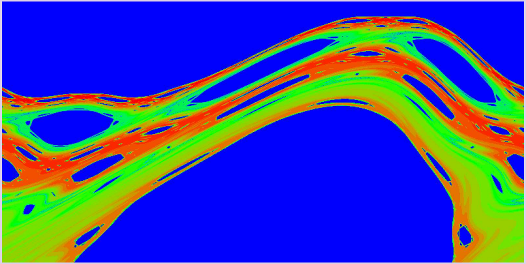
$$\begin{cases} \mathbf{A}_{i,j} = N_i / N_c \\ \sum_i \mathbf{A}_{i,j} = 1 \quad \text{(closed systems)} \end{cases}$$

S.M.Ulam, *A Collection of mathematical problems*, Interscience, **8**, 73 N.Y. (1960)
A rigorous prove for hyperbolic maps:
T.-Y.Li J.Approx. Theory **17**, 177 (1976)
Related works:
Z. Kovacs and T. Tel, Phys. Rev. A 40, 4641 (1989)
M.Blank, G.Keller, and C.Liverani, Nonlinearity **15**, 1905 (2002)
D.Terhesiu and G.Froyland, Nonlinearity **21**, 1953 (2008)

Links to Markov chains: ∞∞∞∞∞∞∞∞∞∞
Contre-example: Hamiltonian systems with invariant curves, e.g. the Chirikov standard map: noise, induced by coarse-graining, destroys the KAM curves and gives homogeneous ergodic eigenvector at $\lambda = 1$

# Ulam method for the Chirikov standard map



Left: spectrum $G\psi = \lambda\psi$, $M \times M/2$ cells; $M = 280$, $N_d = 16609$, exact and Arnoldi method for matrix diagonalization; generalized Ulam method of one trajectory.

Right: modulus of eigenstate of $\lambda_2 = 0.99878...$, $M = 1600$, $N_d = 494964$.

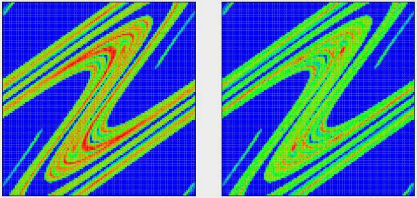
Here $K = K_G$

(Frahm, DS (2010))

# Ulam method for dissipative systems

**Scattering**

$$\begin{cases} \bar{y} = y + K\sin(x + y/2) \\ \bar{x} = x + (y + \bar{y})/2 \pmod{2\pi} \end{cases}$$

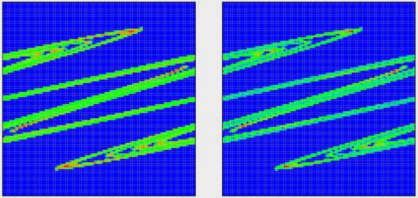


$N = 110 \times 110,\ K = 7, a = 2$

$\lambda_1 = 0.756 \quad \lambda_3 = -0.01 + i0.513$

**Dissipation**

$$\begin{cases} \bar{y} = \eta y + K\sin x \\ \bar{x} = x + \bar{y} \pmod{2\pi} \end{cases}$$



$N = 110 \times 110,\ K = 7, \eta = 0.3$

$\lambda_1 = 1 \quad \lambda_3 = -0.258 + i0.445$

(Ermann, DS (2010))

# Fractal Weyl law

invented for open quantum systems, quantum chaotic scattering:
the number of Gamow eigenstates $N_\gamma$, that have escape rates $\gamma$ in a finite bandwidth $0 \leq \gamma \leq \gamma_b$, scales as

$$N_\gamma \propto \hbar^{-\nu}, \quad \nu = d/2$$

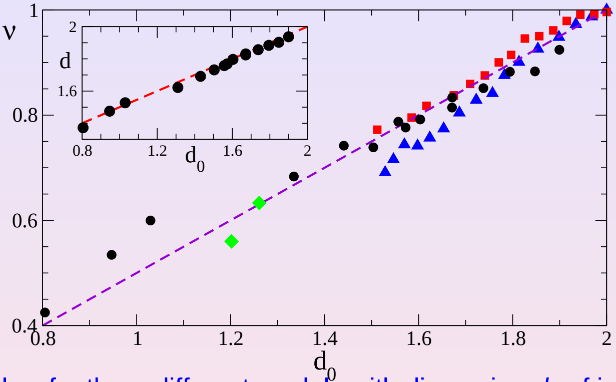
where $d$ is a fractal dimension of a strange invariant set formed by obits non-escaping in the future and in the past

References:
J.Sjostrand, Duke Math. J. **60**, 1 (1990)
M.Zworski, Not. Am. Math. Soc. **46**, 319 (1999)
W.T.Lu, S.Sridhar and M.Zworski, Phys. Rev. Lett. **91**, 154101 (2003)
S.Nonnenmacher and M.Zworski, Commun. Math. Phys. **269**, 311 (2007)

Quantum Chirikov standard map with absorption
F.Borgonovi, I.Guarneri, DLS, Phys. Rev. A **43**, 4517 (1991)
DLS, Phys. Rev. E **77**, 015202(R) (2008)

Perron-Frobenius operators?

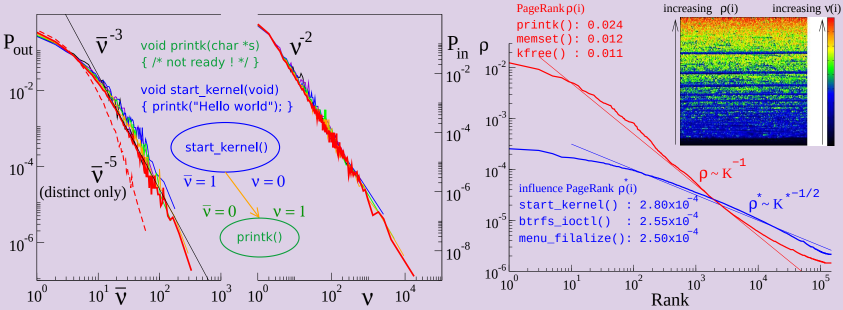
# Fractal Weyl law for Ulam networks




Fractal Weyl law for three different models with dimension $d_0$ of invariant set. The fractal Weyl exponent $\nu$ is shown as a function of fractal dimension $d_0$ of the strange repeller in model 1 and strange attractor in model 2 and Henon map; dashed line shows the theory dependence $\nu = d_0/2$. Inset shows relation between the fractal dimension $d$ of trajectories nonescaping in future and the fractal inv-set dimension $d_0$ for model 1; dashed line is $d = d_0/2 + 1$. (Ermann, DS (2010))

# Linux Kernel Network

Procedure call network for Linux

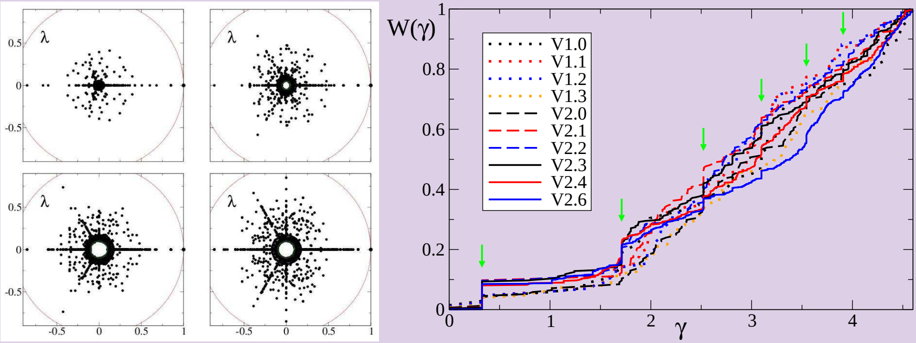



Links distribution (left); PageRank and inverse PageRank (CheiRank) distribution (right) for Linux versions up to 2.6.32 with $N = 285509$ ($\rho \sim 1/j^{\beta}, \beta = 1/(\nu - 1)$).

(Chepelianskii arxiv:1003.5455)

# Fractal Weyl law for Linux Network

Sjöstrand Duke Math J. 60, 1 (1990), Zworski *et al.* PRL 91, 154101 (2003) → quantum chaotic scattering;
Ermann, DS EPJB 75, 299 (2010)→ Perron-Frobenius operators

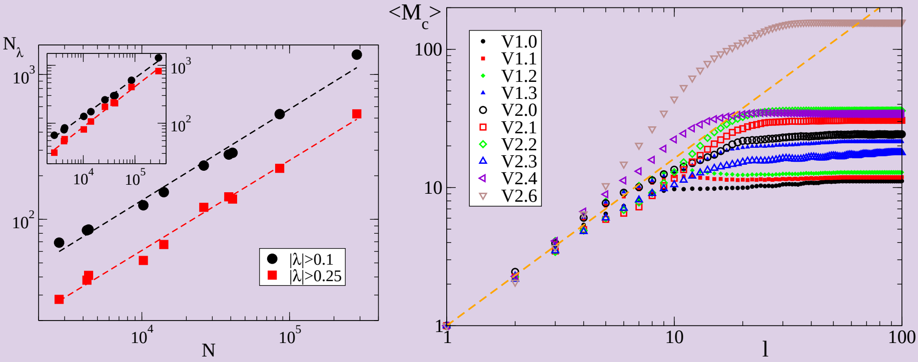



Spectrum of Google matrix (left); integrated density of states for relaxation rate $\gamma = -2 \ln |\lambda|$ (right) for Linux versions, $\alpha = 0.85$.

(Ermann, Chepelianskii, DS (2011))

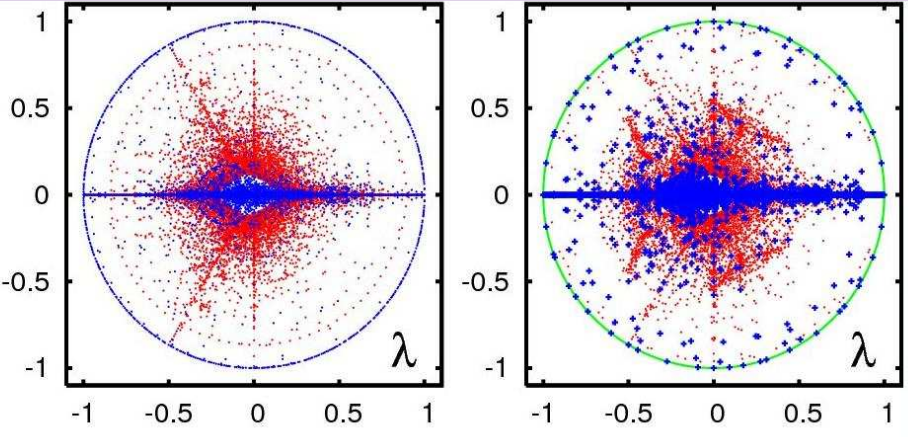
# Fractal Weyl law for Linux Network

Number of states $N_\lambda \sim N^\nu, \quad \nu = d/2 \quad (N \sim 1/\hbar^{d/2})$



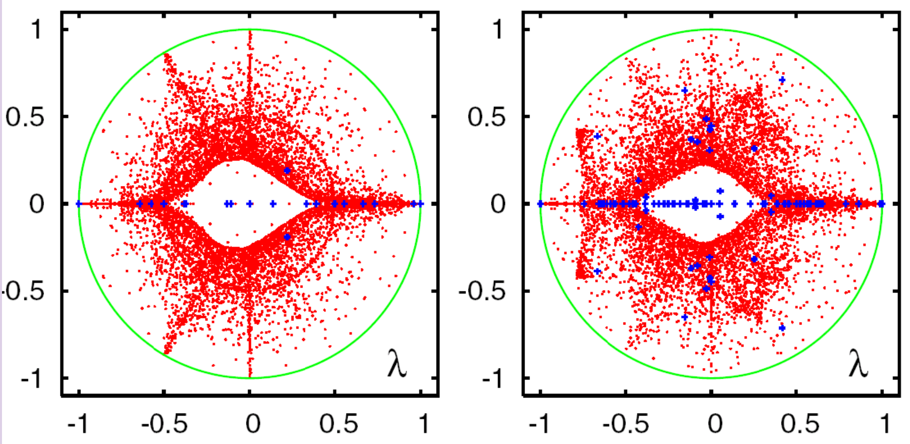

Number of states $N_\lambda$ with $|\lambda| > 0.1; 0.25$ vs. $N$, lines show $N_\lambda \sim N^\nu$ with $\nu \approx 0.65$ (left); average mass $< M_c >$ (number of nodes) as a functon of network distance $l$, line shows the power law for fractal dimension $< M_c > \sim l^d$ with $d \approx 1.3$ (right).

# Spectrum of UK University networks



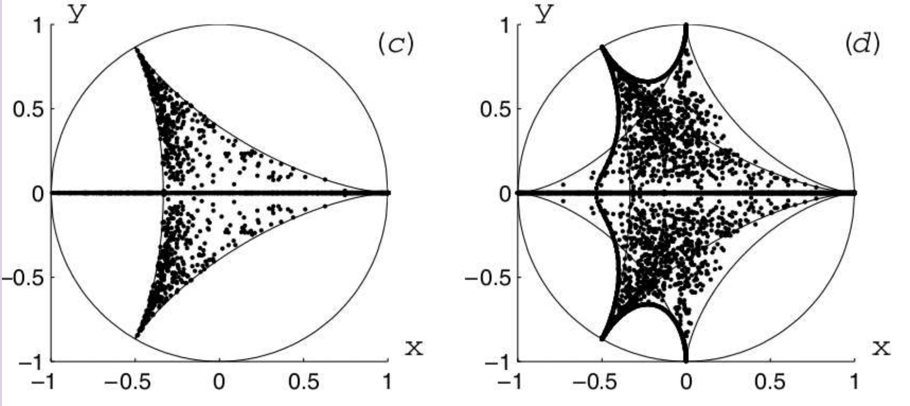

Arnoldi method: Spectrum of Google matrix for Univ. of Cambridge (left) and Oxford (right) in 2006; 20% at $\lambda = 1$ ($N \approx 200000$, $\alpha = 1$). [Frahm, Georgeot, DS arxiv:1105.1062 (2011)]

# Spectrum of UK University networks



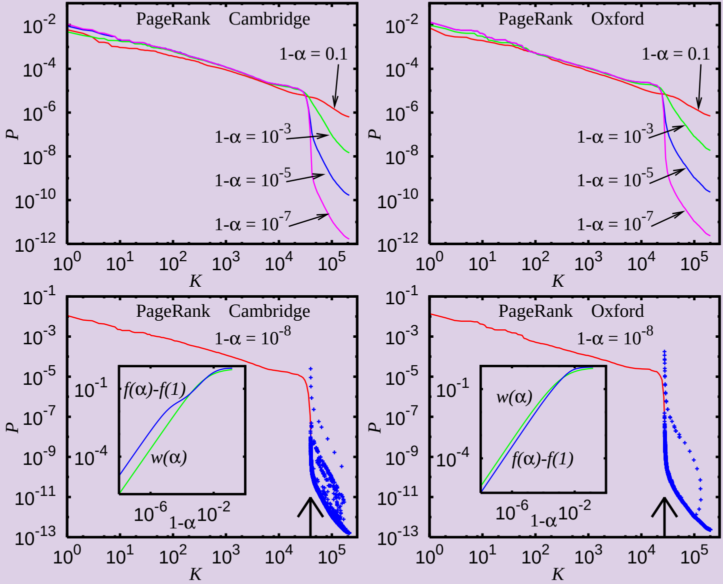

Spectrum of CheiRank Google matrix for Univ. of Cambridge (left) and Oxford (right) in 2006 ($N \approx 200000$, $\alpha = 1$) [Frahm, Georgeot, DS arxiv:1105.1062 (2011)]

# Spectrum of random orthostochastic matrices

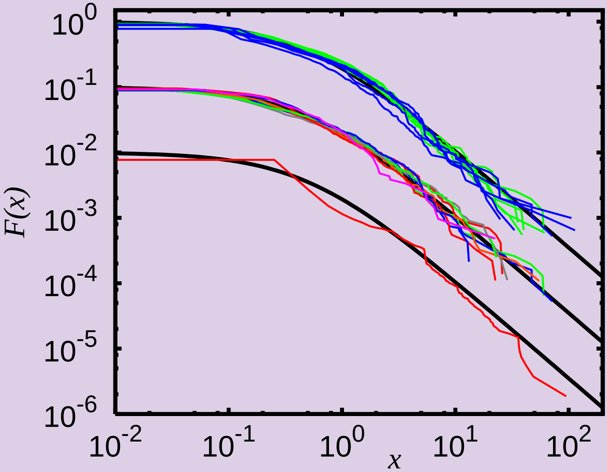



Spectrum $N = 3$ (left), 4 (right) [K.Zyczkowski *et al.* J.Phys. A **36**, 3425 (2003)]

# Universal Emergence of PageRank

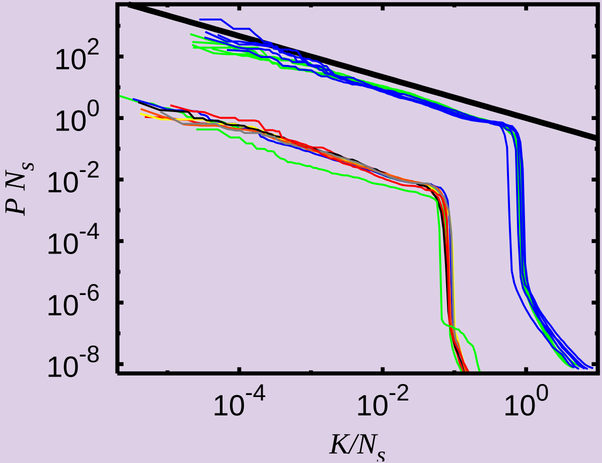


# Invariant subspaces size distribution




$F(x)$ integrated number of invariant subspaces with size larger that $d/d_0$; $x = d/d_0$, $d_0$ is average size of subspaces (top: Cambridge, Oxford 2002-2006; middle: all others; bottom: Wikipedia CheiRank). Curve: $F(x) = 1/(1+2x)^{3/2}$.

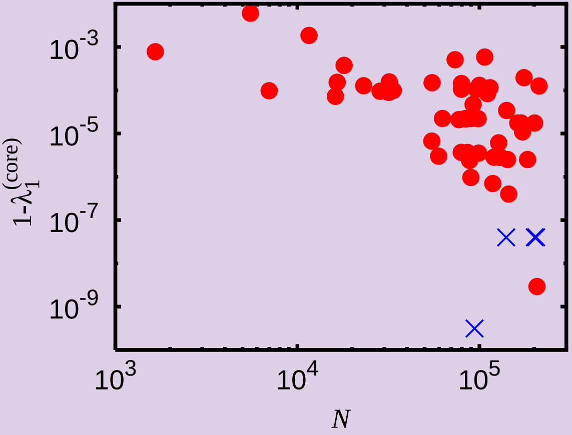
# PageRank at $\alpha \to 1$




Top: Cambridge, Oxford 2002-2006; bottom: all others ($\alpha = 1 - 10^{-8}$).

$P = \frac{1-\alpha}{1-\alpha S}\ \frac{1}{N}\ e$ ; $P = \sum_{\lambda_j=1} c_j\ \psi_j + \sum_{\lambda_j \neq 1} \frac{1-\alpha}{(1-\alpha)+\alpha(1-\lambda_j)}\ c_j\ \psi_j$

# Gap of core space at $\alpha = 1$




Gap vs $N$ for universities Glasgow, Cambridge, Oxford, Edinburgh, UCL, Manchester, Leeds, Bristol and Birkbeck (2002-2006) and Bath,Hull,Keele,Kent,Nottingham, Aberdeen, Sussex, Birmingham, East Anglia, Cardiff, York (2006). Red dots are for gap $> 10^{-9}$ and blue crosses (moved up by $10^9$) are for Cambridge 2002, 2003 and 2005 and Leeds 2006 with gap $< 10^{-16}$; point at $2.91 \cdot 10^{-9}$ is Cambridge 2004.

# Google Matrix Applications

practically to everything ....


more data at
http://www.quantware.ups-tlse.fr/QWLIB/2drankwikipedia/ .../tradecheirank/

# References:

R1. S.Brin and L.Page, *The anatomy of a large-scale hypertextual Web search engine*, Comp. Networks ISDN Systems **30**, 107 (1998)

R2. A.A. Markov, *Rasprostranenie zakona bol'shih chisel na velichiny, zavisyaschie drug ot druga*, Izvestiya Fiziko-matematicheskogo obschestva pri Kazanskom universitete, 2-ya seriya, **15** (1906) 135 (in Russian) [English trans.: *Extension of the limit theorems of probability theory to a sum of variables connected in a chain* reprinted in Appendix B of: R.A. Howard *Dynamic Probabilistic Systems*, volume 1: *Markov models*, Dover Publ. (2007)].

R3. D.Austin, *How Google Finds Your Needle in the Web's Haystack*. AMS Feature Columns, http://www.ams.org/samplings/feature-column/fcarc-pagerank (2008)

R4. Wikipedia articles *PageRank*, *CheiRank*, *Google matrix* (2008-2011)

R5. D.Fogaras, *Where to start browsing the web?*, Lect. Notes Computer Sci. **2877**, 65 (2003)

R6. V.Hrisitidis, H.Hwang and Y.Papakonstantinou, *Authority-based keyword search in databases*, ACM Trans. Database Syst. **33**, 1 (2008)

R7. A.D.Chepelianksii, *Towards physical laws for software architecture* arXiv:1003.5455[cs.SE] (2010)

R8. A.O.Zhirov, O.V.Zhirov and D.L.Shepelyansky, *Two-dimensional ranking of Wikipedia articles*, Eur. Phys. J. B **77**, 523 (2010)

R9. S.M. Ulam, *A Collection of mathematical problems*, Vol. 8 of Interscience tracs in pure and applied mathematics, Interscience, New York, p. 73 (1960).

# References (continued):

R10. K.M.Frahm and D.L.Shepelyansky, *Ulam method for the Chirikov standard map*, Eur. Phys. J. B **76**, 57 (2010)

R11. L.Ermann and D.L.Shepelyansky, *Ulam method and fractal Weyl law for Perron-Frobenius operators*, Eur. Phys. J. B **75**, 299 (2010)

R12. L.Ermann, A.D.Chepelianskii and D.L.Shepelyansky, *Fractal Weyl law for Linux Kernel Architecture*, Eur. Phys. J. B **79**, 115 (2011)

R13. L.Ermann and D.L.Shepelyansky, *Google matrix of the world trade network*, arxiv:1103.5027 (2011)

R14. L.Ermann, A.D.Chepelianskii and D.L.Shepelyansky, *Towards two-dimensional search engines*, arxiv:1106.6215[cs.IR] (2011)

R15. K.M.Frahm, B.Georgeot and D.L.Shepelyansky, *Universal emergence of PageRank*, arxiv:1105.1062[cs.IR] (2011)

**Books, reviews:**

B1. A. M. Langville and C. D. Meyer, *Google's PageRank and beyond: the science of search engine rankings*, Princeton University Press, Princeton (2006)

B2. M. Brin and G. Stuck, *Introduction to dynamical systems*, Cambridge Univ. Press, Cambridge, UK (2002).

B3. E. Ott, *Chaos in dynamical systems*, Cambridge Univ. Press, Cambridge (1993).