# Kolmogorov turbulence, Anderson localization and KAM integrability

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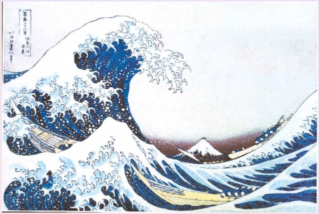
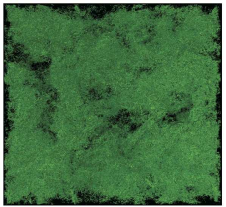
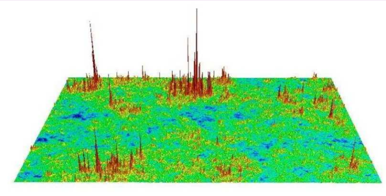
Images: Hokusai, two programs at the Newton Institute, Cambridge 2008







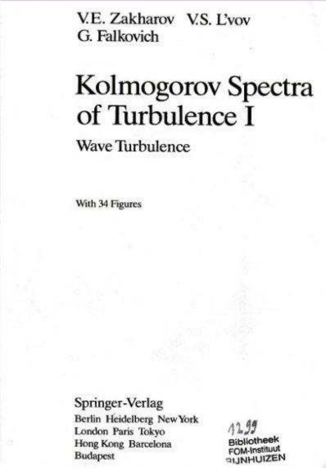
“Through mechanisms still only partially understood, wind transfers energy and momentum to surface water waves.”

A.C.Newell and V.E.Zakharov (PRL 1992)

following arXiv:1203.1130v1 [nlin.CD]

# Kolmogorov and weak wave turbulence

Kolmogorov DAN SSSR **30**, 299; **32**, 19 (1941);
Obukhov Izv. AN SSSR Ser. Geogr. Geofiz., **5(4-5)**, 453 (1941)


V.E. Zakharov V.S. L'vov
G. Falkovich

Kolmogorov Spectra of Turbulence I

Wave Turbulence

With 34 Figures

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Kolmogorov turbulence: energy flow $j$ (current) from large to small spacial scales $E_k \sim j^{2/3} k^{-5/3}$

Concept of weak turbulence:
Zakharov-Filonenko spectrum (1967) $E_k \sim k^{-7/4}$
surface waves on deep water

Random-phase conjecture: "In the theory of weak turbulence nonlinearity of waves is assumed to be small; this enables us, using the hypothesis of the random nature of the phase of individual waves, to obtain the kinetic equation for the mean square of the wave amplitudes"

==> Finite size systems: discrete spectrum of waves
Kartashova (2007), Nazarenko (2011)

Anderson localization (metal-insulator transition), chaos border,
Kolmogorov-Arnold-Moser (KAM) theory, Fermi-Pasta-Ulam (FPU) problem

# Anderson localization: introduction & perspectives

1958 => from the talk of P.W.Anderson at Newton Institute, July 21, 2008
see http://www.newton.ac.uk/programmes/MPA/seminars/072117001.html

"Well, In my country," said alice, still panting a little, "you would generally get to somehere else, if you ran very fast for a long time, as we've been doing". "A slow sort of country!", said the queen. "Now *here,* it takes all the running *you* can do, to stay in the same place."

Perspectives: a)localization in new type of systems; b)effects of interactions
==> nonlinear perturbation of pure point spectrum of Anderson localization

# Chirikov standard map for soliton dynamics

Nonlinear Schrödinger equation => integrable
(Zakharov, Shabat Zh. Eksp. Teor. Fiz. **61**, 118 (1971))

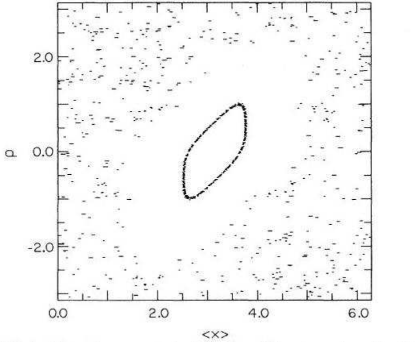



FIG. 1. Two phase-space trajectories with parameters $\beta=25$, $k=0.5$, and $T=2$ (classical $K$ is 2), obtained by numerical in-

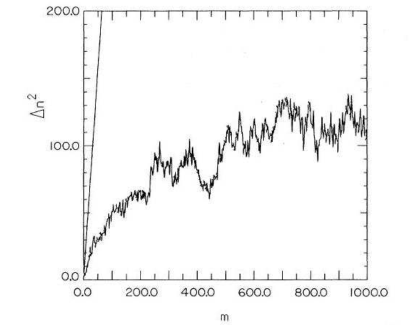



FIG. 4. Plot of the wave packet width in Fourier space $\Delta n^2$ vs number of periods $m$. Here $\beta=10$, $k=2.5$, $T=1$ and classical $K=5$; the initial soliton position and velocity are $x_0=0.2$ and

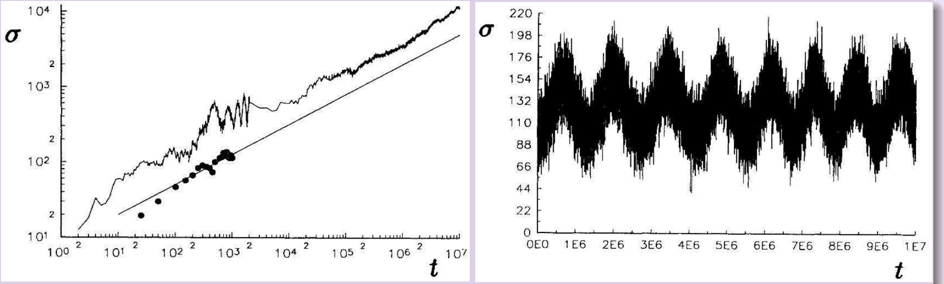
$$i\hbar\frac{\partial}{\partial t}\psi = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \beta|\psi|^2 + k\cos x\ \delta_T(t)\right)\psi$$

$$\bar{p} = p + K\sin x\ ,\quad \bar{x} = x + \bar{p} \qquad (K = kT/m,\ \beta \sim 25 \gg 1)$$

Benvenuto, Casati, Pikovsky, DS (1991)

# Delocalization of quantum chaos by weak nonlinearity

kicked nonlinear rotator (KNR)




$$\psi_n(t+1) = e^{-iT\hat{n}^2/2 - i\beta|\psi_n|^2} e^{-ik\cos\hat{\theta}}\psi_n(t)$$

Left: $k = 5, T = 1, K = 5, \beta = 1$, dots from kicked NSE (previous page); slope $\alpha = 2/5$
Right: $k = 5, T = 1, K = 5, \beta = 0.03$, DS (1993)
( also García-Mata, DS (2009), Lapteva *et al* (2010))

# Nonlinearity and Anderson localization: estimates

$$i\hbar\frac{\partial\psi_n}{\partial t} = E_n\psi_n + \beta|\,\psi_n\,|^2\psi_n + V(\psi_{n+1}+\psi_{n-1})\;;[-W/2 < E_n < W/2]\;\;(DANSE)$$

localization length $\ell \approx 96(V/W)^2$ (1D); $\ln\ell \sim (V/W)^2$ (2D) Amplitudes $C$ in the linear eigenbasis are described by the equation

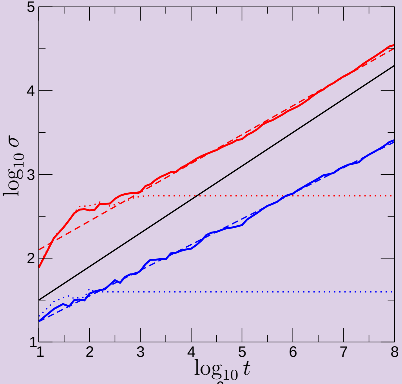
$$i\frac{\partial C_m}{\partial t} = \epsilon_m C_m + \beta\sum_{m_1m_2m_3} U_{mm_1m_2m_3}C_{m_1}C^*_{m_2}C_{m_3}$$

the transition matrix elements are $U_{mm_1m_2m_3} = \sum_n Q^{-1}_{nm}Q_{nm_1}Q^*_{nm_2}Q_{nm_3} \sim 1/\ell^{3d/2}$. There are about $\ell^{3d}$ random terms in the sum with $U \sim \ell^{-3d/2}$ so that we have $idC/dt \sim \beta C^3$. We assume that the probability is distributed over $\Delta n > \ell^d$ states of the lattice basis. Then from the normalization condition we have $C_m \sim 1/(\Delta n)^{1/2}$ and the transition rate to new non-populated states in the basis $\boldsymbol{m}$ is $\Gamma \sim \beta^2|C|^6 \sim \beta^2/(\Delta n)^3$. Due to localization these transitions take place on a size $\ell$ and hence the diffusion rate in the distance $\Delta R \sim (\Delta n)^{1/d}$ of $d-$ dimensional $\boldsymbol{m}-$ space is $d(\Delta R)^2/dt \sim \ell^2\Gamma \sim \beta^2\ell^2/(\Delta n)^3 \sim \beta^2\ell^2/(\Delta R)^{3d}$.
At large time scales $\Delta R \sim R$ and we obtain

$$\Delta n \sim R^d \sim (\beta\ell)^{2d/(3d+2)}t^{d/(3d+2)};\;(\Delta n)^2 \propto t^\alpha;\;\alpha = 2/(3d+2)$$

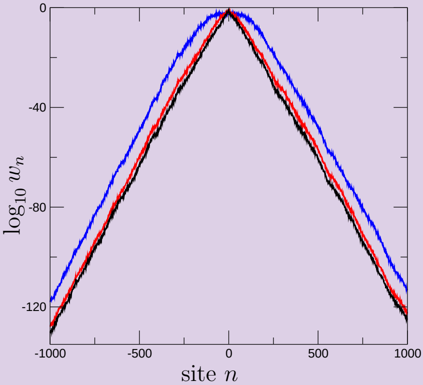
Chaos criterion: $S = \delta\omega/\Delta\omega \sim \beta > \beta_c \sim 1$ here $\delta\omega \sim \beta|\psi_n|^2 \sim \beta/\Delta n$ is nonlinear frequency shift and $\Delta\omega \sim 1/\Delta n$ is spacing between exites eigenmodes
DS (1993); Pikovsky, DS (2008) ($d = 1$); García-Mata, DS (2009) ($d \geq 1$)
Mulansky, Pikovsky (2009) different nonlinearities

# Nonlinearity and Anderson localization (1D)




$W/V = 2, 4$, $\beta = 0, 1$; $\sigma = (\Delta n)^2 \propto t^{\alpha}$;

$\alpha = 2/5$ (theory) 0.34, 0.31 numerics

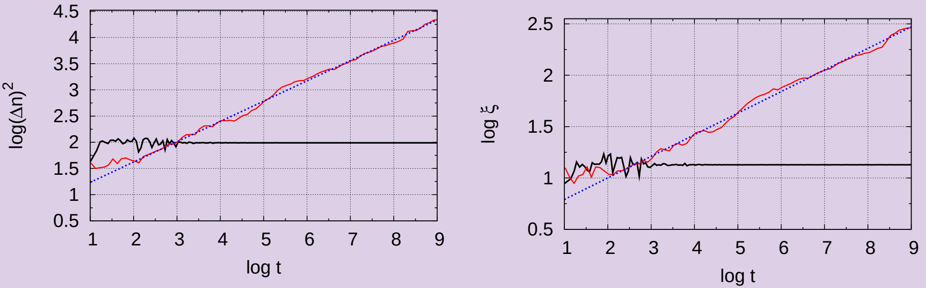



$W/V = 4$, $\beta = 1$, $t = 10^8$, $\beta = 0$

$$i\hbar\frac{\partial\psi_n}{\partial t} = E_n\psi_n + \beta|\psi_n|^2\psi_n + V(\psi_{n+1} + \psi_{n-1})\,;[-W/2 < E_n < W/2]$$

Pikovsky, DS (2008)

# Kicked nonlinear rotator (1D)



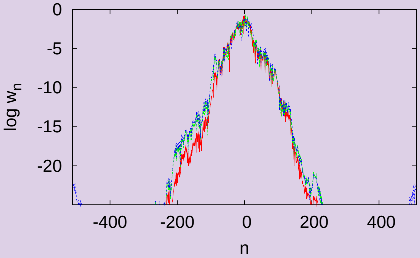

$\alpha_1 = 0.387 \pm 0.003$ (theory 0.4)

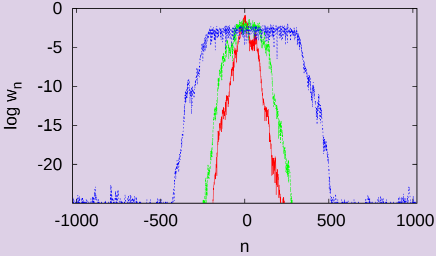
$\nu = 0.210 \pm 0.002$ (theory 0.2); $\xi = (\sum_n |\psi_n|^2)^2 / \sum_n |\psi_n|^4$

$$\psi_n(t+1) = e^{-iT\hat{n}^2/2 - i\beta|\psi_n|^2} e^{-ik\cos\hat{\theta}} \psi_n(t) \quad , (k = 3, T = 2, \beta = 0, 1)$$

DS (1993); García-Mata, DS (2009)

# Kicked nonlinear rotator (1D)



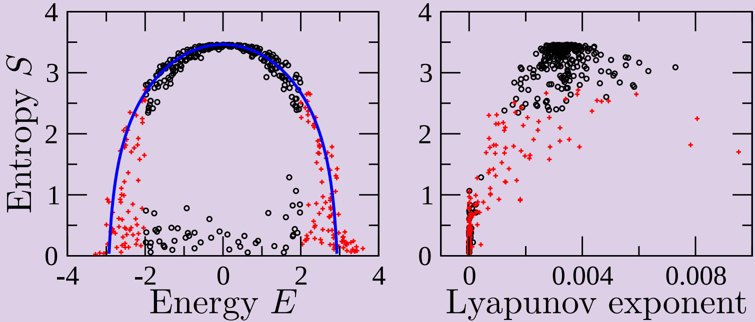




$t = 10^3, 10^6, 10^9; \beta = 0; 1$

García-Mata, DS (2009)

# Dynamical thermalization in DANSE (1D)

starting from Fermi-Pasta-Ulam problem (1955):
regular lattice, delocalized linear modes $\rightarrow$ disorder localized modes




$N = 32$, $W = 4$, $\beta = 1$, $t = 10^7 + 10^6$, initial state: linear eigenmode

Gibbs distribution with temperature $T$ for localized linear modes, $\rho_m = |C_m|^2$:
entropy $S = -\sum_m \rho_m \ln \rho_m$, $\rho_m = Z^{-1} \exp(-\epsilon_m/T)$, $Z = \sum_m \exp(-\epsilon_m/T)$,
$E = T^2 \partial \ln Z/\partial T$, $S = E/T + \ln Z$ . $\langle \ln Z \rangle \approx \ln N + \ln \sinh(\Delta/T) - \ln(\Delta/T)$, $\Delta \approx 3$

Mulansky, Ahnert, Pikovsky, DS (2009)

# Possible experimental tests & applications

- BEC in disordered potential (Aspect, Inguscio)
- kicked rotator with BEC (Phillips, Hoogerland)
- nonlinear wave propagation in disordered media (Segev, Silberberg)
- lasing in random media (Cao)
- energy propagation in complex molecular chains
  (proteins, Fermi-Pasta-Ulam problem)
- NONLINEAR SPIN-GLASS ? Links to Frenkel-Kontorova model?

- OTHER GROUPS:
  S.Aubry *et al.* PRL **100**, 084103 (2008)
  A.Dhar *et al.* PRL **100**, 134301 (2008)
  S.Fishman *et al.* J. Stat. Phys. **131**, 843 (2008) ...
  S.Flach *et al.* PRL **102**, 024101 (2009) ...
  T.Kottos and B.Shapiro, PRE **83**, 062103 (2011)
  W.-M.Wang *et al.* arXiv:0805.4632[math.DS] (2008)
  see also the participant list of the NLSE Workshop
  at the Lewiner Institute, Technion, June 2008 (http://physics.technion.ac.il/ nlse/)

# Experiments on 2D disordered photonic lattices

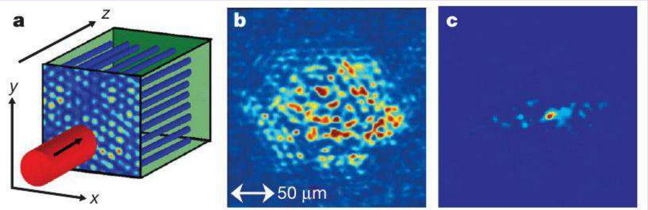
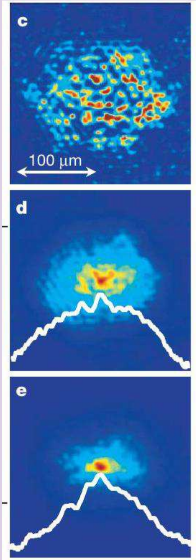



**Figure 1 | Transverse localization scheme. a,** A probe beam entering a disordered lattice, which is periodic in the two transverse dimensions ($x$ and $y$) but invariant in the propagation direction ($z$). In the experiment described here, we use a triangular (hexagonal) photonic lattice with a periodicity of 11.2 μm and a refractive-index contrast of $\sim 5.3 \times 10^{-4}$. The lattice is induced optically, by transforming the interference pattern among three plane waves into a local change in the refractive index, inside a photorefractive SBN:60 ($Sr_{0.6}Ba_{0.4}Nb_2O_6$) crystal. The input probe beam is of 514 nm wavelength and 10.5 μm full-width at half-maximum (FWHM), and it is always launched at the same location, while the disorder is varied in each realization of the multiple experiments. **b,** Experimentally observed diffraction pattern after $L = 10$ mm propagation in the fully periodic hexagonal lattice. **c,** Typical experimentally observed intensity distribution after $L = 10$ mm propagation in one particular realization of the 15% disorder in the lattice.




Segev *et al.* Nature **446**, 52 (2007) (right: disorder growing from top to bottom)

# Experiments on 1D disordered photonic lattices

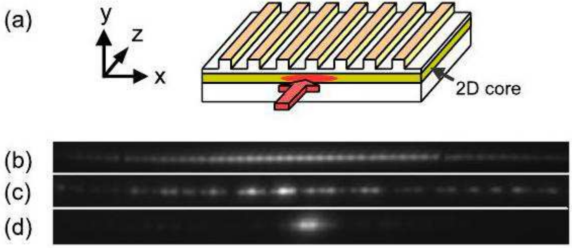



FIG. 1 (color online). (a) Schematic view of the sample used in the experiments. The red arrow indicates the input beam. (b)–(d) Images of output light distribution, when the input beam covers a few lattice sites: (b) in a periodic lattice, (c) in a disordered lattice, when the input beam is coupled to a location which exhibits a high degree of expansion, and (d) in the same disordered lattice when the beam is coupled to a location in which localization is clearly observed.

Silberberg *et al.* PRL **100**, 013906 (2008)

# BEC Experiments in 1D incommensurate lattice

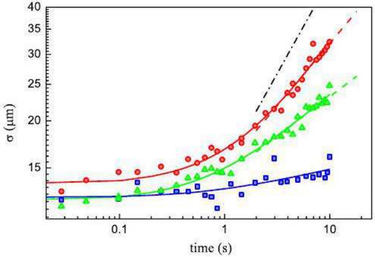



FIG. 1 (color online). Time evolution of the width $\sigma$ for different initial interaction energies: $E_{\rm int} = 0$ (squares), $E_{\rm int} = 1.8J$ (triangles), and $E_{\rm int} = 2.3J$ (circles). The continuous lines are the fit with Eq. (1). The dashed lines show the fitted asymptotic behavior, while the dash-dotted line shows the expected behavior for normal diffusion. The lattice parameters are $J/h = 180$ Hz, $\Delta/J = 4.9$.

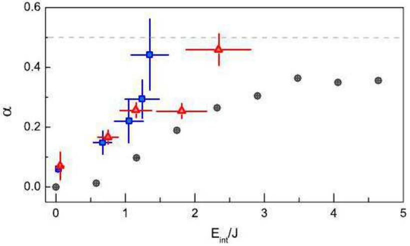



FIG. 2 (color online). Diffusion exponent $\alpha$ vs the initial interaction energy $E_{\rm int}$ in the experiment (triangles and squares) and simulations (circles). The experimental data are for $\Delta/J = 5.3(4)$ and two different values of the tunneling: $J/h = 180$ Hz (triangles) and $J/h = 300$ Hz (squares). The vertical bars are the fitting error of Eq. (1) to the data, while the horizontal bars indicate the statistical error.

$^{39}K$ BEC, $5 \times 10^4$ atoms, optical lattice $V(x) = V_1 \cos^2(k_1 x) + V_2 \cos^2(k_2 x)$,
$\lambda_i = 2\pi/k_i = 1064nm, 859nm$
André-Aubry model (or related Harper model) => hopping $J$ in 1st lattice and potential/"disorder" $\Delta \sim V_2$ of 2d lattice, metalic phase at $J/\Delta > 2$
G.Modugno *et al.* PRL **106**, 230403 (2011)

# Nonlinearity and localization: open problems

- exponent $\alpha \approx 1/3 < 2/5$
  indications on its small decrease at very large times in certain models but not in DANSE
  Flach *et al.*, Mulansky *et al.* (2009-2011)
- different (higher/lower) nonlinearity exponents $|\psi|^\mu$ still give anomalous spreading
  Mulansky, Pikovsky (2009)
- main part of measure is non-chaotic at small local $\beta$
  (zero Lyapunov exponent)
  Pikovsky, Fishman (2011)
- => Arnold diffusion scenario:
  Arnold diffusion in systems with many degrees of freedom
  Chirikov (1979); Chirikov, Vecheslavov (1997)
  spreading over Arnold web of narrow chaotic separatrix layers
  Mulansky *et al.* (2011)

# Fast Arnold diffusion and spreading in nonlinear disorderd lattices

- for the Chirikov standard map the width of separatrix layer drops exponentially with perturbation: $w_s \sim \lambda^3 \exp(-\pi\lambda/2)$, $\lambda \sim 1/\sqrt{K}$
  Arnold diffusion along web of chaotic separatrix layers
  Diffusion rate $\ln D \propto \ln w_s^2 \propto -\lambda$
  (Chirikov Phys. Rep. **52**, 263 (1979))
- Chirikov-Vecheslavov (1997): $w_s \sim K^{2.5}$, $D \sim K^{3/2} w_s^2 \sim K^{\nu_D}$, $\nu_D \approx 6.6$!
  multi-particle standard map: $H = |p|^2/2 - K \sum_{i=1}^{N+1} \cos(x_{i+1} - x_i)\delta_1(t)$
  many degrees of freedom: $2 \leq L = N + 1 \leq 15$
- independent computations of weak chaos measure over all layers: $\mu \propto w_s \propto K^{1.6}$ $(10^{-4} \leq K \leq 0.01)$
  strong chaos measure $\mu_{sch} \sim K$ with Lyapunov exponent $\lambda \sim \sqrt{K}$
  (Mulansky *et al.* (2011))
- slow anomalous spreading:
  $H = \sum_{k=1}^{L} [p_k^2/2 + \eta_k q_k^4/4 + \gamma(q_{k+1} - q_k)^6/6]$; $(0.5 \leq \eta_k \leq 1.5, \gamma \sim 1)$
  Results: $\sigma =< (\Delta k)^2 >\sim t^\alpha$, $\alpha \approx 0.55$;
  $\alpha = 8/(9 + 2\nu_D)$; $\nu_D = 6.6 => \alpha = 0.36$
  (Mulansky *et al.* (2011))

# Fast Arnold diffusion and spreading in nonlinear disorderd lattices

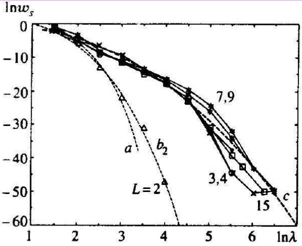
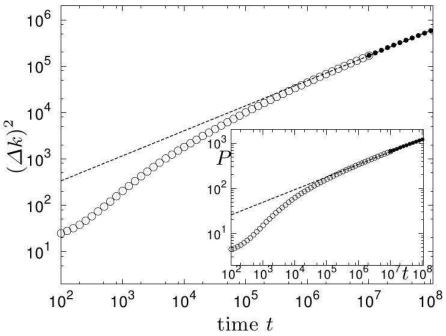
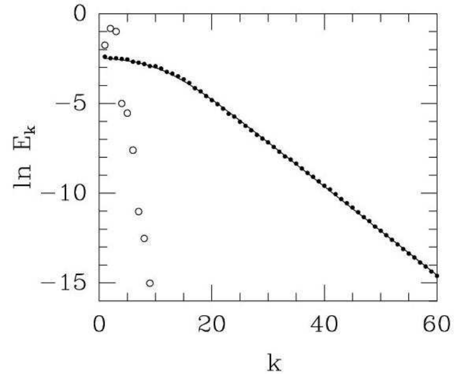



FIG. 1. Summary of numerical data for the model (2.1). Broken solid lines connecting various symbols show computed values of $w_s$ as a function of the adiabaticity parameter $\lambda \equiv 1/\sqrt{K}$ and the resonance dimension $L = N$ indicated by the numbers. Dotted lines represent the theory: (a) small-λ limit, one fitting parameter, Eq. (3.5); $(b_2)$ large-λ limit for $L = 2$, two fitting parameters, Eq. (4.9); (c) intermediate asymptotics, three fitting parameters, Eq. (5.8).




- Left: Chirikov-Vecheslavov (1997); Right: Mulansky *et al.* (2011)
- DANSE: $H_0$ is linear that makes situation more complicated, also more linear modes are coupled by nonlinearity: $\ell \sim 1 \rightarrow t_{spread} \sim 10^8$

# Low energy chaos in the FPU problem




$\alpha$-FPU model:
$H = \sum_{n=0}^{N} [{p_n}^2 + (x_{n+1} - x_n)^2]/2$
$+\alpha \sum_{n=0}^{N} (x_{n+1} - x_n)^3/3$

$q_k = \pi k/(N+1)$;
$\omega_k = 2\sin(q_k/2) \approx q_k - q_k^3/24$
initial energy $E_0$

Chaos border $\alpha\sqrt{E_0} > k^2/N^{3/2}$

Toda lattice

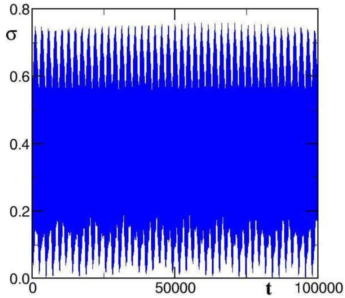
resonant Hamiltonian for long waves

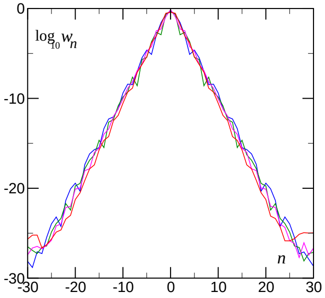
$$\bar{H} = \sum_k \omega_k I_k + \frac{\alpha}{2\sqrt{N+1}} \sum_{k_1,k_2,k_3} (\omega_{k_1}\omega_{k_2}\omega_{k_3} I_{k_1} I_{k_2} I_{k_3})^{1/2} \cos(\theta_{k_3} - \theta_{k_2} - \theta_{k_1}) \delta_{k_3,k_1+k_2}$$

chaos at small $k$ waves but no ergodicity and chaos at high $k$ waves,
no energy flow from small to large spacial scales
(DS (1997))

# Kicked nonlinear Schrödinger equation (KINSE)

momentum states $n$ => spacial coordinate for Anderson localization (Fishman, Grempel, Prange (1982)



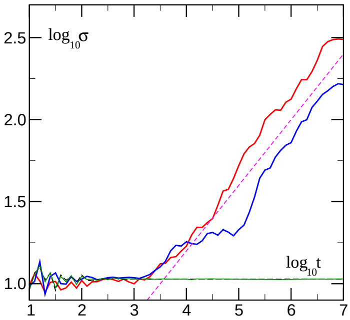



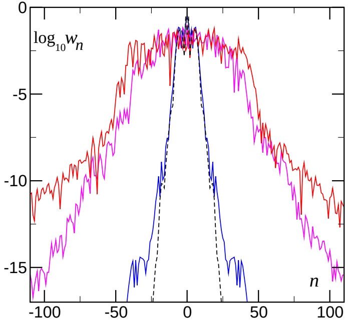

Left: second moment $\sigma =< n^2 >$. Right: probability distribution over linear modes $n$ at $t = 10^3 - 10^6$; $\beta = 1, k = 0.3, T = 2, K = kT = 0.6$.

$$i\hbar\partial\psi/\partial\tau = -\partial^2\psi/2\partial^2 x + \beta|\psi|^2\psi - k\cos x\ \psi\sum_{m=-\infty}^{\infty}\delta(\tau - mT)$$

KAM+Anderson: a small wind does not generate turbulence, thus, no energy flow from large to small spacial scales

# Energy flow in KINSE



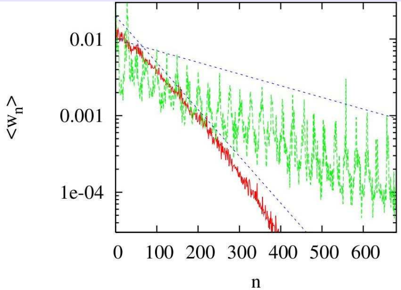



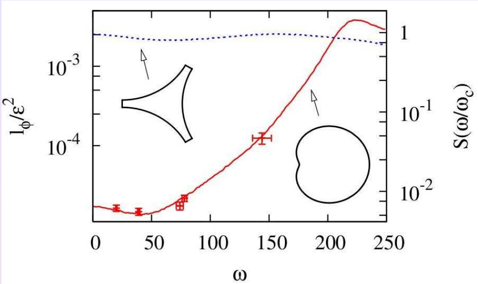

Left: second moment $\sigma = < n^2 >$, $k = 3$, $T = 2$, $K = kT = 6$, $\beta = 1$ (red), $0.5$ (blue), $0.05$ (green), 0 (black dashed); slope $\alpha = 0.4$

Right: probability distribution over linear modes $n$ at $t = 10^7$; $\beta = 1$ (red), $0.5$ (magenta), $0.05$ (blue), 0 (black dashed)

numerical fits give $\alpha = 0.346 \pm 0.014 (\beta = 0.5)$, $0.438 \pm 0.007 (\beta = 1)$
Thus, the behaviour is similar to the models of DANSE and KNR
Energy flow to high modes above a certain chaos border: $\beta > \beta_c \sim 1/10$
(a similarity with Anderson transition)

# Photonic localization in Sinai billiard



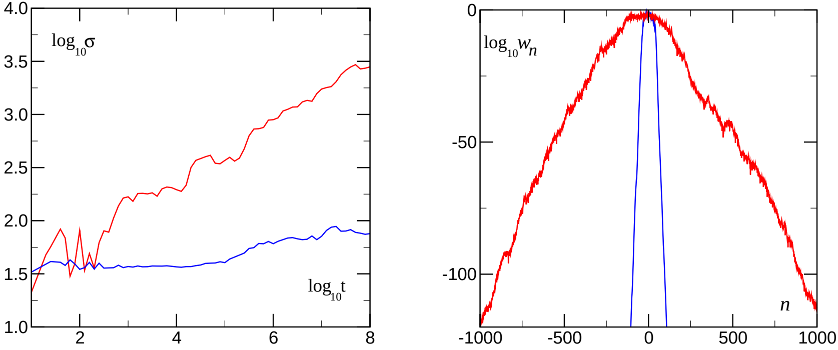




Left: probability localization over billiard eigenstates $n$ at two microwave amplitude driving $\epsilon$ (theory is shown by dashed line)

Right: dependence of rescaled localization length $\ell_\phi/\epsilon^2$ on microwave frequency $\omega$; classical spectral denssity of perturbation $S(\omega/\omega_c) \propto x_\omega^2$ is shown by curves for two chaotic billiards

waves in a chaotic billiard with *ac*-driving $V(t) = \epsilon x \sin(\omega t)$
localization length in energy: $\ell_\phi = \pi\epsilon^2 R^2 S(\omega/\omega_c)/\hbar\omega_c\Delta$
(measured in a number of photons)
(Prosen, DS (2005))

# Kolmogorov turbulence in Sinai billiard




Left: DANSE plus static Stark field $\delta E_n = f|n|$: $W = 4$, $V = 1$, $\beta = 1$, $f = 0$ (red), $f = 0.5$ (blue)

Right: probability distribution at $t = 10^8$

NSE in Sinai billiard

$i\partial\psi/\partial\tau = -\Delta\psi/2 + V(x,y)\psi + \beta|\psi|^2\psi + F\sin(\omega\tau)x\psi$

Conjecture: no energy flow to high modes

(DS (2012))

# Discussion

The conditions for emergence of Kolmogorov turbulence, and related weak wave turbulence, in finite size systems are analyzed by analytical methods and numerical simulations of simple models. The analogy between Kolmogorov energy flow from large to small spacial scales and conductivity in disordered solid state systems is proposed. It is argued that the Anderson localization can stop such an energy flow. The effects of nonlinear wave interactions on such a localization are analyzed. The results obtained for finite size system models show the existence of an effective chaos border between the Kolmogorov-Arnold-Moser (KAM) integrability at weak nonlinearity, when energy does not flow to small scales, and developed chaos regime emerging above this border with the Kolmogorov turbulent energy flow from large to small scales.

Energy flow from large to small spacial scales only above a certain chaos border

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