Kolmogorov turbulence facing Anderson localization and KAM integrability

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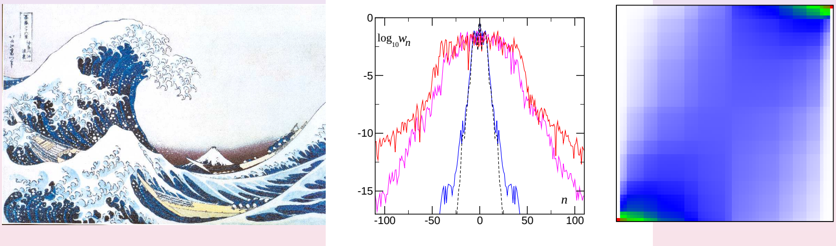
Images: Hokusai, kicked NSE rotator, quantum Gibbs theory



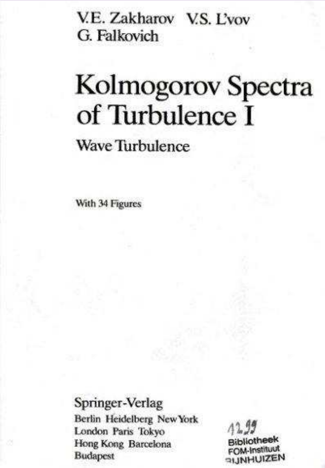
"Through mechanisms still only partially understood, wind transfers energy and momentum to surface water waves."

A.C.Newell and V.E.Zakharov (PRL 1992)

following arXiv:1203.1130v1 [nlin.CD]; L.Ermann, DS (in preparation)

Kolmogorov and weak wave turbulence

Kolmogorov DAN SSSR **30**, 299; **32**, 19 (1941);
Obukhov Izv. AN SSSR Ser. Geogr. Geofiz., **5(4-5)**, 453 (1941)



V.E. Zakharov V.S. L'vov
G. Falkovich

Kolmogorov Spectra of Turbulence I

Wave Turbulence

With 34 Figures

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Kolmogorov turbulence: energy flow (current) from large to small spacial scales $E_k \sim k^{-5/3}$

Concept of weak turbulence:
Zakharov-Filonenko spectrum (1967) $E_k \sim k^{-7/4}$
surface waves on deep water

Random-phase conjecture: “In the theory of weak turbulence nonlinearity of waves is assumed to be small; this enables us, using the hypothesis of the random nature of the phase of individual waves, to obtain the kinetic equation for the mean square of the wave amplitudes”

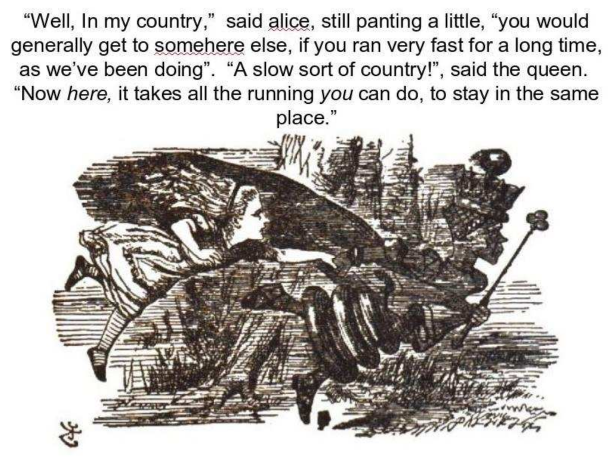
V.L'vov lectures NGU 1976

==> Finite size systems: discrete spectrum of waves
Nazarenko (2011)

Anderson localization (metal-insulator transition), chaos border,
Kolmogorov-Arnold-Moser (KAM) theory, Fermi-Pasta-Ulam (FPU) problem

Anderson localization: introduction & perspectives

1958 => from the talk of P.W.Anderson at Newton Institute, July 21, 2008
see http://www.newton.ac.uk/programmes/MPA/seminars/072117001.html



“Well, In my country,” said alice, still panting a little, “you would generally get to somehere else, if you ran very fast for a long time, as we’ve been doing”. “A slow sort of country!”, said the queen. “Now *here,* it takes all the running *you* can do, to stay in the same place.”

Perspectives: a)localization in new type of systems; b)effects of interactions
==> nonlinear perturbation of pure point spectrum of Anderson localization

Chirikov standard map for soliton dynamics

Nonlinear Schrödinger equation => integrable
(Zakharov, Shabat Zh. Eksp. Teor. Fiz. **61**, 118 (1971))

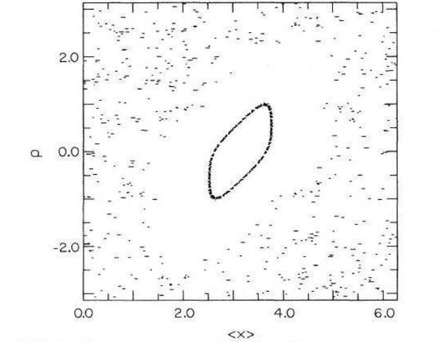


FIG. 1. Two phase-space trajectories with parameters $\beta=25$, $k=0.5$, and $T=2$ (classical K is 2), obtained by numerical in-

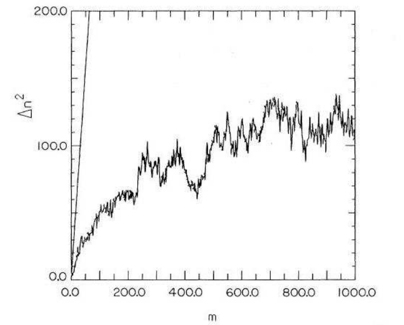


FIG. 4. Plot of the wave packet width in Fourier space Δn^2 vs number of periods m. Here $\beta=10$, $k=2.5$, $T=1$ and classical $K=5$; the initial soliton position and velocity are $x_0=0.2$ and

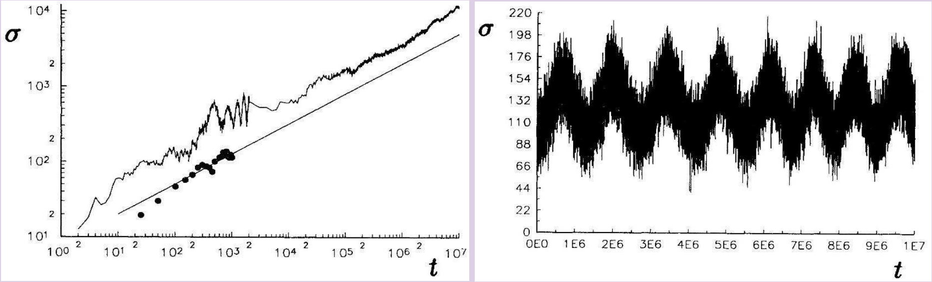
$$i\hbar\frac{\partial}{\partial t}\psi=\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}-\beta|\psi|^2+k\cos x\ \delta_T(t)\right)\psi$$

$$\bar{p}=p+K\sin x\ ,\quad \bar{x}=x+\bar{p}\qquad (K=kT/m,\ \beta\sim 25\gg 1)$$

Benvenuto, Casati, Pikovsky, DS (1991)

Delocalization of quantum chaos by weak nonlinearity

kicked nonlinear rotator (KNR)



$$\psi_n(t+1) = e^{-iT\hat{n}^2/2-i\beta|\psi_n|^2} e^{-ik\cos\hat{\theta}} \psi_n(t)$$

Left: $k = 5, T = 1, K = 5, \beta = 1$, dots from kicked NSE (previous page); slope $\alpha = 2/5$
Right: $k = 5, T = 1, K = 5, \beta = 0.03$, DS (1993)
(also García-Mata, DS (2009), Lapteva *et al* (2010))

Nonlinearity and Anderson localization: estimates

$$i\hbar\frac{\partial\psi_n}{\partial t} = E_n\psi_n + \beta|\,\psi_n\,|^2\psi_n + V(\psi_{n+1}+\psi_{n-1})\ ; [-W/2 < E_n < W/2]\ \ (DANSE)$$

localization length $\ell \approx 96(V/W)^2$ (1D); $\ln \ell \sim (V/W)^2$ (2D) Amplitudes C in the linear eigenbasis are described by the equation

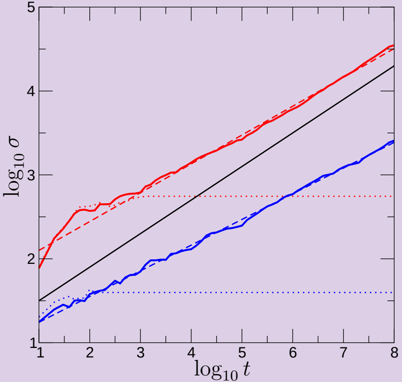
$$i\frac{\partial C_m}{\partial t} = \epsilon_m C_m + \beta\sum_{m_1m_2m_3} U_{mm_1m_2m_3} C_{m_1} C^*_{m_2} C_{m_3}$$

The transition matrix elements are $U_{mm_1m_2m_3} = \sum_n Q^{-1}_{nm} Q_{nm_1} Q^*_{nm_2} Q_{nm_3} \sim 1/\ell^{3d/2}$.
We have $idC/dt \sim \beta C^3$. The transition rate is $\Gamma \sim \beta^2|C|^6 \sim \beta^2/(\Delta n)^3$. Diffusive spreading: $\Delta R \sim (\Delta n)^{1/d}$ of $d-$ dimensional $\boldsymbol{m}-$ space is
$d(\Delta R)^2/dt \sim \ell^2\Gamma \sim \beta^2\ell^2/(\Delta n)^3 \sim \beta^2\ell^2/(\Delta R)^{3d}$.
At large time scales $\Delta R \sim R$ and we obtain

$$\Delta n \sim R^d \sim (\beta\ell)^{2d/(3d+2)} t^{d/(3d+2)};\ (\Delta n)^2 \propto t^\alpha;\ \alpha = 2/(3d+2)$$

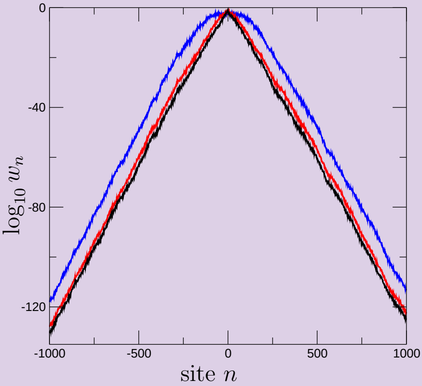
Chaos criterion: $S = \delta\omega/\Delta\omega \sim \beta > \beta_c \sim 1$ here $\delta\omega \sim \beta|\psi_n|^2 \sim \beta/\Delta n$ is nonlinear frequency shift and $\Delta\omega \sim 1/\Delta n$ is spacing between exites eigenmodes
DS (1993); Pikovsky, DS (2008) ($d = 1$); García-Mata, DS (2009) ($d \geq 1$)
Mulansky, Pikovsky (2009) different nonlinearities

Nonlinearity and Anderson localization (1D)



$W/V = 2, 4$, $\beta = 0, 1$; $\sigma = (\Delta n)^2 \propto t^{\alpha}$;

$\alpha = 2/5$ (theory) 0.34, 0.31 numerics



$W/V = 4$, $\beta = 1$, $t = 10^8$, $\beta = 0$

$$i\hbar \frac{\partial \psi_n}{\partial t} = E_n \psi_n + \beta |\psi_n|^2 \psi_n + V(\psi_{n+1} + \psi_{n-1})\ ; [-W/2 < E_n < W/2]$$

Pikovsky, DS (2008)

Possible experimental tests & applications

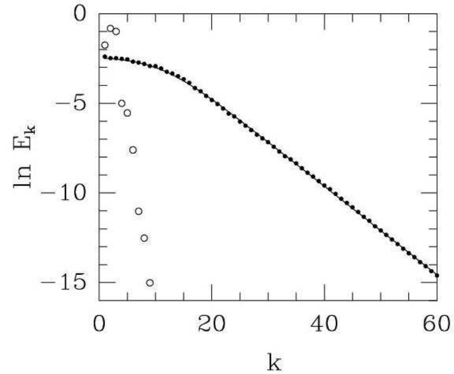
- BEC in disordered potential (Aspect, Inguscio)
- kicked rotator with BEC (Phillips, Hoogerland)
- nonlinear wave propagation in disordered media (Segev, Silberberg)
- lasing in random media (Cao)
- energy propagation in complex molecular chains (proteins, Fermi-Pasta-Ulam problem)
- Quantum Gibbs distribution instead of classical Boltzmann one ?

- OTHER GROUPS:
  S.Aubry *et al.* PRL **100**, 084103 (2008)
  A.Dhar *et al.* PRL **100**, 134301 (2008)
  S.Fishman *et al.* J. Stat. Phys. **131**, 843 (2008) ...
  S.Flach *et al.* PRL **102**, 024101 (2009) ...
  T.Kottos and B.Shapiro, PRE **83**, 062103 (2011)
  W.-M.Wang *et al.* arXiv:0805.4632[math.DS] (2008)
  see also the participant list of the NLSE Workshop
  at the Lewiner Institute, Technion, June 2008 (http://physics.technion.ac.il/ nlse/)

Nonlinearity and localization: open problems

- exponent $\alpha \approx 1/3 < 2/5$
  indications on its small decrease at very large times in certain models but not in DANSE
  Flach *et al.*, Mulansky *et al.* (2009-2011)
- different (higher/lower) nonlinearity exponents $|\psi|^\mu$ still give anomalous spreading
  Mulansky, Pikovsky (2009)
- main part of measure is non-chaotic at small local β
  (zero Lyapunov exponent)
  Pikovsky, Fishman (2011)
- => Arnold diffusion scenario:
  Arnold diffusion in systems with many degrees of freedom
  Chirikov (1979); Chirikov, Vecheslavov (1997)
  spreading over Arnold web of narrow chaotic separatrix layers
  Mulansky *et al.* (2011)

Low energy chaos in the FPU problem



α-FPU model:
$H = \sum_{n=0}^{N} [{p_n}^2 + (x_{n+1} - x_n)^2]/2$
$+\alpha \sum_{n=0}^{N} (x_{n+1} - x_n)^3/3$

$q_k = \pi k/(N+1)$;
$\omega_k = 2\sin(q_k/2) \approx q_k - q_k^3/24$
initial energy E_0

Chaos border $\alpha\sqrt{E_0} > k^2/N^{3/2}$

Toda lattice

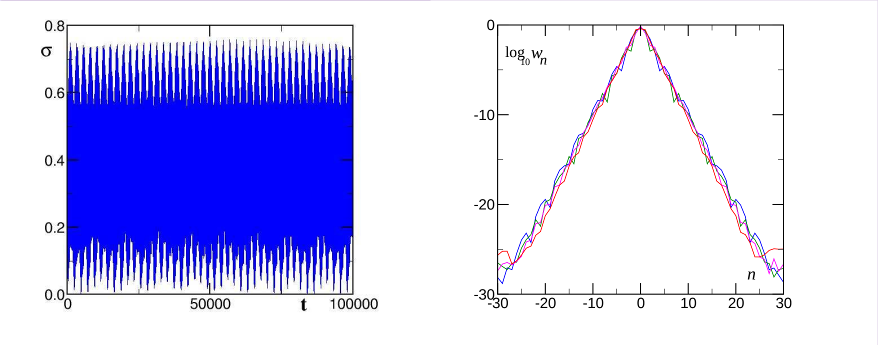
resonant Hamiltonian for long waves

$$\bar{H} = \sum_k \omega_k I_k + \frac{\alpha}{2\sqrt{N+1}} \sum_{k_1,k_2,k_3} (\omega_{k_1}\omega_{k_2}\omega_{k_3} I_{k_1} I_{k_2} I_{k_3})^{1/2} \cos(\theta_{k_3} - \theta_{k_2} - \theta_{k_1}) \delta_{k_3,k_1+k_2}$$

chaos at small k waves but no ergodicity and chaos at high k waves,
no energy flow from small to large spacial scales
(DS (1997))

Kicked nonlinear Schrödinger equation (KINSE)

momentum states *n* => spacial coordinate for Anderson localization (Fishman, Grempel, Prange (1982)

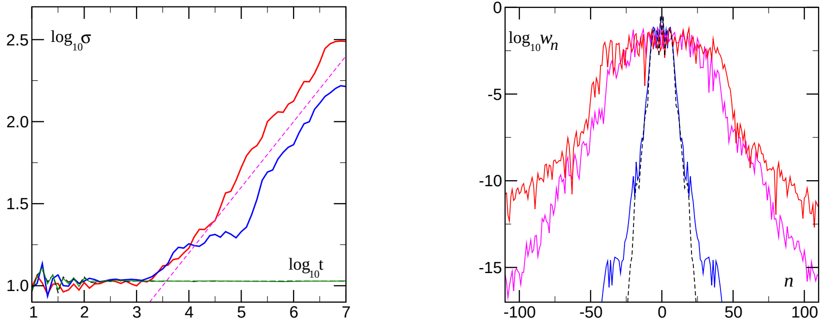


Left: second moment $\sigma = < n^2 >$. Right: probability distribution over linear modes *n* at $t = 10^3 - 10^6$; $\beta = 1, k = 0.3, T = 2, K = kT = 0.6$.

$$i\hbar\partial\psi/\partial\tau = -\partial^2\psi/2\partial^2 x + \beta|\psi|^2\psi - k\cos x\ \psi\sum_{m=-\infty}^{\infty}\delta(\tau - mT)$$

KAM+Anderson: a small wind does not generate turbulence, thus, no energy flow from large to small spacial scales

Energy flow in KINSE

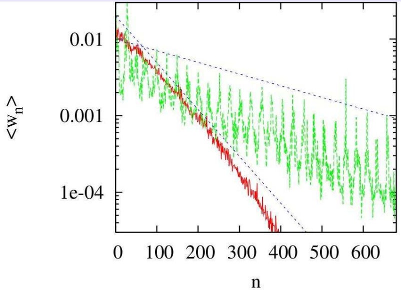


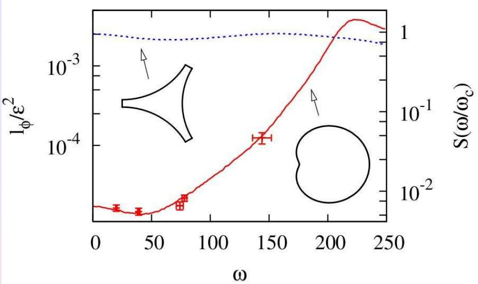
Left: second moment $\sigma = < n^2 >$, $k = 3$, $T = 2$, $K = kT = 6$, $\beta = 1$ (red), 0.5 (blue), 0.05 (green), 0 (black dashed); slope $\alpha = 0.4$

Right: probability distribution over linear modes n at $t = 10^7$; $\beta = 1$ (red), 0.5 (magenta), 0.05 (blue), 0 (black dashed)

numerical fits give $\alpha = 0.346 \pm 0.014 (\beta = 0.5), 0.438 \pm 0.007 (\beta = 1)$
Thus, the behaviour is similar to the models of DANSE and KNR
Energy flow to high modes above a certain chaos border: $\beta > \beta_c \sim 1/10$
(a similarity with Anderson transition)

Photonic localization in Sinai billiard



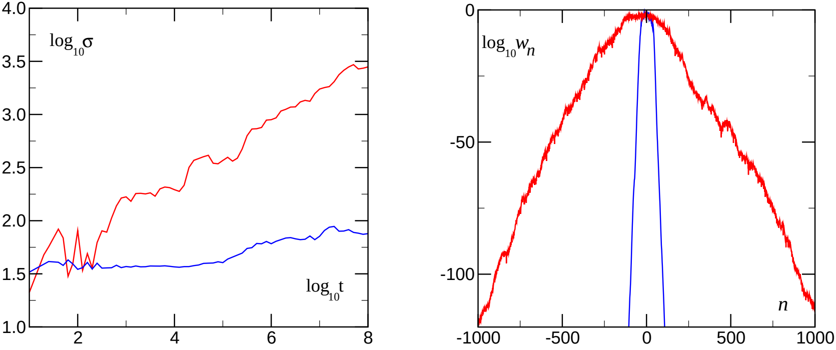


Left: probability localization over billiard eigenstates n at two microwave amplitude driving ϵ (theory is shown by dashed line)

Right: dependence of rescaled localization length ℓ_ϕ/ϵ^2 on microwave frequency ω; classical spectral denssity of perturbation $S(\omega/\omega_c) \propto x_\omega^2$ is shown by curves for two chaotic billiards

waves in a chaotic billiard with *ac*-driving $V(t) = \epsilon x \sin(\omega t)$
localization length in energy: $\ell_\phi = \pi\epsilon^2 R^2 S(\omega/\omega_c)/\hbar\omega_c\Delta$
(measured in a number of photons)
(Prosen, DS (2005))

Kolmogorov turbulence in Sinai billiard



Left: DANSE plus static Stark field $\delta E_n = f|n|$: $W = 4$, $V = 1$, $\beta = 1$, $f = 0$ (red), $f = 0.5$ (blue)

Right: probability distribution at $t = 10^8$

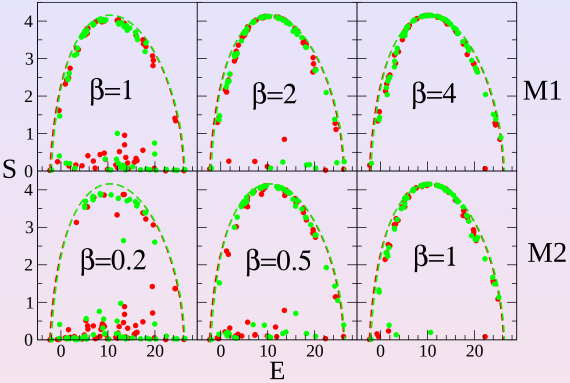
NSE in Sinai billiard

$$i\partial\psi/\partial\tau = -\Delta\psi/2 + V(x,y)\psi + \beta|\psi|^2\psi + F\sin(\omega\tau)x\psi$$

Conjecture: no energy flow to high modes

(DS (2012))

Quantum Gibbs in nonlinear classical lattices



2d DANSE + NSE term; $W = 2, f = 1, N = 8 \times 8, t = 10^6$

$i\partial\psi_{n_x n_y}/\partial t = E_{n_x n_y}\psi_{n_x n_y} + (\psi_{n_x+1 n_y} + \psi_{n_x-1 n_y} + \psi_{n_x n_y+1} + \psi_{n_x n_y-1}) + \beta| \psi_{n_x n_y} |^2\psi_{n_x n_y}$

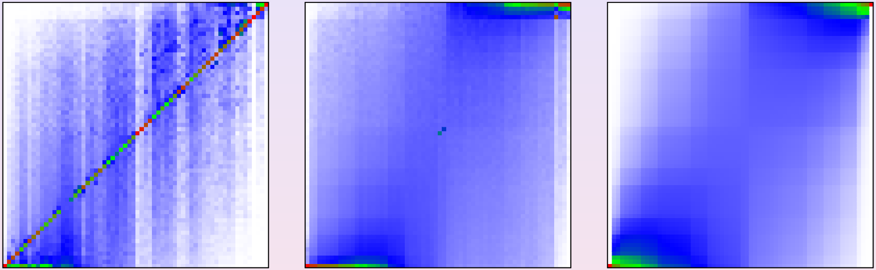
M1: $E_{n_x n_y} = \delta E_{n_x n_y} + f(n_x^2 + n_y^2), -W/2 \le \delta E_{n_x n_y} \le W/2$

M2: $\beta \to \beta(n_x^2 + n_y^2)$ (3 disorder realisations are shown)

Quantum Gibbs anzats: $\rho_m = Z^{-1}\exp(-\epsilon_m/T)$; $Z = \sum_m \exp(-\epsilon_m/T)$

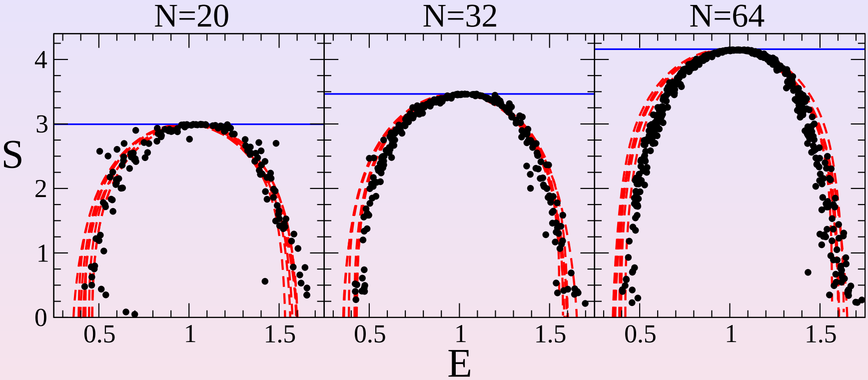
$S = -\sum_m \rho_m \ln \rho_m$; $E = T^2\partial \ln Z/\partial T$; $\partial S/\partial E = 1/T$

Quantum Gibbs in nonlinear classical lattices



Quantum Gibbs probability (color): $W = 2, f = 1$, $N = 8 \times 8$, $t = 10^6$;
$\beta = 1$ (left), 4 (center), theory (right)

Quantum Gibbs in Klein-Gordon lattice



1d Klein-Gordon lattice: $W = 2$, $\beta = 1$, $t = 10^8$
$H = \sum_l [(p_l^2 + \tilde{\epsilon}_l u_l^2)/2 + \beta u_l^2/4 + (u_{l+1} - u_l)^2/(2W)]$

Disorder: $1/2 \leq \tilde{\epsilon}_l \leq 3/2$ (7 realisations are shown)

Discussion

The conditions for emergence of Kolmogorov turbulence, and related weak wave turbulence, in finite size systems are analyzed by analytical methods and numerical simulations of simple models. The analogy between Kolmogorov energy flow from large to small spacial scales and conductivity in disordered solid state systems is proposed. It is argued that the Anderson localization can stop such an energy flow. The effects of nonlinear wave interactions on such a localization are analyzed. The results obtained for finite size system models show the existence of an effective chaos border between the Kolmogorov-Arnold-Moser (KAM) integrability at weak nonlinearity, when energy does not flow to small scales, and developed chaos regime emerging above this border with the Kolmogorov turbulent energy flow from large to small scales.

Energy flow from large to small spacial scales only above a certain chaos border

Emergence of quantum Gibbs distribution from dynamical thermalization in nonlinear classical lattices

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