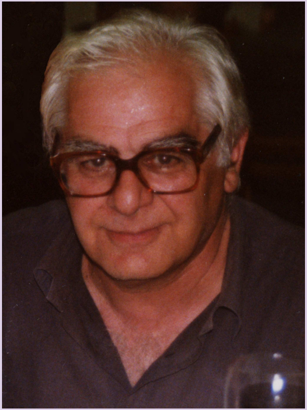
Spectral properties of Google matrix

Dima Shepelyansky
www.quantware.ups-tlse.fr/dima

- (1973 - 1984) On the history of Bohigas-Giannoni-Schmit (BGS) conjecture
- (1906) Markov vs Wigner (1955)
- (1990) Fractal Weyl law and quantum chaotic scattering
- (1991) World Wide Web
- (1998) Brin and Page: Google matrix, search engines
- (2001) Wikipedia, Social networks: (2004) Facebook, (2006) Twitter ...



Towards applications and further developments of quantum chaos ...

On the history of BGS conjecture

Zaslavskii, Filonenko (Krasnoyarsk 1973): $P(s) \sim s^{c/h}$

Statistical properties of the energy spectrum of "gliding" electrons with mixed classical trajectories

G. M. Zaslavskiĭ and N. N. Filonenko

Physics Institute, Siberian Division, USSR Academy of Sciences
(Submitted January 15, 1973)
Zh. Eksp. Teor. Fiz. **65**, 643–656 (August 1973)

An investigation is made of the statistical properties of the distribution of distances between energy levels in the quasiclassical approximation, for a finite system with mixed classical trajectories in phase space. The model is that of electrons "drifting" in a magnetic field along a periodically corrugated surface which is convex at all points in the region of motion of the electrons. Quantization conditions in the quasiclassical approximation are obtained. Estimates are presented for the probability of a given spacing ΔE between the levels. A Gaussian distribution is obtained for large values of ΔE; for small values of ΔE the probability is mainly a power function of ΔE. The exponent depends on the mixing properties of the trajectories.

McDonald, Kaufman (Berkeley Phys. Rev. Lett. 1979)

VOLUME 42 30 APRIL 1979 NUMBER 18

Spectrum and Eigenfunctions for a Hamiltonian with Stochastic Trajectories

Steven W. McDonald and Allan N. Kaufman
Physics Department and Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720
(Received 20 February 1979)

Quantum stochasticity (the nature of wave functions and eigenvalues when the short-wave-limit Hamiltonian has stochastic trajectories) is studied for the two-dimensional Helmsholtz equation with "stadium" boundary. The eigenvalue separations have a Wigner

Casati, Valz-Gris, Guarneri (Milan Lett. Nuovo Cimento 1980) non-conclusive

On the Connection between Quantization of Nonintegrable Systems and Statistical Theory of Spectra (*).

G. CASATI and F. VALZ-GRIS

Istituto di Fisica dell'Università - Via Celoria 16, 20133 Milano, Italia

I. GUARNIERI

Istituto di Matematica dell'Università - Pavia, Italia

On the history of BGS conjecture (1984)

Bohigas-Giannoni-Schmit (Orsay 1984): RMT distribution

PHYSICAL REVIEW LETTERS

VOLUME 52 — 2 JANUARY 1984 — NUMBER 1

Characterization of Chaotic Quantum Spectra and Universality of Level Fluctuation Laws

O. Bohigas, M. J. Giannoni, and C. Schmit

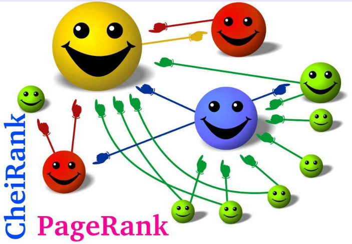
Division de Physique Théorique, Institut de Physique Nucléaire, F-91406 Orsay Cedex, France

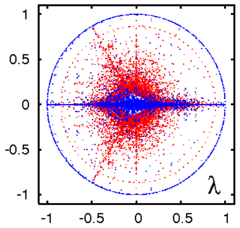
(Received 2 August 1983)

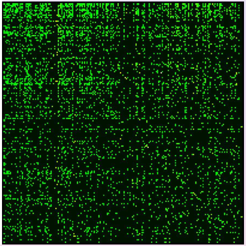
It is found that the level fluctuations of the quantum Sinai's billiard are consistent with the predictions of the Gaussian orthogonal ensemble of random matrices. This reinforces the belief that level fluctuation laws are universal.

(1906) Markov vs Wigner (1955)

Collaboration: L.Ermann, K.Frahm, B.Georgeot, O.Zhirov + A.Chepelianskii
V.Kandiah, Y.-H.Eom support → EC FET Open grant NADINE







1945: Nuclear physics → Wigner (1955)→ Random Matrix Theory
1991: WWW, small world social networks → Markov (1906) → Google matrix

Despite the importance of large-scale search engines on the web, very little academic research has been done on them.

S.Brin and L.Page, Comp. Networks ISDN Systems **30**, 107 (1998)

Fractal Weyl law

invented for open quantum systems, quantum chaotic scattering:
the number of Gamow eigenstates N_γ, that have escape rates γ in a finite bandwidth $0 \leq \gamma \leq \gamma_b$, scales as

$$N_\gamma \propto \hbar^{-\nu} \propto N^\nu, \quad \nu = d/2$$

where d is a fractal dimension of a strange invariant set formed by obits non-escaping in the future and in the past (N is matrix size)

References:
J.Sjostrand, Duke Math. J. **60**, 1 (1990)
M.Zworski, Not. Am. Math. Soc. **46**, 319 (1999)
W.T.Lu, S.Sridhar and M.Zworski, Phys. Rev. Lett. **91**, 154101 (2003)
S.Nonnenmacher and M.Zworski, Commun. Math. Phys. **269**, 311 (2007)

Resonances in quantum chaotic scattering:
three disks, quantum maps with absorption

Perron-Frobenius operators, Ulam method for dynamical maps, Ulam networks, dynamical maps, strange attractors

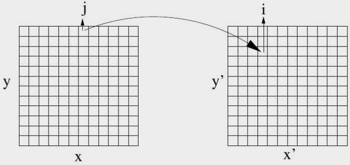
Linux kernel network $d = 1.3$, $N \leq 285509$;
Phys. Rev. up to 2009 $d \approx 1$, $N = 460422$

Ulam networks

Ulam conjecture (method) for discrete approximant of Perron-Frobenius operator of dynamical systems

Discretized phase-space:
Adjacency matrix $\mathbf{A} = P(j \to i)$

$N = N_x \times N_y$ cells.



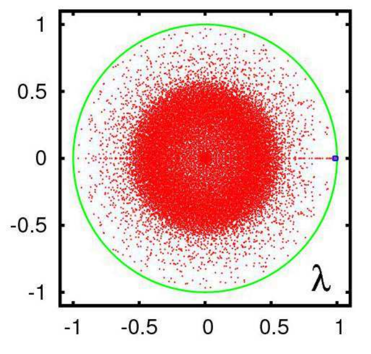
N_c: traj. from cell j
N_i: traj. to call i

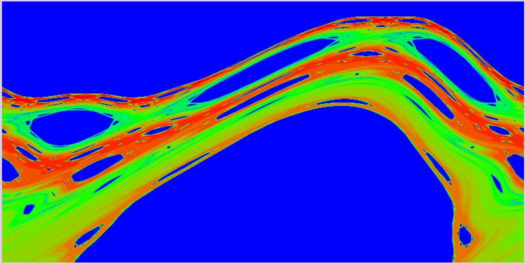
$$\begin{cases} \mathbf{A}_{i,j} = N_i/N_c \\ \sum_i \mathbf{A}_{i,j} = 1 \quad \text{(closed systems)} \end{cases}$$

S.M.Ulam, *A Collection of mathematical problems*, Interscience, **8**, 73 N.Y. (1960)
A rigorous prove for hyperbolic maps:
T.-Y.Li J.Approx. Theory **17**, 177 (1976)
Related works:
Z. Kovacs and T. Tel, Phys. Rev. A 40, 4641 (1989)
M.Blank, G.Keller, and C.Liverani, Nonlinearity **15**, 1905 (2002)
D.Terhesiu and G.Froyland, Nonlinearity **21**, 1953 (2008)

Links to Markov chains: ∞∞∞∞∞∞∞∞∞∞
Contre-example: Hamiltonian systems with invariant curves, e.g. the Chirikov standard map: noise, induced by coarse-graining, destroys the KAM curves and gives homogeneous ergodic eigenvector at $\lambda = 1$

Ulam method for the Chirikov standard map



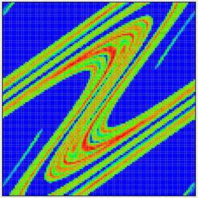


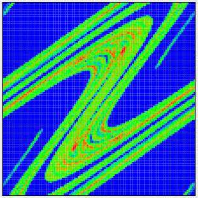
Left: spectrum $G\psi = \lambda\psi$, $M \times M/2$ cells; $M = 280$, $N_d = 16609$, exact and Arnoldi method for matrix diagonalization; generalized Ulam method of one trajectory.
Right: modulus of eigenstate of $\lambda_2 = 0.99878...$, $M = 1600$, $N_d = 494964$.
Here $K = K_G$
(Frahm, DS (2010))

Ulam method for dissipative systems

Scattering

$$\begin{cases} \bar{y} = y + K\sin(x + y/2) \\ \bar{x} = x + (y + \bar{y})/2 \pmod{2\pi} \end{cases}$$



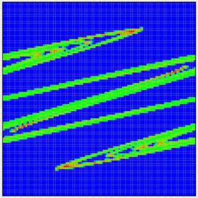


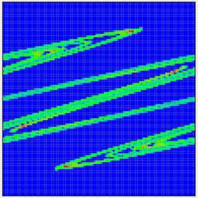
$N = 110 \times 110,\ K = 7, a = 2$

$\lambda_1 = 0.756 \quad \lambda_3 = -0.01 + i0.513$

Dissipation

$$\begin{cases} \bar{y} = \eta y + K\sin x \\ \bar{x} = x + \bar{y} \pmod{2\pi} \end{cases}$$



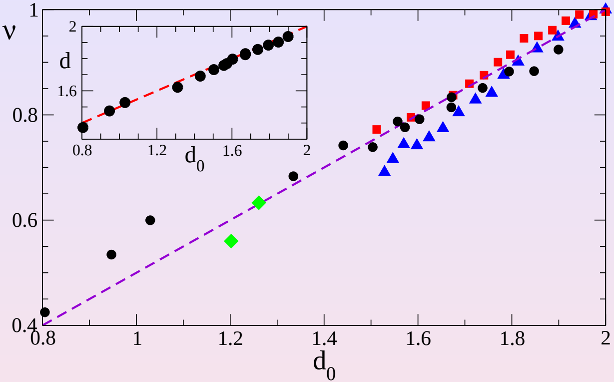


$N = 110 \times 110,\ K = 7, \eta = 0.3$

$\lambda_1 = 1 \quad \lambda_3 = -0.258 + i0.445$

(Ermann, DS (2010))

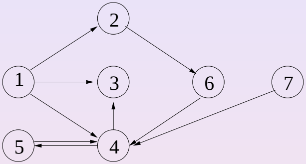
Fractal Weyl law for Ulam networks



Fractal Weyl law for three different models with dimension d_0 of invariant set. The fractal Weyl exponent ν is shown as a function of fractal dimension d_0 of the strange repeller in model 1 and strange attractor in model 2 and Henon map; dashed line shows the theory dependence $\nu = d_0/2$. Inset shows relation between the fractal dimension d of trajectories nonescaping in future and the fractal inv-set dimension d_0 for model 1; dashed line is $d = d_0/2 + 1$. (Ermann, DS (2010))

How Google works

Markov chains (1906) and Directed networks



Weighted adjacency matrix

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For a directed network with N nodes the adjacency matrix $\mathbf{A}$ is defined as $A_{ij} = 1$ if there is a link from node j to node i and $A_{ij} = 0$ otherwise. The weighted adjacency matrix is

$$S_{ij} = A_{ij} / \sum_k A_{kj}$$

In addition the elements of columns with only zeros elements are replaced by $1/N$.

How Google works

Google Matrix and Computation of PageRank

$\mathbf{P} = \mathbf{SP} \Rightarrow \mathbf{P}$= stationary vector of $\mathbf{S}$; can be computed by iteration of $\mathbf{S}$.

To remove convergence problems:

- Replace columns of 0 (dangling nodes) by $\frac{1}{N}$:

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & \frac{1}{7} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{7} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{7} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{7} & 0 & 1 & 1 & 1 \\ 0 & 0 & \frac{1}{7} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{7} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{7} & 0 & 0 & 0 & 0 \end{pmatrix}; \mathbf{S}^* = \begin{pmatrix} \frac{1}{7} & 1 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{1}{7} \\ \frac{1}{7} & 0 & 0 & 0 & 0 & 1 & \frac{1}{7} \\ \frac{1}{7} & 0 & 0 & 0 & 0 & 0 & \frac{1}{7} \\ \frac{1}{7} & 0 & \frac{1}{2} & 0 & 1 & 0 & \frac{1}{7} \\ \frac{1}{7} & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{7} \\ \frac{1}{7} & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{7} \\ \frac{1}{7} & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{7} \end{pmatrix}.$$

- To remove degeneracies of $\lambda = 1$, replace $\mathbf{S}$ by **Google matrix**

  $\mathbf{G} = \alpha\mathbf{S} + (1-\alpha)\frac{\mathbf{E}}{N}$; $GP = \lambda P$ => Perron-Frobenius operator

- α models a random surfer with a random jump after approximately 6 clicks (usually $\alpha = 0.85$); PageRank vector => P at $\lambda = 1$ ($\sum_j P_j = 1$).

- CheiRank vector P^*: $G^* = \alpha\mathbf{S}^* + (1-\alpha)\frac{\mathbf{E}}{N}$, $G^*P^* = P^*$
  ($\mathbf{S}^*$ with inverted link directions)
  Fogaras (2003) ... Chepelianskii arXiv:1003.5455 (2010) ...

Real directed networks

Real networks are characterized by:

- **small world property**: average distance between 2 nodes $\sim \log N$
- **scale-free property**: distribution of the number of ingoing or outgoing links $\rho(k) \sim k^{-\nu}$

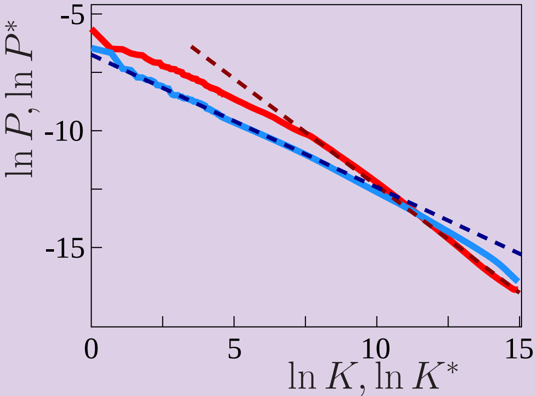
PageRank vector for large WWW:

- $P(K) \sim 1/K^{\beta}$, where K is the ordered rank index
- number of nodes N_n with PageRank P scales as $N_n \sim 1/P^{\nu}$ with numerical values $\nu = 1 + 1/\beta \approx 2.1$ and $\beta \approx 0.9$.
- PageRank $P(K)$ on average is proportional to the number of ingoing links
- CheiRank $P^*(K^*) \sim 1/K^{*\beta}$ on average is proportional to the number of outgoing links ($\nu \approx 2.7; \beta = 1/(\nu - 1) \approx 0.6$)
- WWW at present: $\sim 10^{11}$ web pages

Donato *et al.* EPJB **38**, 239 (2004)

Wikipedia ranking of human knowledge

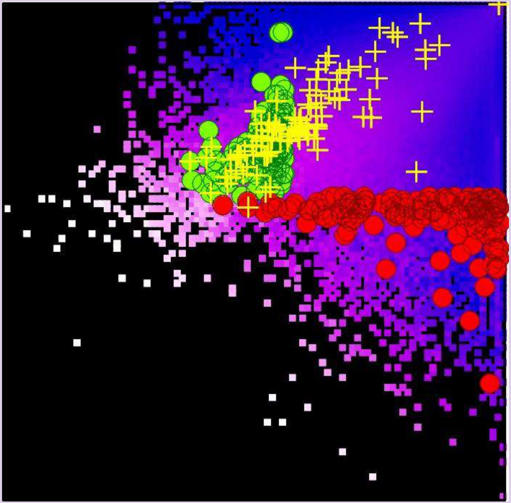
Wikipedia English articles $N = 3282257$ dated Aug 18, 2009



Dependence of probability of PagRank P (red) and CheiRank P^* (blue) on corresponding rank indexes K, K^*; lines show slopes $\beta = 1/(\nu - 1)$ with $\beta = 0.92; 0.57$ respectively for $\nu = 2.09; 2.76$.

[Zhirov, Zhirov, DS EPJB **77**, 523 (2010)]

Two-dimensional ranking of Wikipedia articles



Density distribution in plane of PageRank and CheiRank indexes $(\ln K, \ln K^*)$: 100 top personalities from PageRank (green), CheiRank (red) and Hart book (yellow)

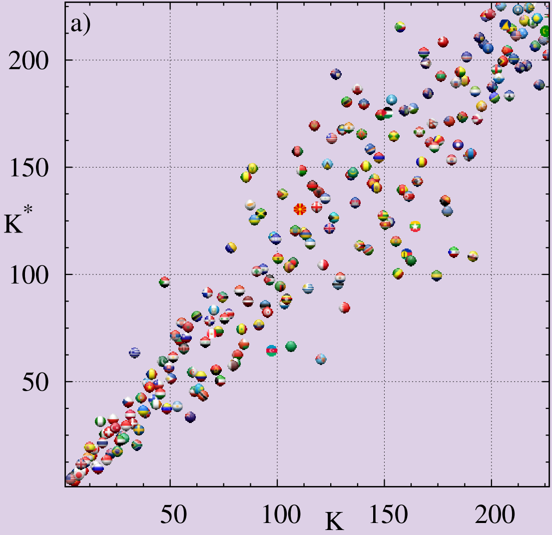
Wikipedia ranking of persons

Persons EN-Wikipedia Aug 2009: PageRank: 1. Napoleon I of France, 2. George W. Bush, 3. Elizabeth II of the United Kingdom, 4. William Shakespeare, 5. Carl Linnaeus, 6. Adolf Hitler, 7. Aristotle, 8. Bill Clinton, 9. Franklin D. Roosevelt, 10. Ronald Reagan.
2DRank: 1. Michael Jackson, 2. Frank Lloyd Wright, 3. David Bowie, 4. Hillary Rodham Clinton, 5. Charles Darwin, 6. Stephen King, 7. Richard Nixon, 8. Isaac Asimov, 9. Frank Sinatra, 10. Elvis Presley.
CheiRank: 1. Kasey S. Pipes, 2. Roger Calmel, 3. Yury G. Chernavsky, 4. Josh Billings (pitcher), 5. George Lyell, 6. Landon Donovan, 7. Marilyn C. Solvay, 8. Matt Kelley, 9. Johann Georg Hagen, 10. Chikage Oogi.

9 languages Wiki-s => www.quntware.ups-tlse.fr/QWLIB/wikiculturenetwork/
Persons 9-Wikipedia 2012: PageRank: 1. Napoleon, 2. Jesus, 3. Carl Linnaeus, 4. Aristotle, 5. Adolf Hitler, 6. Julius Caesar, 7. Plato, 8. Charlemagne , 9. William Shakespeare, 10. Pope John Paul II.
2DRank: 1. Michael Jackson, 2. Adolf Hitler, 3. Julius Caesar, 4. Pope Benedict XVI, 5. Wolfgang Amadeus Mozart, 6. Pope John Paul II, 7. Ludwig van Beethoven, 8. Bob Dylan, 9. William Shakespeare, 10. Alexander the Great
==> Top PageRank Universities: 1. Harvard, 2. Oxford, 3. Cambridge, 4. Columbia, 5. Yale, 6. MIT, 7. Stanford, 8. Berkeley, 9. Princeton, 10. Cornell.

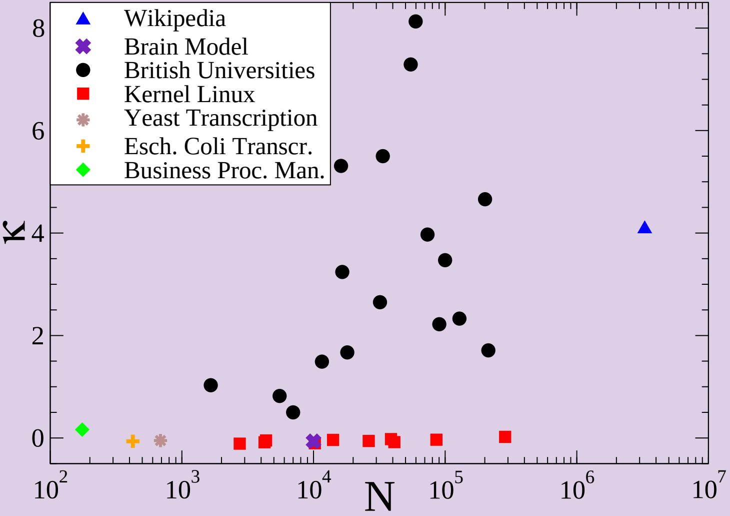
Ranking of World Trade

UN COMTRADE database 2008: All commodities



Ermann, DS arxiv:1103.5027 (2011)

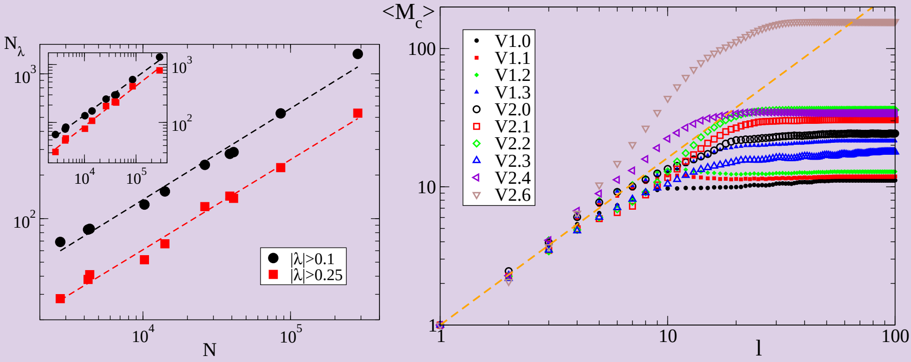
Correlator of PageRank and CheiRank



$\kappa = N \sum_i P(K(i)) P^*(K^*(i)) - 1$

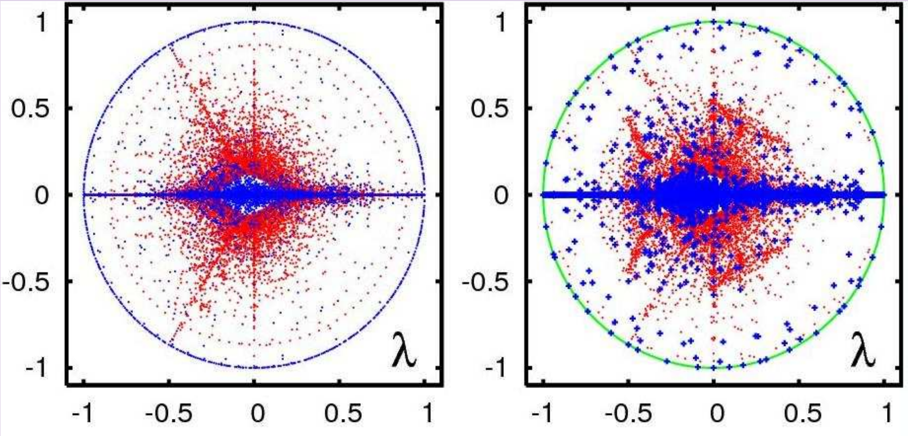
Fractal Weyl law for Linux Network

Number of states $N_\lambda \sim N^\nu, \quad \nu = d/2 \quad (N \sim 1/\hbar^{d/2})$



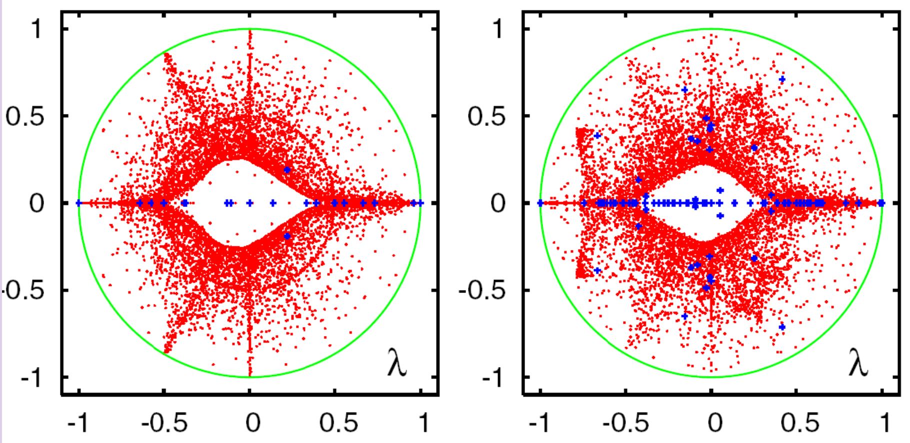
Number of states N_λ with $|\lambda| > 0.1; 0.25$ vs. N, lines show $N_\lambda \sim N^\nu$ with $\nu \approx 0.65$ (left); average mass $< M_c >$ (number of nodes) as a functon of network distance l, line shows the power law for fractal dimension $< M_c > \sim l^d$ with $d \approx 1.3$ (right).

Spectrum of UK University networks



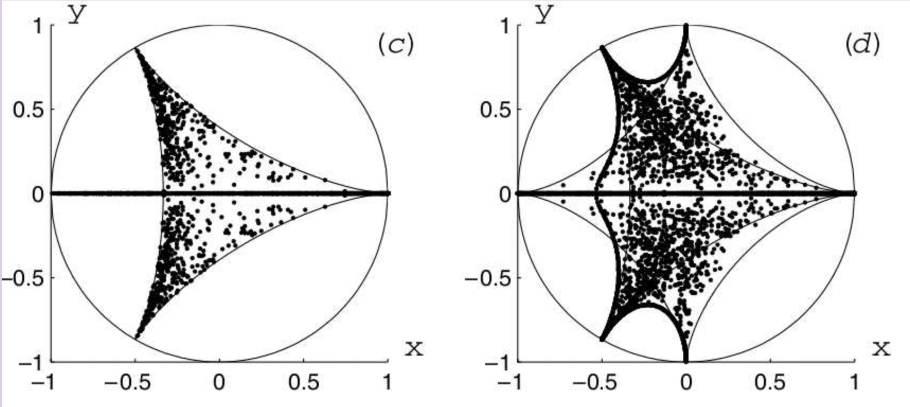
Arnoldi method: Spectrum of Google matrix for Univ. of Cambridge (left) and Oxford (right) in 2006; 20% of sub-spaces ($N \approx 200000$, $\alpha = 1$). [Frahm, Georgeot, DS arxiv:1105.1062 (2011)]

Spectrum of UK University networks



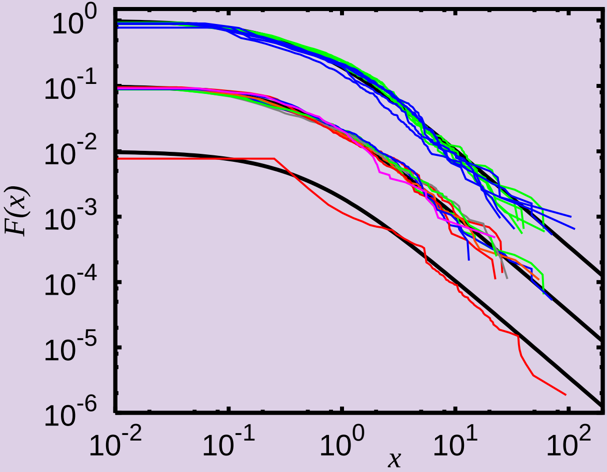
Spectrum of CheiRank Google matrix for Univ. of Cambridge (left) and Oxford (right) in 2006 ($N \approx 200000$, $\alpha = 1$) [Frahm, Georgeot, DS arxiv:1105.1062 (2011)]

Spectrum of random orthostochastic matrices



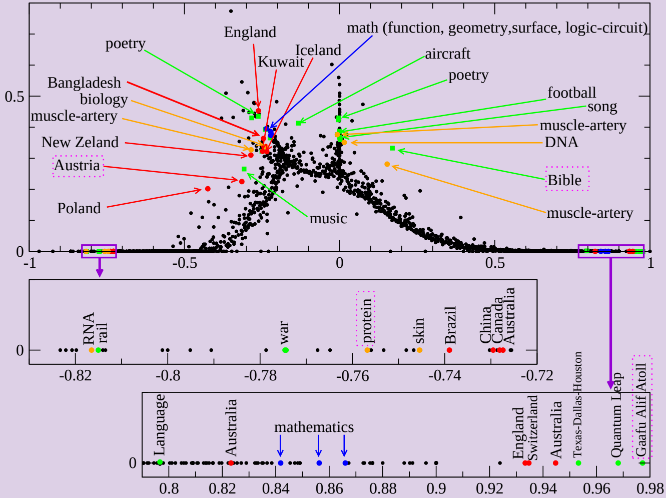
Spectrum $N = 3$ (left), 4 (right) [K.Zyczkowski *et al.* J.Phys. A **36**, 3425 (2003)]

Invariant subspaces size distribution



$F(x)$ integrated number of invariant subspaces with size larger that d/d_0; $x = d/d_0$, d_0 is average size of subspaces (top: Cambridge, Oxford 2002-2006; middle: all others; bottom: Wikipedia CheiRank). Curve: $F(x) = 1/(1+2x)^{3/2}$.

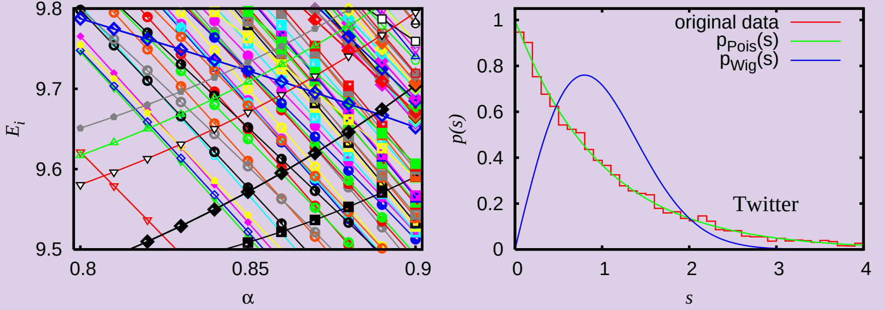
Wikipedia spectrum and eigenstates



Spectrum S of EN Wikipedia, Aug 2009, $N = 3282257$. Eigenvalues-communities are labeled by most repeated words following word counting of first 1000 nodes.

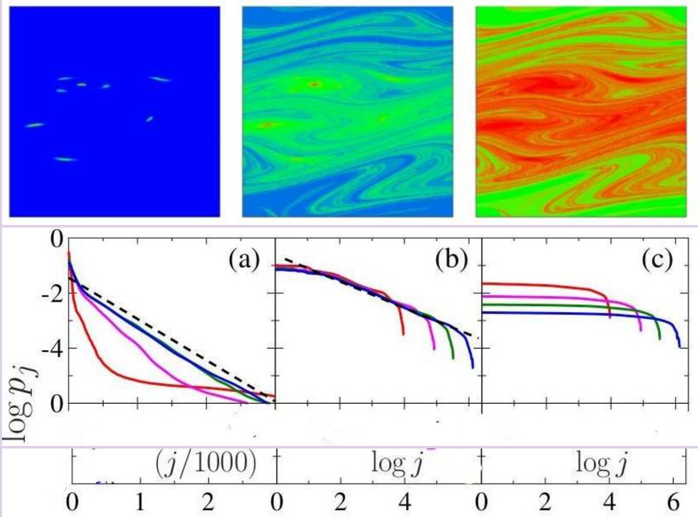
(Ermann, Frahm, DS 2013)

Poisson statistics of Twitter network 2009



Left panel: Dependence of certain top PageRank levels $E_i = -\ln(P_i)$ on the damping factor α for entire Twitter $N \approx 4.1 \times 10^7$. Data points on curves with one color correponds to the same node i. *Right panel:* Histogramm of unfolded level spacing statistics for Twitter. The Poisson distribution $p_{\rm Pois}(s) = \exp(-s)$ and the Wigner surmise $p_{\rm Wig}(s) = \frac{\pi}{2}\, s\, \exp(-\frac{\pi}{4}\, s^2)$ are also shown for comparison. (Frahm, DS 2014)

Anderson delocalization of PageRank ?



Ulam network of dynamical map $\alpha = 1; 0.95; 0.85$

Oriol Bohigas LXX Celebration

Dima Shepelyansky
www.quantware.ups-tlse.fr/dima

Boris Chirikov LXX Celebration, Toulouse, July 17, 1998

- Caro Oriol,
  during my visit to Novosibirsk for the Chirikov Memorial Seminar at the Budker Institute of Nuclear Physics (23.05.2008) I found in the archive of Chirikov his correspondence letters with you and his recommendation letter which is partially reproduced below. I fully support the opinion of Boris and wish you health and further chaotic creative research.
  Amities, Dima

  P.S. electronic copies of your correspondence with Boris are available at

  www.quantware.ups-tlse.fr/chirikov/archive/bohigas.pdf

From the Letter to Achim Richter on TUD Honorary Doctorate (dated around 12 - 18 Jan 2001): "... This was especially important in his [Bohigas] pioneering research of a deep relation between the already well developed theory of random matrices, a purely statistical one, and the underlaying chaotic dynamics, the brand-new, that time, quantum chaos. This put the firm foundations of the contemporary statistical theory of complex quantum systems. ... The Honorary Doctorate would be a fair recognition of his [Bohigas] important contribution to physics in the last century, and a strong stimulus for farther research in the new millennium.

École de sciences avanceés de Luchon

École de sciences avancées de Luchon

School of advanced sciences of Luchon

Network analysis and applications
Session I, June 21 - July 5, 2014

School Organizers:
András Benczúr (Budapest), Andreas Kaltenbrunner (Barcelona), Dima Shepelyansky (Toulouse)

Scope: In past ten years, modern societies developed enormous communication and social networks. New characterization tools of complex networks provide possibilities for information retrieval in social networks, communication, economy, gene, protein and other networks. The interdisciplinary approaches based on complex networks and Markov chains allow to obtain advanced results in such diverse fields as physics, computer science and bioinformatics. The school will present lectures of world leading experts in these fields.

http://www.quantware.ups-tlse.fr/ecoleluchon2014/

Selected References:

Fractal Weyl law and Ulam networks:
R1. L.Ermann and D.L.Shepelyansky, "Ulam method and fractal Weyl law for Perron-Frobenius operators", Eur. Phys. J. B v.75, p.299-304 (2010)
R2. L.Ermann, A.D.Chepelianskii and D.L.Shepelyansky, "Fractal Weyl law for Linux Kernel Architecture", Eur. Phys. J. B v.79, p.115-120 (2011)
PageRank-CheiRank:
R3. A.O.Zhirov, O.V.Zhirov and D.L.Shepelyansky, *Two-dimensional ranking of Wikipedia articles*, Eur. Phys. J. B **77**, 523 (2010)
R4. L.Ermann and D.L.Shepelyansky, "Google matrix of the world trade network", Acta Physica Polonica A v.120(6A), pp. A158-A171 (2011)
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