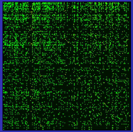
Ulam networks, fractal Weyl law and Anderson localization



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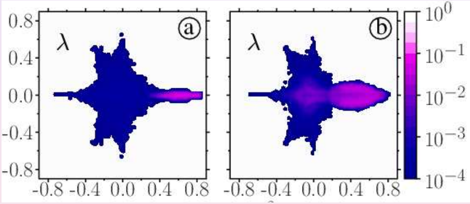
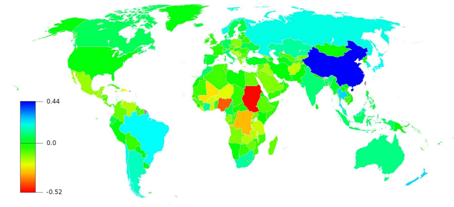




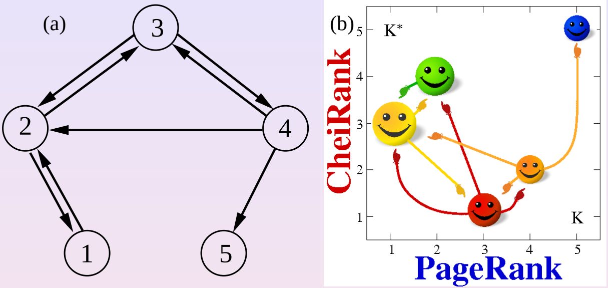
* Markov (1906) $\to$ Brin and Page (1998)
* Ulam networks and fractal Weyl law, Anderson transition on directed networks
* Applications: multiproduct world trade network (UN COMTRADE + OECD-WTO), Wikipedia ranking

Support: EC FET Open project NADINE

Refs. at www.quantware.ups-tlse.fr/FETNADINE/ + arXiv:1409.0428

Google matrix construction rules

Markov chains (1906) and Directed networks



For a directed network with N nodes the adjacency matrix **A** is defined as $A_{ij} = 1$ if there is a link from node j to node i and $A_{ij} = 0$ otherwise. The weighted adjacency matrix is

$$S_{ij} = A_{ij} / \sum_k A_{kj}$$

In addition the elements of columns with only zeros elements are replaced by $1/N$.

Google matrix construction rules

Google Matrix and Computation of PageRank

$\mathbf{P} = \mathbf{SP} \Rightarrow$ **P**= stationary vector of **S**; can be computed by iteration of **S**.

To remove convergence problems:

- Replace columns of 0 (dangling nodes) by $\frac{1}{N}$:

$$S = \begin{pmatrix} 0 & 1/2 & 1/3 & 0 & 1/5 \\ 1 & 0 & 1/3 & 1/3 & 1/5 \\ 0 & 1/2 & 0 & 1/3 & 1/5 \\ 0 & 0 & 1/3 & 0 & 1/5 \\ 0 & 0 & 0 & 1/3 & 1/5 \end{pmatrix} \qquad S^* = \begin{pmatrix} 0 & 1/3 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/2 & 1/3 & 0 & 1 & 0 \\ 0 & 1/3 & 1/2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- To remove degeneracies of $\lambda = 1$, replace **S** by **Google matrix**

  $\mathbf{G} = \alpha\mathbf{S} + (1-\alpha)\frac{\mathbf{E}}{N}$; $\quad GP = \lambda P \quad$ => Perron-Frobenius operator

- α models a random surfer with a random jump after approximately 6 clicks (usually $\alpha = 0.85$); PageRank vector => P at $\lambda = 1$ ($\sum_j P_j = 1$).

- CheiRank vector P^*: $G^* = \alpha\mathbf{S}^* + (1-\alpha)\frac{\mathbf{E}}{N}$, $G^*P^* = P^*$
  ($\mathbf{S}^*$ with inverted link directions)
  Fogaras (2003) ... Chepelianskii arXiv:1003.5455 (2010) ...

Real directed networks

Real networks are characterized by:

- **small world property**: average distance between 2 nodes $\sim \log N$
- **scale-free property**: distribution of the number of ingoing or outgoing links $\rho(k) \sim k^{-\nu}$

PageRank vector for large WWW:

- $P(K) \sim 1/K^{\beta}$, where K is the ordered rank index
- number of nodes N_n with PageRank P scales as $N_n \sim 1/P^{\nu}$ with numerical values $\nu = 1 + 1/\beta \approx 2.1$ and $\beta \approx 0.9$.
- PageRank $P(K)$ on average is proportional to the number of ingoing links
- CheiRank $P^*(K^*) \sim 1/K^{*\beta}$ on average is proportional to the number of outgoing links ($\nu \approx 2.7; \beta = 1/(\nu - 1) \approx 0.6$)
- WWW at present: $\sim 10^{11}$ web pages

Donato *et al.* EPJB **38**, 239 (2004)

Fractal Weyl law

invented for open quantum systems, quantum chaotic scattering:
the number of Gamow eigenstates N_γ, that have escape rates γ in a finite bandwidth $0 \leq \gamma \leq \gamma_b$, scales as

$$N_\gamma \propto \hbar^{-\nu} \propto N^\nu, \quad \nu = d/2$$

where d is a fractal dimension of a strange invariant set formed by obits non-escaping in the future and in the past (N is matrix size)

References:
J.Sjostrand, Duke Math. J. **60**, 1 (1990)
M.Zworski, Not. Am. Math. Soc. **46**, 319 (1999)
W.T.Lu, S.Sridhar and M.Zworski, Phys. Rev. Lett. **91**, 154101 (2003)
S.Nonnenmacher and M.Zworski, Commun. Math. Phys. **269**, 311 (2007)

Resonances in quantum chaotic scattering:
three disks, quantum maps with absorption

Perron-Frobenius operators, Ulam method for dynamical maps, Ulam networks, dynamical maps, strange attractors

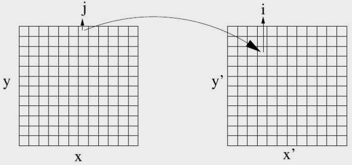
Linux kernel network $d = 1.3$, $N \leq 285509$;
Phys. Rev. up to 2009 $d \approx 1$, $N = 460422$

Ulam networks

Ulam conjecture (method) for discrete approximant of Perron-Frobenius operator of dynamical systems

Discretized phase-space:
Adjacency matrix $\mathbf{A} = P(j \to i)$

$N = N_x \times N_y$ cells.



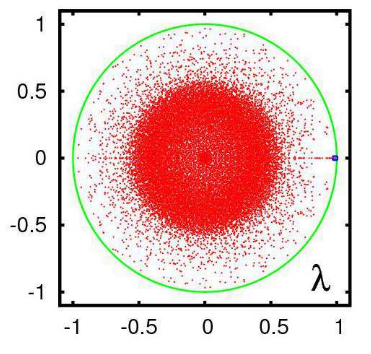
N_c: traj. from cell j
N_i: traj. to call i

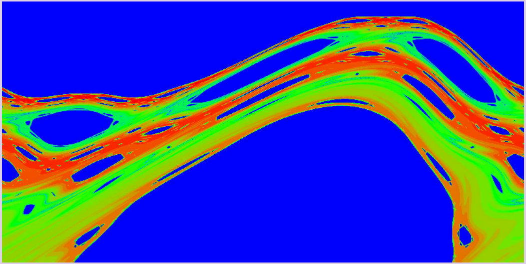
$$\begin{cases} \mathbf{A}_{i,j} = N_i/N_c \\ \sum_i \mathbf{A}_{i,j} = 1 \quad \text{(closed systems)} \end{cases}$$

S.M.Ulam, *A Collection of mathematical problems*, Interscience, **8**, 73 N.Y. (1960)
A rigorous prove for hyperbolic maps:
T.-Y.Li J.Approx. Theory **17**, 177 (1976)
Related works:
Z. Kovacs and T. Tel, Phys. Rev. A 40, 4641 (1989)
M.Blank, G.Keller, and C.Liverani, Nonlinearity **15**, 1905 (2002)
D.Terhesiu and G.Froyland, Nonlinearity **21**, 1953 (2008)

Links to Markov chains: $\infty\infty\infty\infty\infty\infty\infty\infty\infty\infty$
Contre-example: Hamiltonian systems with invariant curves, e.g. the Chirikov standard map: noise, induced by coarse-graining, destroys the KAM curves and gives homogeneous ergodic eigenvector at $\lambda = 1$

Ulam method for the Chirikov standard map





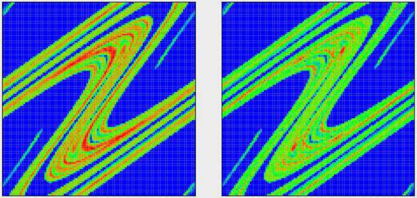
$\bar{y} = y + K \sin x, \ \bar{x} = x + \bar{y} \ \ (mod 2\pi); \ K = 0.971635...$
Left: spectrum $G\psi = \lambda\psi$, $M \times M/2$ cells; $M = 280$, $N_d = 16609$, exact and Arnoldi method for matrix diagonalization; generalized Ulam method of one trajectory.
Right: modulus of eigenstate of $\lambda_2 = 0.99878...$, $M = 1600$, $N_d = 494964$.
Here $K = K_G$

Ulam method for dissipative systems

Strange repellers and strange attractors

Scattering

$$\left\{ \begin{array}{lcl} \bar{y} & = & y + K \sin(x + y/2) \\ \bar{x} & = & x + (y + \bar{y})/2 \quad (\mathrm{mod} 2\pi) \end{array} \right.$$

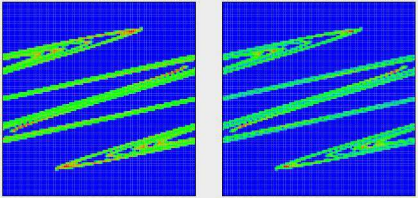


$N = 110 \times 110,\ K = 7, a = 2$

$\lambda_1 = 0.756 \qquad \lambda_3 = -0.01 + i0.513$

Dissipation

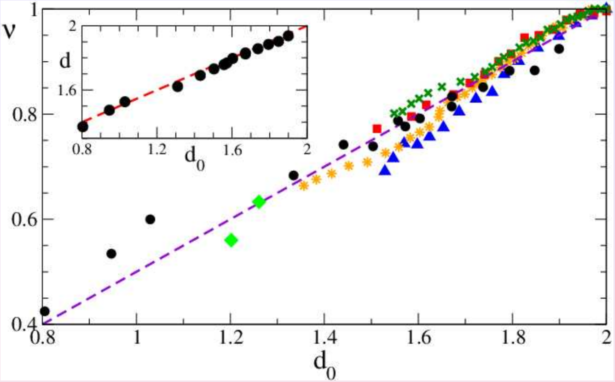
$$\left\{ \begin{array}{lcl} \bar{y} & = & \eta y + K \sin x \\ \bar{x} & = & x + \bar{y} \quad (\mathrm{mod} 2\pi) \end{array} \right.$$



$N = 110 \times 110,\ K = 7, \eta = 0.3$

$\lambda_1 = 1 \qquad \lambda_3 = -0.258 + i0.445$

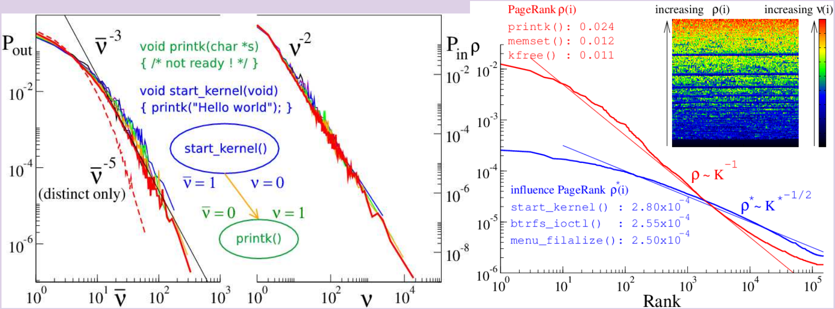
Fractal Weyl law for Ulam networks



Fractal Weyl law for three different models with dimension d_0 of invariant set. The fractal Weyl exponent ν is shown as a function of fractal dimension d_0 of the strange repeller in model 1 and strange attractor in model 2 and Henon map; dashed line shows the theory dependence $\nu = d_0/2$. Inset shows relation between the fractal dimension d of trajectories nonescaping in future and the fractal inv-set dimension d_0 for model 1; dashed line is $d = d_0/2 + 1$.

Linux Kernel Network

Procedure call network for Linux

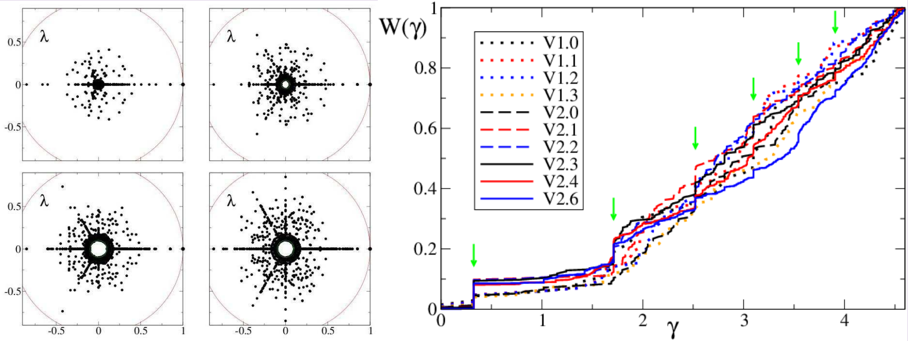


Links distribution (left); PageRank and inverse PageRank (CheiRank) distribution (right) for Linux versions up to 2.6.32 with $N = 285509$ ($\rho \sim 1/j^{\beta}, \beta = 1/(\nu - 1)$).

(Chepelianskii arxiv:1003.5455)

Fractal Weyl law for Linux Network

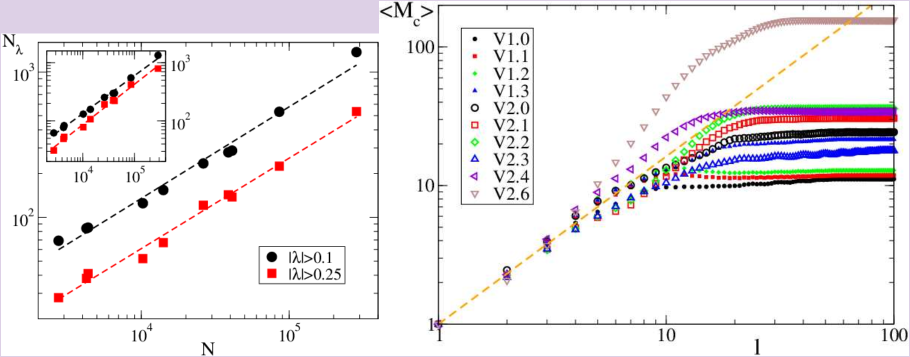
Sjöstrand Duke Math J. 60, 1 (1990), Zworski *et al.* PRL 91, 154101 (2003) $\rightarrow$ quantum chaotic scattering;
Ermann, DS EPJB 75, 299 (2010)$\rightarrow$ Perron-Frobenius operators



Spectrum of Google matrix (left); integrated density of states for relaxation rate $\gamma = -2\ln|\lambda|$ (right) for Linux versions, $\alpha = 0.85$.

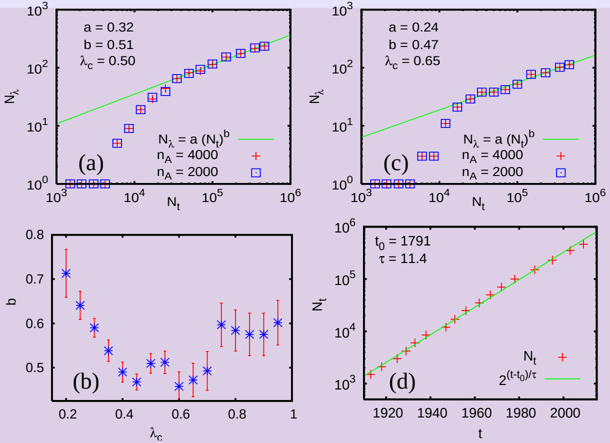
Fractal Weyl law for Linux Network

Number of states $N_\lambda \sim N^\nu$, $\nu = d/2$ $(N \sim 1/\hbar^{d/2})$



Number of states N_λ with $|\lambda| > 0.1; 0.25$ vs. N, lines show $N_\lambda \sim N^\nu$ with $\nu \approx 0.65$ (left); average mass $< M_c >$ (number of nodes) as a functon of network distance l, line shows the power law for fractal dimension $< M_c > \sim l^d$ with $d \approx 1.3$ (right).

Fractal Weyl law for Physical Review network



Panel (a) (or (c)): shows the number N_λ of eigenvalues with $\lambda_c \leq \lambda \leq 1$ for $\lambda_c = 0.50$ (or $\lambda_c = 0.65$) versus the network size N_t (up to time t). The green line shows the fractal Weyl law $N_\lambda = a\,(N_t)^b$ with parameters $a = 0.32 \pm 0.08$ ($a = 0.24 \pm 0.11$) and $b = 0.51 \pm 0.02$ ($b = 0.47 \pm 0.04$) obtained from a fit in the range $3 \times 10^4 \leq N_t < 5 \times 10^5$. *Panel (b):* exponent b with error bars obtained from the fit $N_\lambda = a\,(N_t)^b$ in the range $3 \times 10^4 \leq N_t < 5 \times 10^5$ versus cut value λ_c. *Panel (d):* effective network size N_t versus cut time t (in years). The green line shows the exponential fit $2^{(t-t_0)/\tau}$ with $t_0 = 1791 \pm 3$ and $\tau = 11.4 \pm 0.2$. Thus $N = 463348$, $N_\ell = 4684496$, $d \approx 1$, $b = \nu \approx 0.5$.

Anderson transition on directed networks

Anderson (1958) metal-insulator transition for electron transport in disordered solids

$H = \epsilon_n \psi_n + V(\psi_{na+1} + \psi_{n-1}) = E\psi_n \; ; \quad -W/2 < \epsilon_n < W/2$

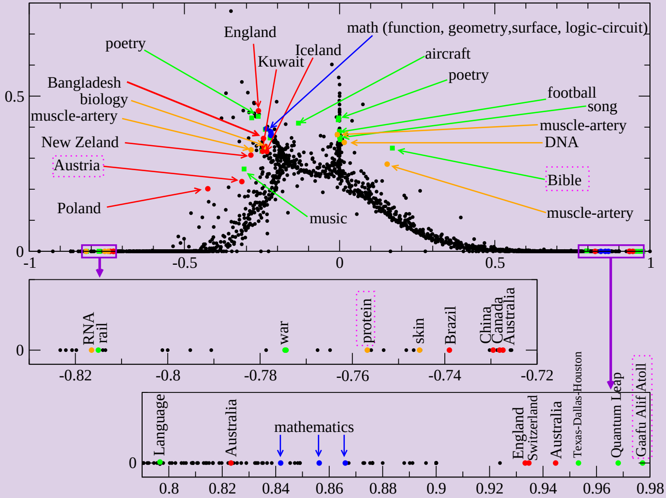
In dimensions $d = 1, 2$ all eigenstates are exponentially localized, insulating phase. At $d = 3$ for $W > 16.5V$ all eigenstates are exponentially localized, for $W < 16.5V$ there are metalic delocalized states, mobility edge, metalic phase

Random Matrix Theory - RMT (Wigner (1955)) for Hermitian and unitary matrices (quantum chaos, many-body quantum systems, quantum computers)

Google matrix, Markov chains, Perron-Frobenium operators:
=> complex spectrum of eigenvalues; new field of research
Can we have the Anderson transition for Google matrix? All the world would go blind if PageRank is delocalized What are good RMT models of Google matrix? Subspaces and core

$$\mathbf{S} = \begin{pmatrix} \mathbf{S_{ss}} & \mathbf{S_{sc}} \\ 0 & \mathbf{S_{cc}} \end{pmatrix}$$

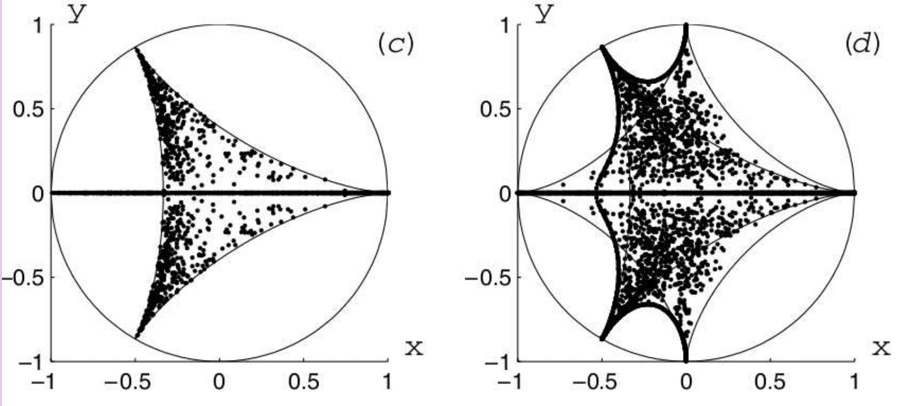
Wikipedia spectrum and eigenstates



Spectrum S of EN Wikipedia, Aug 2009, $N = 3282257$. Eigenvalues-communities are labeled by most repeated words following word counting of first 1000 nodes.

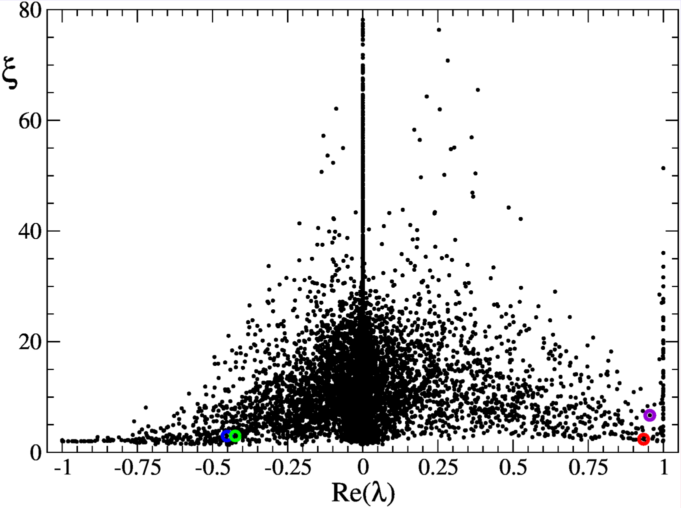
(Ermann, Frahm, DS 2013)

Spectrum of random orthostochastic matrices



Spectrum $N = 3$ (left), 4 (right) [K.Zyczkowski *et al.* J.Phys. A **36**, 3425 (2003)]

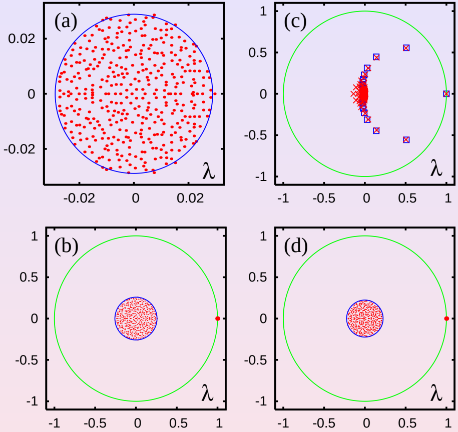
Localization features



Multiproduct world trade network $N = N_p \times N_c = 61 \times 227 = 13847$ (year 2008): small IPR values ξ. Small gaps in S of directed networks.

Random Matrix Models of directed networks

random matrix elements of G with sum equat unity in each column ($N = 400$)



(a) N positive random elements with unit sum in each column;
(c) triangular matrix with random elements;
(b),(d) $Q = 20$ nonzero elements in each column
- blue circle is theory with radius $\sim 1/\sqrt{N}, 1/\sqrt{Q}$

Anderson type models for directed networks

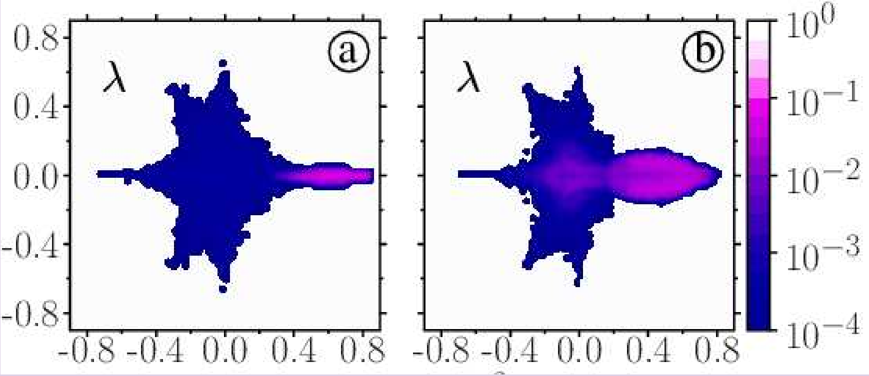
We use the usual Anderson model with diagonal disorder terms W_i and transitions V to nearby sites on a lattice in dimension d:

$$W_{\mathbf{i}}\psi_{\mathbf{i}} + V\psi_{\mathbf{i+1}} + V\psi_{\mathbf{i-1}} = \lambda\psi_{\mathbf{i}} \ , \tag{1}$$

where indexes in bold are vectors in d-dimensional space. On this we construct the matrices S and G for $d = 2, 3$. The matrix S is constructed as follows: each transition matrix element, corresponding to V terms, in the Anderson model in dimension d is replaced by a random number ε_i uniformly distributed in the interval $[0, \varepsilon_{max}/2d]$, the diagonal element $W_{\mathbf{i}}$ is replaced by unity minus the sum of all ε_i over $2d$ nearby sites $(1 - \sum_{i=1}^{2d} \varepsilon_i)$. The asymmetric matrix S constructed in this way belongs to the Google matrix class.

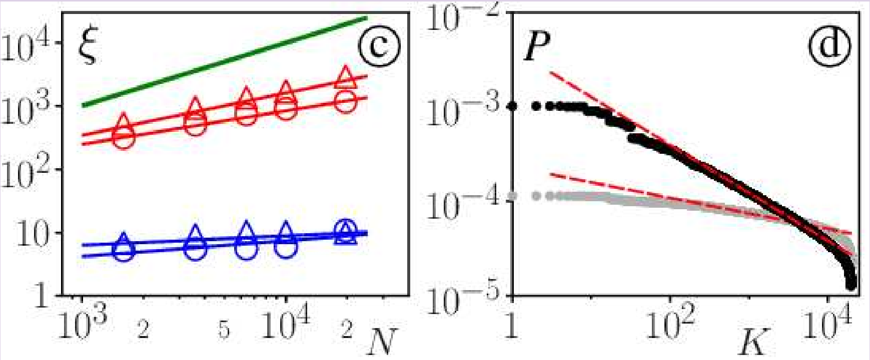
By replacing matrix elements in the model AD2 by blocks B of size 4×4 we obtain the **model AD2Z**. In a similar way we obtain the **model AD2ZS** with block shortcuts. In this case we restrict our studies only for dimension $d = 2$.

Anderson transition on directed networks



Panels show distribution of IPR values ξ (number of nodes contributing to an eigenstate) on the plane λ or eigenvalues of G matrix for two nodels of directed networks with disorder; color shows the ratio ξ/N, $\alpha = 0.85$.

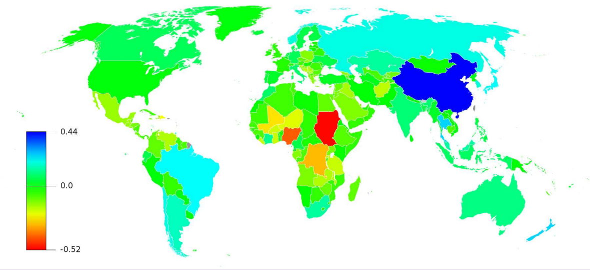
Anderson transition on directed networks

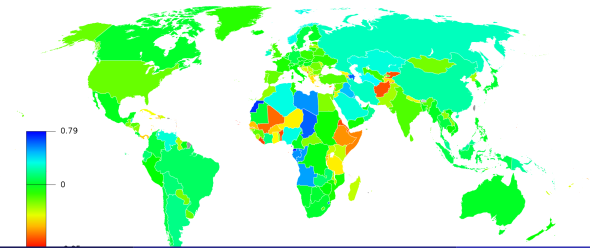


Panel (c): dependence of ξ on N for AD2Z with triangles for states with λ located in the delocalized domain $\mathrm{Re}\lambda \in (0.3, 0.85)$ (red triangles, fit gives $\nu = 0.67$) and in the localized domain $\mathrm{Re}\lambda < -0.5$ (blue triangles, $\nu = 0.15$); for AD2ZS at $\delta = 0.25$ with circles for states with λ located in the delocalized domain $\mathrm{Re}\lambda \in (0.2, 0.85)$ (red circles, $\nu = 0.53$) and in the quasi-localized domain $\mathrm{Re}\lambda < -0.5$ (blue circles, $\nu = 0.25$); fits are shown by lines, green line shows $\xi = N$. Panel (d): dependence of PageRank probability P on PageRank index K for models AD2Z (gray symbols) and AD2ZS at $\delta = 0.25$ (black symbols); the fits for the range $K \in (100, 6000)$ are shown by dashed lines with $\beta = 0.16$ (AD2Z) and $\beta = 0.51$ (AD2ZS) for the parameters of panels (a,b).

CheiRank-PageRank balance (2008)

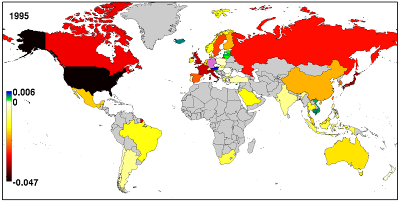
$B_c = (P_c{}^* - P_c)/(P_c{}^* + P_c)$ (top - CheiRank-PageRank; bottom -Export-Import volume; multiproduct world trade $N_c = 227, N_p = 61, N = 13847$ ==> Ermann lecture)

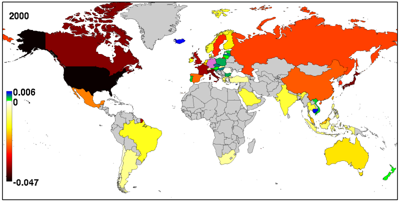


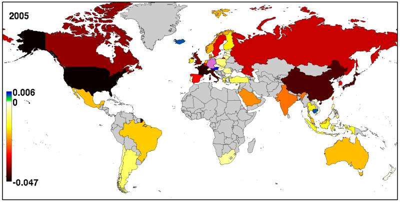


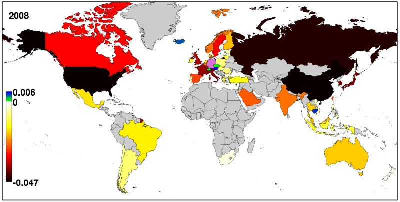
WNEA-WTO-OECD of DEU

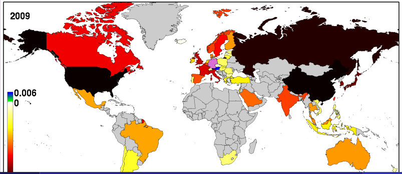
CheiRank-PageRank balance sensitivity to labor cost in Gernamy



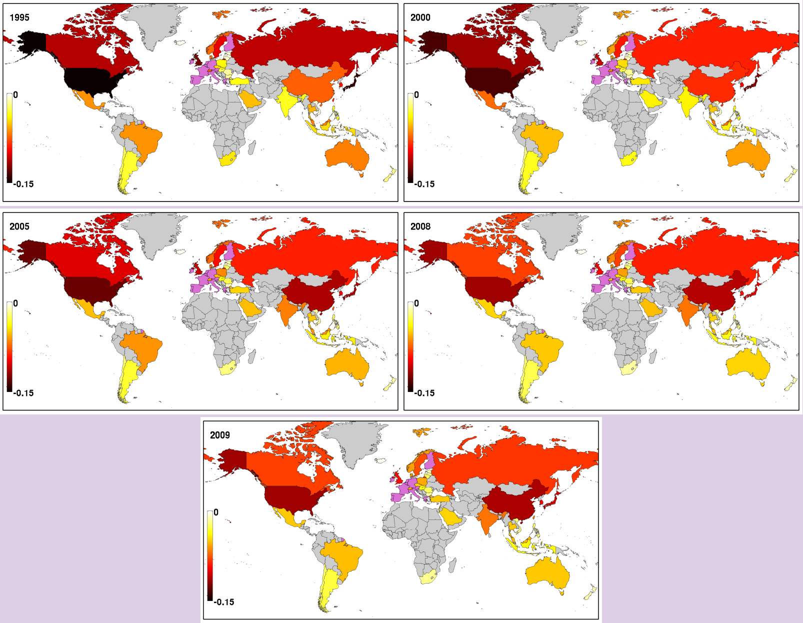






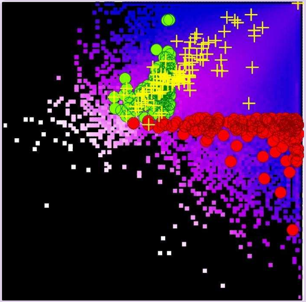


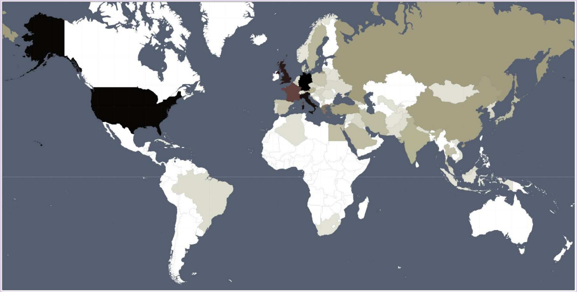
WNEA-WTO-OECD for EUROZONE



Top historical figures of 24 Wikipedia editions

2DRanking of Wikipedia articles; top 100 historical figures;
comparison with historical studies of M.Hart (37 and 43 percent overlap)
35 centures and all countries by birth place; 17 millions wiki-articles

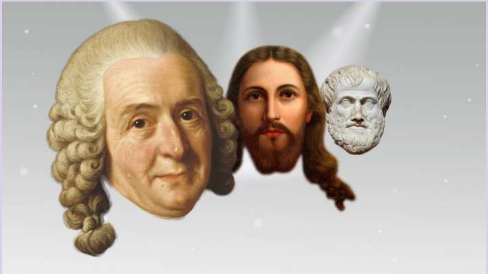




A.Zhirov, O.Zhirov, DLS EPJB (2010); Y.-H.Eom, K.M.Frahm, A.Benczur, DLS EPJB (2013); Y.-H.Eom, DLS PLoS ONE (2013), Y.-H.Eom,P.Aragon, D.Laniado, A.Kaltenbrunner, S.Vigna, DLS arXiv2014 - PLoS ONE (2015)

Top historical figures of 24 Wikipedia editions

Top global PageRank historical figures: Carl Linnaeus, Jesus, Aristotle ...



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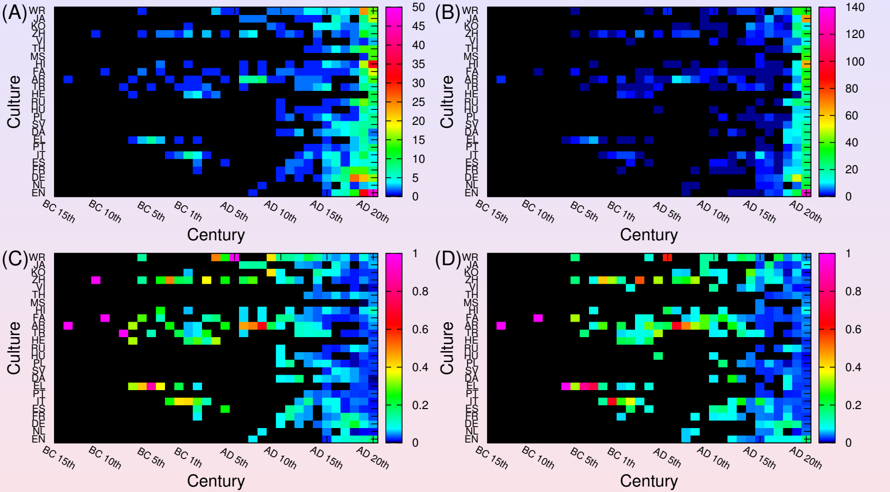
And the winner of Wikipedia's influence list is ... an 18th century botanist. Hear hear

Carl Linnaeus is hardly a household name, but the Swedish doctor who created a global naming system for species deserves this accolade

Patrick Barkham
theguardian.com, Friday 13 June 2014 09.00 BST
Jump to comments (51)

Media highlights: The Guardian, The Independent, The Washington Post, France24, EC CORDIS, Uppsala Universitet: “Carl Linnaeus ranked most influential person of all time” ... (about 20 countries)
Competitors: MIT Pantheon project http://pantheon.media.mit.edu (2014); Stony-Brook NY http://www.whoisbigger.com/ (2014)

Time evolution of 35 centuries



Birth date distribution of historical figures from the global PageRank list (A,C, 1045 persons) and 2DRank list (B,D, 1616 persons). Each historical figure is attributed to her/his own language according to her/his birth place as described in the paper (if the birth place is not among our 24 languages then a person is attributed to the remaining world (WR)). Color in panels (A,B) shows the total number of persons for a given century, while in panels (C,D) color shows a percent for a given century (normalized to unity in each column).

Further applications of Markov chains and Google matrix ?

Google matrix power