Interlinks of dynamical thermalization Kolmogorov turbulence, KAM and Anderson localization

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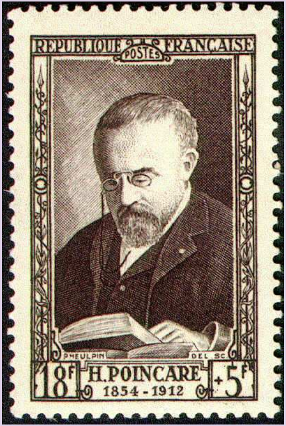
It is not so much important to be rigorous as to be right
A.N.Kolmogorov (from Chirikov Phys Rep (1979))

* Boltzmann - Loschmidt dispute:
irreversible statistical laws from reversible dynamical equations

Sitzungsberichte der Akademie der Wissebschaften, Wien,
II **73**, 128 (1876); **75**, 67 (1877)

Dynamical chaos (mathematics, physics)



* Poincaré (1893);
* Chirikov resonance overlap criterion
for plasma experiments at Kurchatov Inst. (1959)

Chirikov talk (celebrating his 70th at Toulouse)

CHAOS in SIBERIA
A story for history

Short (40') account of a long (40 yrs) work
Boris Chirikov & many coworkers in
Budker Institute of Nuclear Physics
& elsewhere

My (accidental) start
(≈ 1958, Kurchatov Institute, Moscow)

a simple–looking but turned–out rich
Budker's problem: single particle confinement in
Budker's (adiabatic) magnetic trap
(for the great END – controlled nuclear fusion !)

Toulouse
16 July 1998

1

BACKGROUND (retrospectively)
Revival of intensive studies
into nonlinear dynamics,
surprising rediscovery of chaos (stochasticity) after
Boltzmann ... Poincare ...

- new applications:
  strong focusing accelerators,
  controlled nuclear fusion
- computers, NUMERICAL EXPERIMENTS !!

– Lehmer, 1951
pseudorandom number generators after
Galton Board (= Lorentz gas in external field !)

– Goward and Hine (CERN), 1953, accelerators

– Fermi, Pasta and Ulam, 1955
foundations of statistical mechanics

– Symon and Sessler, 1956, accelerators

– Kolmogorov, 1954, KAM–integrability
(in spite of Poincare theorem !)

2

Chirikov talk (celebrating his 70th at Toulouse)

instructive examples — SIMPLE CHAOS:

most complex dynamics

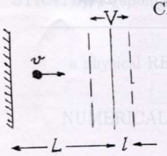
BUT simple statistics

- MICROTRON, Vekler, 1944

$$E_{n+1} = E_n + V_0 \sin\phi_n$$
$$\phi_{n+1} = \phi_n + E_{n+1}/E_0, \quad F = 2 \text{ freedoms}$$
$$K = V_0/E_0 \gg 1$$
$$E_0 = \omega_0/2\pi\Omega, \quad e=m=c=1$$

- FERMI ACCELERATION, Ulam, 1961

Chirikov, Zaslavsky, 1964



$$\frac{v^2}{V^2} < \left(\frac{4L}{l}\right) !$$

(adiabatics: $\frac{v^2}{V^2} \lesssim \left(\frac{L}{l}\right)^2$)

- COMET HALLEY

diffusion backwards in time

$t_L \sim 10\, Myrs$?

Chirikov, Vecheslavov, 1989

10

Fermi's problem (1923)

ergodicity of nonlinear oscillation

$$H = \sum_{i=1}^{N} \frac{\dot{x}_i^2}{2} + \frac{(x_i - x_{i-1})^2}{2} + (x_i - x_{i-1})^p, \quad p = 3; 4$$
$$N \gg 1$$

Fermi, Pasta, Ulam, 1955

nonergodic (generally) → solitons — Kruskal, Miura, Zabusky, 196[?]

→ dynamical chaos — Chirikov, Izrailev, 1966

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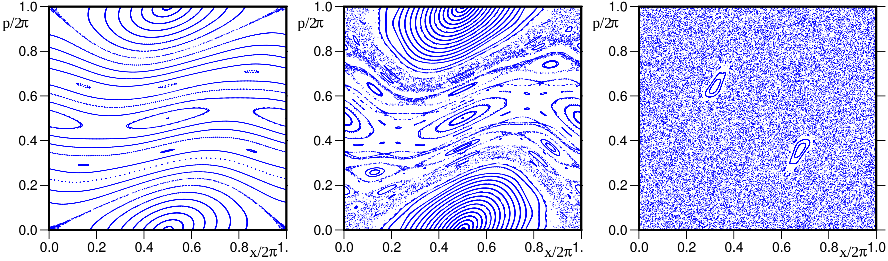
De Luca, Lichtenberg, Lieberman, 1995

Shepelyansky, 1997

Chirikov standard map (1969, 1979)



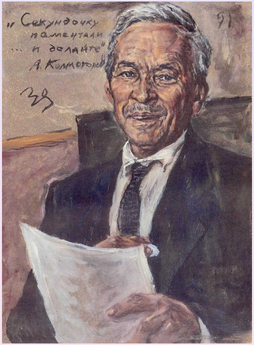
$\bar{p} = p + K \sin x$, $\bar{x} = x + \bar{p}$; $K = 0.5; 0.971635; 5$

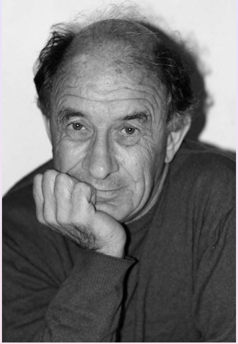
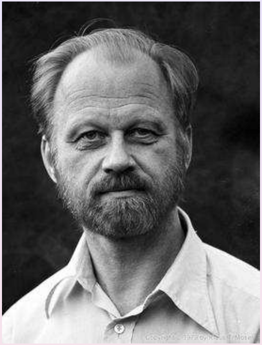
related systems: Frenkel-Kontorova model (1937); Veksler microtron (1944); Ulam map for Fermi acceleration (1961), Kepler map for microwave ionization of Rydberg atoms, Halley comet, dark matter dynamics in the Solar system ... (see Scholarpedia 3(3):3550 (2008))

Diffusion in momentum and energy $< p^2 >= Dt \approx K^2 t/2$ for $K > 0.971635...$ $(E = p^2/2)$

Chirikov, DS Scholarpedia (2008-20017)

Kolmogorov-Arnold-Moser theory (1954-1963)

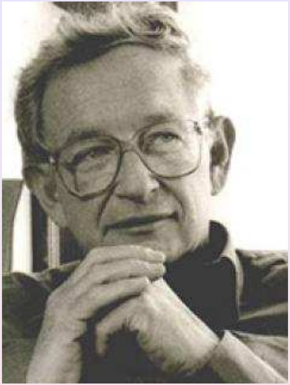


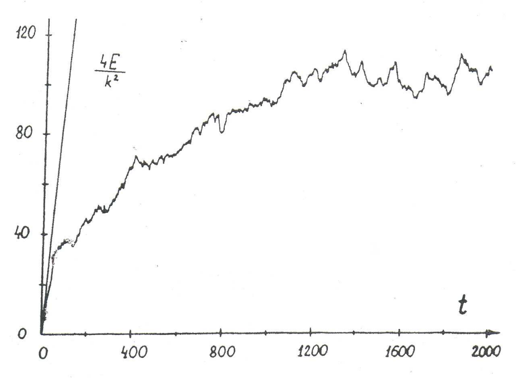



Chirikov criterion (1959) (e.g. for Chirikov standard map $K < K_c$)
term “KAM theory” coined by Chirikov (INP preprint 1969)

Quantum localization of chaos

Quantum Chirikov standard map (kicked rotator)

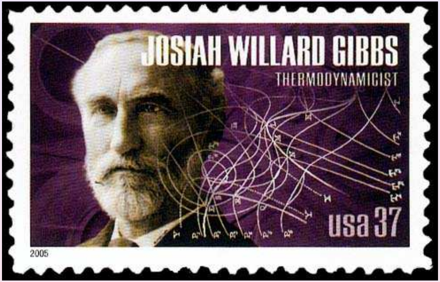




Chirikov, Izrailev, DS (1981) $\rightarrow$ Raizen cold atom experiment (1995):
$\bar{\psi} = \exp(-ik\cos x)\exp(-iTn^2/2)\psi$; $k = K/\hbar$, $T = \hbar$, $n = -i\partial/\partial x$;
localization length: $\ell \approx \Delta n \approx D \sim K^2/\hbar^2$

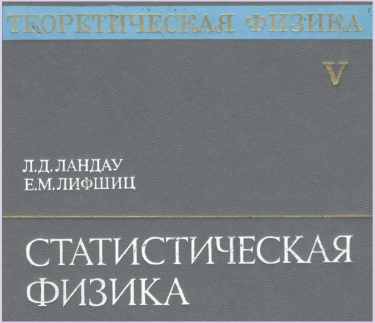
Analogy with Anderson localization (AL) in disordered solids (1958; Nobel prize 1977) shown by Fishman, Grempel, Prange (1982)

Thermalization (classical/quantum)



J.W.Gibbs (1874-1878) ; M.Planck (1900) $\rightarrow \hbar$

Energy equipartition over degrees of freedom



§ 44] ЗАКОН РАВНОРАСПРЕДЕЛЕНИЯ 151

Соответственно теплоемкость $c_p = c_v + 1$ равна

$$c_p = \frac{l+2}{2}. \qquad (44,2)$$

Таким образом, чисто классический идеальный газ должен обладать постоянной теплоемкостью. Формула (44,1) позволяет при этом высказать следующее правило: на каждую переменную в энергии $\varepsilon(p, q)$ молекулы приходится по равной доле 1/2 в теплоемкости c_v газа ($k/2$ в обычных единицах), или, что то же, по равной доле $T/2$ в его энергии. Это правило называют *законом равнораспределения*.

Имея в виду, что от поступательных и вращательных степеней свободы в энергию $\varepsilon(p, q)$ входят только соответствующие им импульсы, мы можем сказать, что каждая из этих степеней свободы вносит в теплоемкость вклад, равный 1/2. От каждой же колебательной степени свободы в энергию $\varepsilon(p, q)$ входит по две переменных (координата и импульс), и ее вклад в теплоемкость равен 1.

Nauka Moscow (1976)

Dynamical thermalization in finite isolated systems?

Fermi (1923); Fermi-Pasta-Ulam (FPU) problem (1955)

Chirikov criterion for onset of chaos (1959)

Novosibirsk => FPU: Izrailev, Chirikov Dokl. Akad. Nauk SSSR 166: 57 (1966)

Integrability of nonlinear Schrödinger equation
Zakharov, Shabat, Zh. Eksp. Teor. Fiz. 61: 118 (1971) + Toda lattice (1967)

Fermi-Pasta-Ulam problem (1955)


“In January 1951, Ulam and Teller came up with the Teller-Ulam design, which is the basis for all thermonuclear weapons. ...
"After the H-bomb was made," Bethe recalled, "reporters started to call Teller the father of the H-bomb. For the sake of history, I think it is more precise to say that Ulam is the father, because he provided the seed, and Teller is the mother, because he remained with the child. As for me, I guess I am the midwife." “
Wikipedia

Virtual visit of Ulam to Akademgorodok (1967)

[redacted] Новосибирск,90

ноября 6

ПРЕДСЕДАТЕЛЮ ПРЕЗИДИУМА СИБИРСКОГО ОТДЕЛЕНИЯ
АН СССР

академику М.А.ЛАВРЕНТЬЕВУ

Глубокоуважаемый Михаил Алексеевич!

Прошу Вас рассмотреть вопрос о приглашении в Академгородок проф.С.М.Улама в качестве гостя Академии наук СССР.
Необходимые документы прилагаются.

ДИРЕКТОР ИНСТИТУТА Г.И.БУДКЕР

invitation Ulam never got

Meeting of Chirikov and Ulam at Mathematical Congress, Moscow 1966
Invitation of Ulam to Novosibirsk for summer 1967 (letter of Budker to Lavrentiev Nov 1966 (initiated by Chirikov)) from Archive of Chirikov
http://www.quantware.ups-tlse.fr/chirikov/archive/ulamprog.pdf

Ulam method (1960) and Ulam networks (2010)

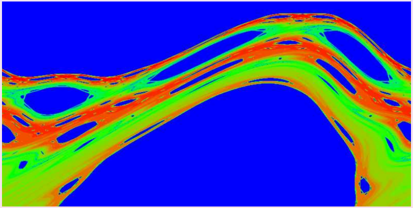
Outside note:

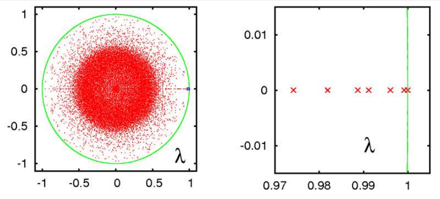
Ulam method for the Chirikov standard map → Markov chains, Google matrix

World Wide Web, PageRank algorithm → Brin, Page (1998);

Ulam networks → DS, Zhirov; Ermann, DS (2010);

Chirikov standard map → Frahm, DS (2010)





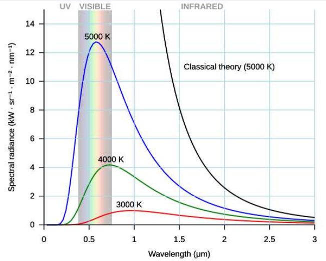
Google matrix analysis of directed networks: WWW, world trade, Wikipedia ...

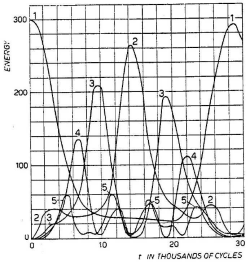
PageRank, CheiRank

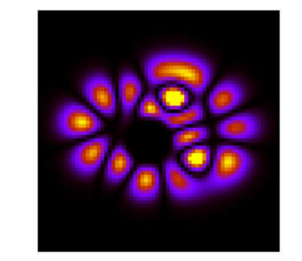
Chepelianski (2010);

Ermann, Frahm, DS Rev Mod Phys (2015)

Dynamical thermalization in finite systems







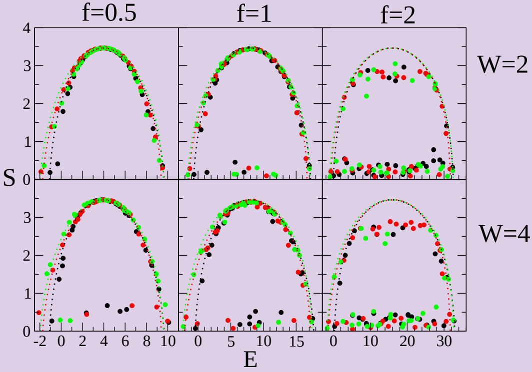
Planck's law (1900); Fermi-Pasta-Ulam problem (1955);
Bose-Einstein condensate (BEC) in Sinai-oscillator trap (2016) (left to right)
Ketterle group PRL (1995) => BEC in Sinai oscillator (not understood; 3d)

Quantum Gibbs ansatz instead of equipartition

DANSE model (1D) $i\hbar\partial\psi_n/\partial t = E_n\psi_n + \beta|\psi_n|^2\psi_n + V(\psi_{n+1}+\psi_{n-1})$

$$H = \sum_{m=1}^{N} \epsilon_m |C_m|^2 + \beta \sum_{m_1 m_2 m_3 m} V_{m_1 m_2 m_3 m} C_{m_1} C_{m_2} C^*{}_{m_1} C^*{}_m$$

with $\sum_m |C_m|^2 = 1$, $V_{mm' m_1 m_1'} \sim \ell^{-3/2}$, $E_n = f|n-n_0| + \delta E_n$, $|\delta E_n| < W/2$



Quantum Gibbs ansatz: $\rho_m - Z^{-1}\exp(-\epsilon_m/T)$, $Z = \sum_m \exp(-\epsilon_m/T)$

$S(E) = -\sum_m \rho_m \ln \rho_m$; $\rho_m = |C_m|^2$ (Ermann, DS NJP (2013))

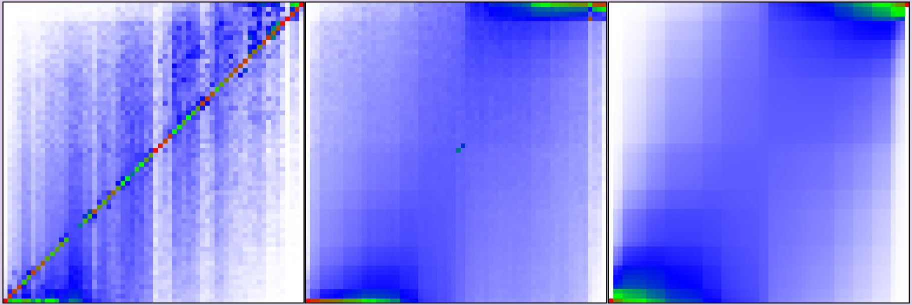
Dynamical thermalization in nonlinear chains

$$i\frac{\partial \psi_{n_x n_y}}{\partial t} = E_{n_x n_y}\psi_{n_x n_y} + \beta|\psi_{n_x n_y}|^2\psi_{n_x n_y} + (\psi_{n_x+1 n_y} + \psi_{n_x-1 n_y} + \psi_{n_x n_y+1} + \psi_{n_x n_y-1}). \quad (6)$$

Periodic boundary conditions are used for the $N \times N$ square lattice with $-N/2 \leqslant n_x, n_y \leqslant N/2$. However, here we use the extended version of this model assuming that

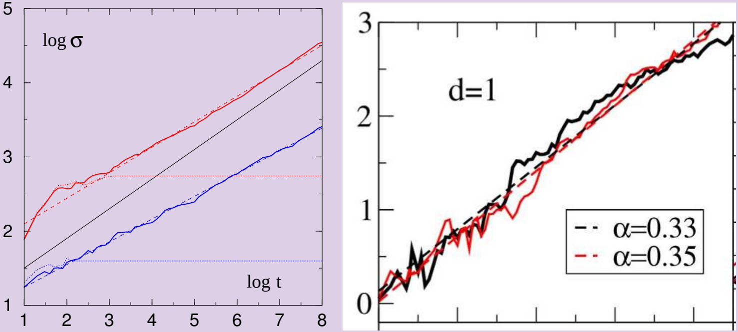
$$E_{n_x n_y} = \delta E_{n_x n_y} + f(n_x^2 + n_y^2), \quad -W/2 \leqslant \delta E_{n_x n_y} \leqslant W/2 \ (M3). \quad (7)$$

This is the $M3$ model with random values of energies $\delta E_{n_x n_y}$ in a given interval.



Nonlinear chains with disorder: 8×8 sites, $f = 1$, $W = 2$, $\beta = 1, 4$ (left, center); quantum Gibbs anzats (right); $t = 2 \times 10^6$ (Ermann, DS NJP (2013))

KAM and spreading in infinite nonlinear chains



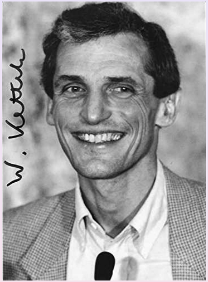
Left: DANSE at disorder $W = 2(red), 4(blue)$; $\beta = 1, 0$; $f = 0$, $t \leq 10^8$;
Right: kicked rotator with nonlinear phase shift $t \leq 10^9$; random (black) and $Tn^2/2$ (red) rotational phases, $\beta = 1$ (log-log scale)
DS (1993; 2012); Pikovsky, DS (2008); Ermann, DS JPA (2014)
Flach *et al.* JPA (2014)

Speading: $\sigma =< n^2 > \propto \ t^\alpha$, $\alpha \approx 0.34 \approx 2/5 < 1/2$ (random phases)
KAM border for $\beta \ll \beta_c \sim 1$?
J.Bourgain, W.-M.Wang J. Eur. Math. Soc. (2008)

Bose-Einstein condensates (BEC) with cold atoms

BEC dynamics => Gross-Pitaevskii equation (GPE)

Ketterle BEC experiment (1995))



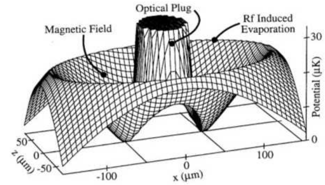


FIG. 1. Adiabatic potential due to the magnetic quadrupole field, the optical plug, and the rf. This cut of the three-dimensional potential is orthogonal to the propagation direction (y) of the blue-detuned laser. The symmetry axis of the quadrupole field is the z axis.

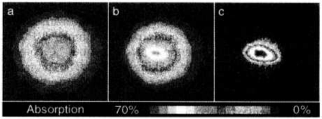


FIG. 2 (color). Two-dimensional probe absorption images, after 6 ms time of flight, showing evidence for BEC. (a) is the velocity distribution of a cloud cooled to just above the transition point, (b) just after the condensate appeared, and (c) after further evaporative cooling has left an almost pure condensate. (b) shows the difference between the isotropic thermal distribution and an elliptical core attributed to the expansion of a dense condensate. The width of the images is 870 μm. Gravitational acceleration during the probe delay displaces the cloud by only 0.2 mm along the z axis.

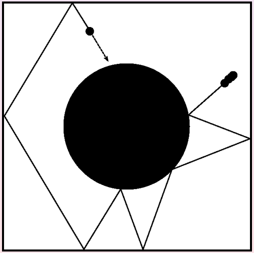
PRL (1995) (Nobel prize 2001)

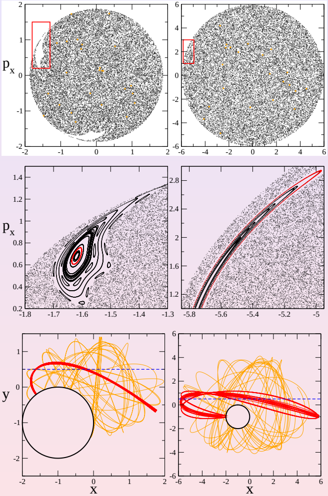
Sinai billiard → Sinai oscillator for BEC

Sinai billiard (1963; 1970) (Abel prize 2014)

$H = ({p_x}^2 + {p_y}^2)/2 + x^2/2 + y^2$

$r_d = 1, x_d = y_d = -1; E = 2; 18 \quad \rightarrow$





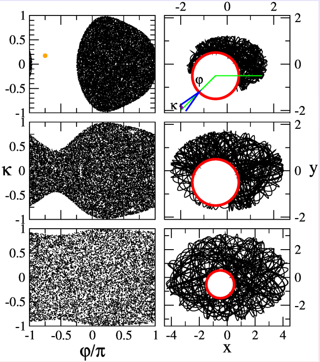
BEC dymanics in Sinai oscillator with GPE

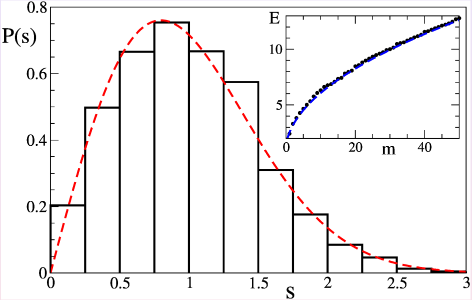
Ermann, Vergini, DS (2015-2017)

Classical and quantum chaos in Sinai oscillator

Poincaré sections (left)

$H = (p_x{}^2 + p_y{}^2)/2 + x^2/2 + y^2$; $r_d = 1$; $\hbar = 1$; $x_d = y_d = -0.5$; $E = 1.5; 3; 10$;





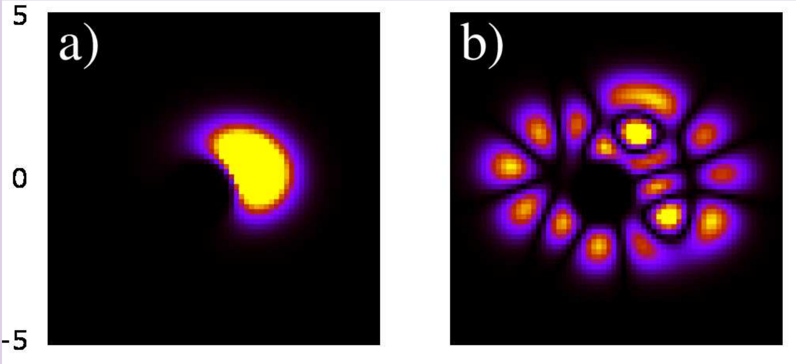
Bohigas-Giannoni-Schmit conjecture (1984); Ullmo Scholarpedia (2016)

Wigner-Dyson statistics of lowest 2500 energy levels unfolded (right)

Random matrix theory (Wigner (1967)); quantum chaos (e.g. Haake (2010))

Ermann, Vergini, DS (2015-2017)

Quantum chaos eigenstates in Sinai oscillator



Eigenstates at $\beta = 0$; ground state $m = 1$ and $m = 24$

Bose-Einstein THERMALIZATION anzats:

$\rho_m = 1/[\exp(E_m - E_g - \mu)/T) - 1]$;

$\rho_m =< |\psi_m|^2 >$, energy $\sum_m E_m \rho_m = E$,
entropy $S = -\sum_m \rho_m \ln \rho_m \rightarrow S(E)$

BEC in Sinai oscillator with GPE

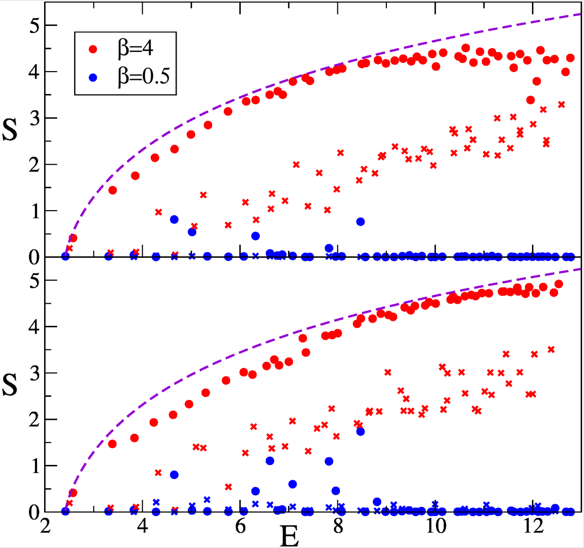
Gross-Pitaevskii equation (GPE or NSE) (Pitaevskii, Stringari (2003))

The BEC evolution in the Sinai oscillator trap is described by the GPE, which reads as

$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\vec{\nabla}^2\psi(\vec{r},t) + \left[\frac{m}{2}(\omega_x^2x^2+\omega_y^2y^2)+V_d(x,y)\right]\psi(\vec{r},t) + \beta|\psi(\vec{r},t)|^2\psi(\vec{r},t). \quad (2)$$

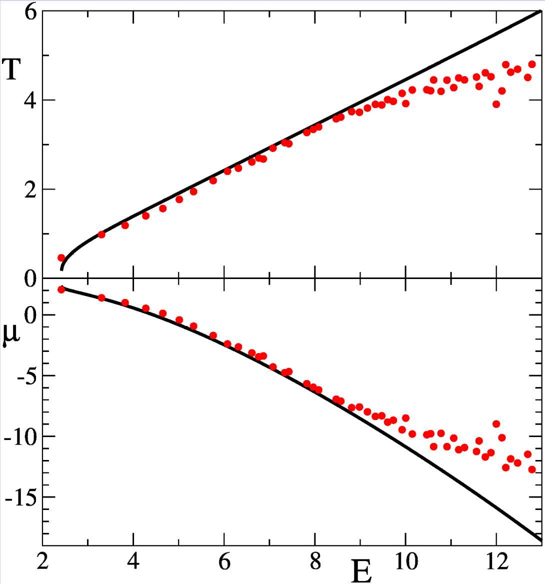
Here in (2), we use the same oscillator and disk parameters as in (1) and take $\hbar = 1$. The wave function is normalized to unity $W = \int |\psi(x,y)|^2 dx\, dy = 1$. Then, the parameter β describes the nonlinear interactions of atoms in BEC. All

Bose-Einstein anzats for dynamical thermalization



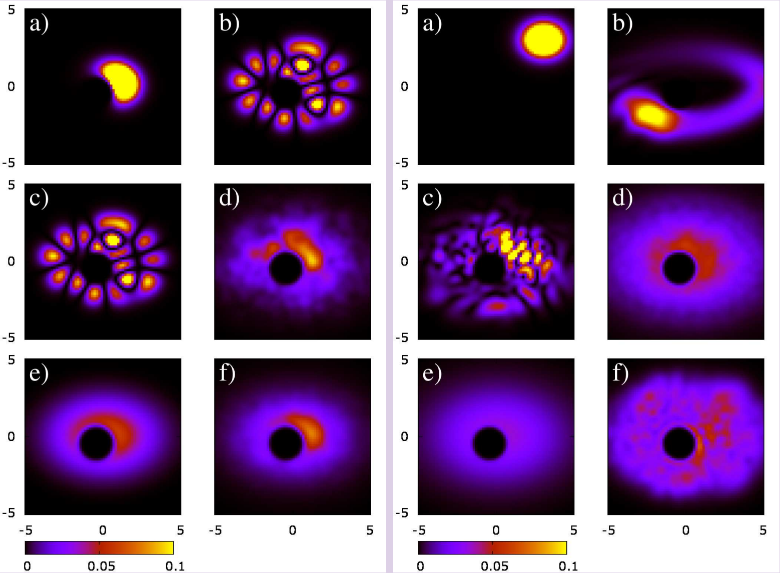
first 50 states; Sinai osc (dots), no disk (X); $500/1500 < t < 1500/2500$ (top/bottom); Bose-Einstein anzats (dashed) → no energy equipartition

Bose-Einstein anzats



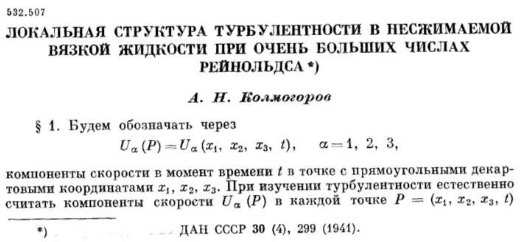
temperature and chemical potential dependence on energy ($\beta = 4$)

BEC time evolution



$\beta = 4$ various initial states

Kolmogorov turbulence defeated by Anderson localization



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ЛОКАЛЬНАЯ СТРУКТУРА ТУРБУЛЕНТНОСТИ В НЕСЖИМАЕМОЙ ВЯЗКОЙ ЖИДКОСТИ ПРИ ОЧЕНЬ БОЛЬШИХ ЧИСЛАХ РЕЙНОЛЬДСА *)

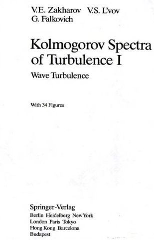
А. Н. Колмогоров

§ 1. Будем обозначать через

$$U_\alpha(P) = U_\alpha(x_1, x_2, x_3, t), \qquad \alpha = 1, 2, 3,$$

компоненты скорости в момент времени t в точке с прямоугольными декартовыми координатами x_1, x_2, x_3. При изучении турбулентности естественно считать компоненты скорости $U_\alpha(P)$ в каждой точке $P = (x_1, x_2, x_3, t)$

*) ДАН СССР 30 (4), 299 (1941).



V.E. Zakharov V.S. L'vov G. Falkovich

Kolmogorov Spectra of Turbulence I

Wave Turbulence

With 34 Figures

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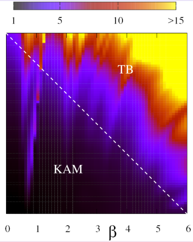


Kolmogorov (1941) - energy flow over space scales $E_k \propto 1/k^{5/3}$;
NSU $\rightarrow$ Zakharov-Filonenko (1967) capilary waves $E_k \propto 1/k^{7/4} \rightarrow$:
"In the theory of weak turbulence nonlinearity of waves is assumed to be small; this enables us, using the hypothesis of the random nature of the phase of individual waves, to obtain the kinetic equation for the mean square of the wave amplitudes."
extentions $\rightarrow$ Zakharov, L'vov, Falkovich(1992); Nazarenko(2011)

BEC in Sinai-oscillator trap with driving

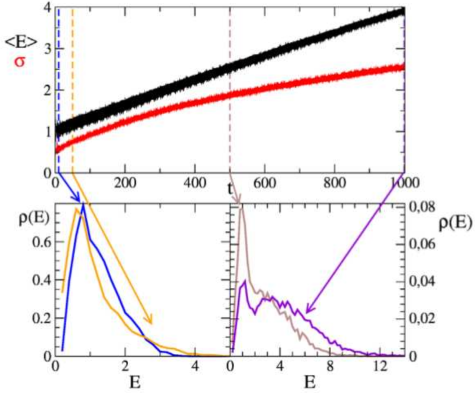


FIG. 1. Classical time evolution of average energy $\langle E \rangle$ and its standard deviation σ for $f = 0.4$. The data are obtained from 10^4 trajectories with random initial conditions at $\langle E \rangle = 1$ and $\sigma = 0.5$. Top panel: $\langle E(t) \rangle$ and $\sigma(t)$ are shown by black and red (gray) curves, respectively. Bottom panels show probability distribution of trajectories $\rho(E, t)$ for (a) $t = 10, 50$ [blue (black),

$H = H_S(x, y, p_x, p_y) + fx \sin \omega t$ (classical dynamics)

Energy flow to high modes

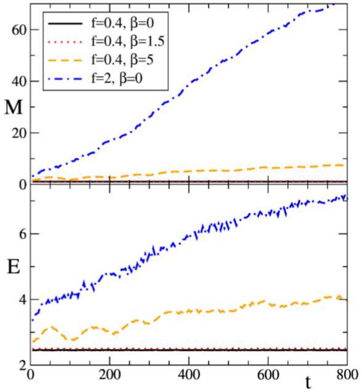


FIG. 2. Time evolution of M (top panel) and energy E (bottom panel) for GPE (2) averaged over time intervals $\Delta t = 1$. The initial state is the ground state of (2) at $\beta = 0$, $f = 0$ [see Fig. 5(a) in Ref. [28]]. Both panels show the cases of $f = 0.4, \beta = 0$ (black solid lines), $f = 0.4$, $\beta = 1.5$ [red (gray) dotted lines], $f = 0.4$, $\beta = 5$ [orange(gray) dashed lines], $f = 2, \beta = 0$ [blue (gray) dot-dashed lines].

$M = \sum_k k\rho_k$; $\ell_\phi \approx 2\pi\rho_c(D/\omega)^2 \approx 2f^2{\omega_x}^2 E^{3/2}/\omega^4$ (ground state);
Anderson photonic localization $\rho_k \propto \exp(-2E/\omega\ell_\phi)$ for $\beta = 0$, $f < f_c \approx 1.5$

Energy flow to high modes

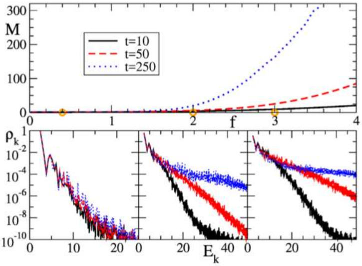


FIG. 3. Top panel shows M as a function of driven force f for linear case ($\beta = 0$). Bottom panels show probability distribution ρ_k, averaged over time interval $\Delta t = 5$, as a function of eigenenergies E_k with $t = 10$ in black solid lines, $t = 50$ in the red (gray) dashed lines, and $t = 250$ in blue (gray) dotted lines. Left, center, and right bottom panels show the cases of $f = 0.4$, 2, and 3, respectively [highlighted with orange (gray) circles in top panel].

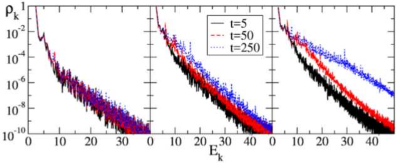


FIG. 4. Same as in bottom panels of Fig. 3 for $f = 0.3$, $\beta = 1.5$ (left panel); $f = 0.5, \beta = 5$ (center panel); $f = 1, \beta = 5$ (right panel).

M and probability distributions ρ_k; left $\rightarrow \beta = 0$;
right $f = 0.3, \beta = 1.5$, $f = 0.5, \beta = 5$, $f = 1, \beta = 5$

Turbulence phase diagram

$$f_c r_d/\hbar\omega_x \approx 1.5[1 - \beta_c/(6\hbar\omega_x r_d{}^2)] \qquad (3)$$

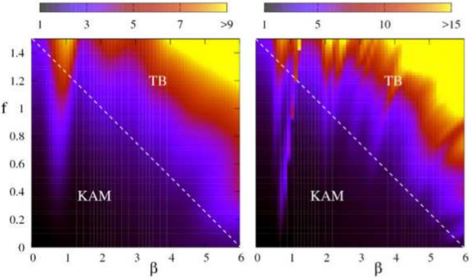
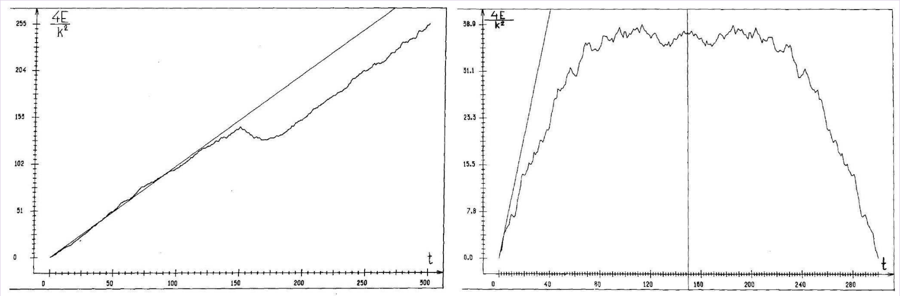


FIG. 5. Number of modes M shown by color (grayness) in the plane of parameters f and β (average is done in the time intervals $100 \leq t \leq 150$ and $250 \leq t \leq 300$ in left and right panels, respectively). The approximate separation of KAM or insulator phase (KAM) and delocalized turbulent or metallic phase (TB) is shown by the white line (3).

Thus there is a stability domain where the Kolmogorov flow from large to small scales is defeated by the Anderson localization and KAM-integrability

Quantum version of Boltzmann-Loschmidt dispute

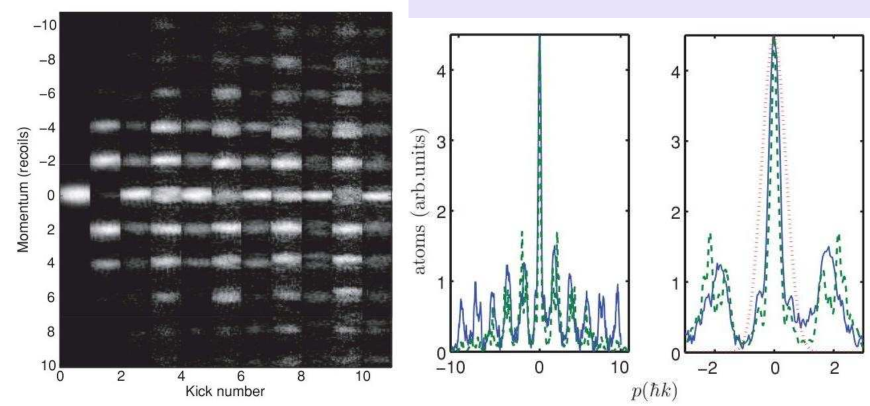
* Time reversal for the Chirikov standard map



BESM-6 computation, rescaled energy or squared momentum vs. time t:
$K = 5$, $\hbar = 0$ (left), $\hbar = 1/4$ (right), DS (1983)

proposal for BEC in kicked optical lattice (Martin, Georgeot, DS (2008))
$k = K/\hbar, \hbar = T = 4\pi + \epsilon(forward), 4\pi - \epsilon(back)$;
$\cos x \to -\cos x$ by a π shift in x

Time reversal of atomic matter waves by Hoogerland group (2011)



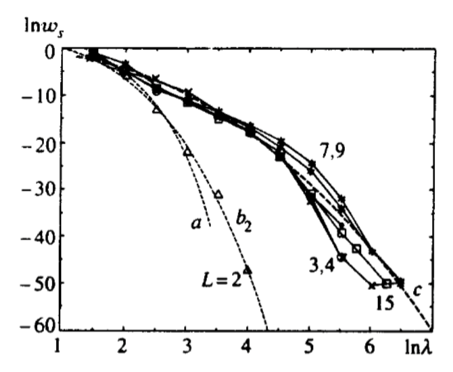
$k = 2 - 3$, $T = 4\pi \pm \epsilon$ (Ullah, Hoogerland (2011))

- Ultracold ^{67}Rb atoms BEC: 10^4 atoms, $T_{cool} = 50nK$, $\lambda = 2\pi/k_L = 760nm$, $\epsilon = 1$; $5 + 5$ kicks; right panel shows zoom near initial distribution shown by red dotted curve (initial/final width is $0.43/0.21$ recoils; full/dashed curve for experiment/numerics).

Discussion (open problems)

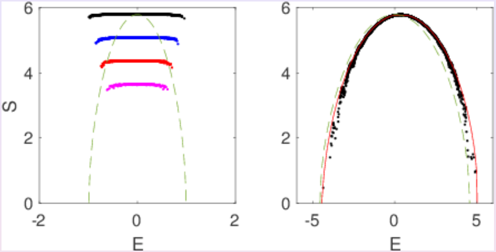
Classical systems with many freedoms:

- Quantum Gibbs instead of equipartition
- KAM border for pure-point spectrum of Anderson localization
- fast Arnold diffusion with many freedoms (fig, $\lambda = 1/\sqrt{K}$)
  $D \sim {w_s}^2 \propto K^{6.6}$ coupled standard maps (Chirikov, Vecheslavov (1997))
  chaos measure of triplet resonances $\mu \sim K$ (Mulansky et al (2011))



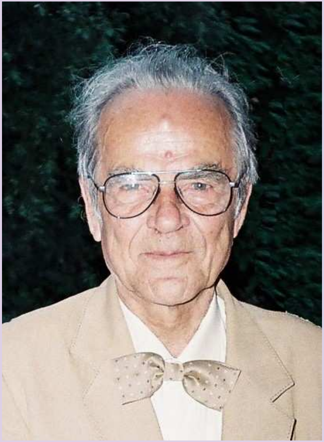
Discussion (open problems)

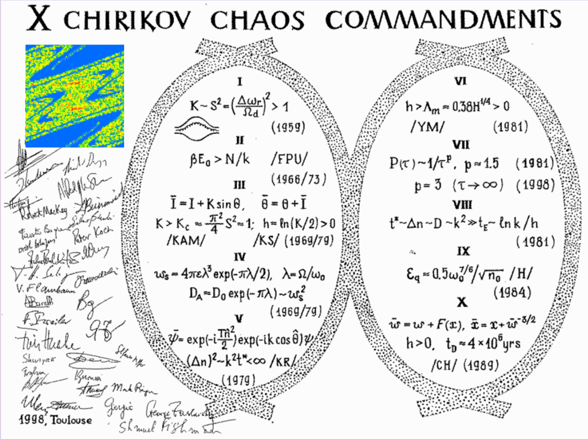
Quantum systems with many freedoms:



- Dynamical thermalization in quantum many-body systems:
  Åberg criterion PRL (1990); Jacquod, DS (1997)
  Two-body random interaction model (TBRIM) for
  quantum dots and SYK black holes ..., (Kolovsky, DS (2017))
  quantum dot $\rightarrow \delta E \approx g^{2/3}\Delta$; SYK black hole $\rightarrow$?
- Time reversal of atoms, waves, sound ...

Boris Chirikov - Sputnik of Chaos





Full References (in order of appearance):

Archive of Boris Chirikov:
http://www.quantware.ups-tlse.fr/chirikov/archive.html
Talk of Boris Chirikov at Int. Conf.
"Classical Chaos and its Quantum Manifestations", Toulouse July 16-18, 1998:
slides: http://www.quantware.ups-tlse.fr/dima/introd/chirikov1998tlse.pdf
video: http://www.quantware.ups-tlse.fr/dima/conf98/Chirikovcelebration1998.mov
Photos of scientists and stamps are from Wikipedia and other open internet sources

R1. J. Loschmidt, *Uber den Zustand des Warmegleichgewichts eines Systems von Korpern mit Rucksicht auf die Schwerkraft*, Sitzungsberichte der Akademie der Wissenschaften, Wien, Vol. II **73**, 128-142 (1876)
R2. L. Boltzmann, *Uber die Beziehung eines allgemeine mechanischen Satzes zum zweiten Haupsatze der Warmetheorie*, Sitzungsberichte der Akademie der Wissenschaften, Wien, Vol. II **75**, 67-73 (1877)
R3. H. Poincaré, *Les méthodes nouvelles de mécanique céleste* , Gauthier-Villars, Paris (1893) [New Methods of Celestial Mechanics, 3 vols. English trans., Ed. D.Goroff, AIP Press (1967); ISBN 1-56396-117-2]
R4. B.V.Chirikov, *Resonance processes in magnetic traps*, At. Energ. **6**, 630 (1959) (in Russian) [English translation: J. Nucl. Energy Part C: Plasma Phys. **1**, 253 (1960)]

Full References (in order of appearance - cont):

R5. B.V.Chirikov, *Research concerning the theory of nonlinear resonance and stochasticity*, Preprint N 267, Institute of Nuclear Physics, Novosibirsk (1969) (in Russian (here)) [Engl. Transl., CERN Trans. 71 - 40, Geneva, October (1971), Translated by A.T.Sanders

R6. B.V.Chirikov, *A universal instability of many-dimensional oscillator systems*, Phys. Rep. **52**, 263 (1979)

R7. B.Chirikov, D.Shepelyansky, *Chirikov standard map*, Scholarpedia **3(3)**, p.3550 (2008)

R8. A.N.Kolmogorov, *On the conservation of conditionally periodic motions under small perturbation of the Hamiltonian*, Dokl. Akad. Nauk. SSR **98**, 527-530 (1954)

R9. J.K.Moser, *On invariant curves of area-preserving mappings of an annulus*, Nach. Akad. Wiss. Gettingen, Math. Phys. Kl. II **1**, 1-20 (1962)

R10. V.I.Arnold, *Proof of a Theorem by A. N. Kolmogorov on the invariance of quasi-periodic motions under small perturbations of the Hamiltonian*, Russian Math. Survey **18**, 13-40 (1963); *Small divisor problems in classical and Celestial Mechanics*, Russian Math. Survey **18**, 85-191 (1963)

R11. P.W. Anderson, *Absence of Diffusion in Certain Random Lattices*, Phys. Rev. **109**, 1492 (1958)

Full References (in order of appearance - cont):

R12. B.V.Chirikov, F.M.Izrailev and D.L.Shepelyansky, *Dynamical stochasticity in classical and quantum mechanics*, Sov. Scient. Rev. C **2**, 209 (1981) [Section C - Mathematical Physics Reviews, Ed. S.P.Novikov vol.2, Harwood Acad. Publ., Chur, Switzerland (1981)]
R13. S.Fishman, D.R.Grempel and R.E.Prange, *Chaos, quantum recurrences, and Anderson localization*, Phys. Rev. Lett. **49**, 509 (1982)
R14. F.L.Moore, J.C.Robinson, C.F.Bharucha, B.Sundaram and M.G.Raizen, *Atom optics realization of the quantum δ-kicked rotor*, Phys. Rev. Lett. **75**, 4598 (1995)
R15. J.W.Gibbs, *Method of Geometrical Representation of the Thermodynamic Properties of Substances by Means of Surfaces,* Transactions of the Connecticut Academy of Arts and Sciences **2**, 382-404 (Dec. 1873)
R16. M.Planck, *Uber eine Verbesserung der Wien'schen Spectralgleichung*, Verhandlungen der Deutschen Physikalischen Gesellschaft. **2**, 202-204 (1900); Translated in D. ter Haar, *On an Improvement of Wien's Equation for the Spectrum*, The Old Quantum Theory, Pergamon Press. 79-81 (1967)
R17. L.D.Landau and E.M.Lifshitz, *Statistical physics v.V*, Nauka, Moskva (1976)
R18. E.Fermi, *Dimostrazione che in generale un sistema meccanico normale È QUASI a ergodico*, Nuovo Cimento (1923) **25**, 267 (1923), https://doi.org/10.1007/BF02959600 ; ibid. **26**, 105 (1923)

Full References (in order of appearance - cont):

R19. E. Fermi, J. Pasta, S. Ulam, and M. Tsingou, *Studies of nonlinear problems. I*, Los Alamos Report No. LA-1940, 1955 (unpublished); E. Fermi, Collected Papers, University of Chicago Press, Chicago, **2**, 978 (1965)
R20. F.M.Izrailev and B.V.Chirikov, *Statistical properties of a non-linear string*, Dokl. Akad. Nauk SSSR **166(1)**, 57 (1966) [in Russian] (English trans. Sov. Phys. - Doklady **11(1)**, 30 (1966))
R21. V.E.Zakharov, A.B.Shabat, *Exact theory of two-dimensional self-focusing and one-dinemsional self-modulation of waves in nonlinear media*, Zh. Eksp. Teor. Fiz. **61**, 118 (1971)
R22. M.Toda, *Studies of a non-linear lattice*, Phys. Rep. **18**, 1 (1975)
R23. S.M. Ulam, *A Collection of mathematical problems*, Vol. 8 of Interscience tracs in pure and applied mathematics, Interscience, New York, p.73 (1960).
R24. D.L.Shepelyansky and O.V.Zhirov, *Google matrix, dynamical attractors and Ulam networks*, Phys. Rev. E **81**, 036213 (2010)
R25. K.M.Frahm and D.L.Shepelyansky, *Ulam method for the Chirikov standard map*, Eur. Phys. J. B **76**, 57 (2010)
R26. A.D.Chepelianksii, *Towards physical laws for software architecture* arXiv:1003.5455[cs.SE] (2010)
R27. L.Ermann and D.L.Shepelyansky, *Ulam method and fractal Weyl law for Perron-Frobenius operators*, Eur. Phys. J. B **75**, 299 (2010)

Full References (in order of appearance - cont):

R28. S.Brin and L.Page, *The anatomy of a large-scale hypertextual Web search engine*, Comp. Networks ISDN Systems **30**, 107 (1998)
R29. L.Ermann, K.M.Frahm and D.L.Shepelyansky, *Google matrix analysis of directed networks*, Rev. Mod. Phys. **87**, 1261 (2015)
R30. L.Ermann and D.L.Shepelyansky, *Quantum Gibbs distribution from dynamical thermalization in classical nonlinear lattices*, New J. Phys. **15**, 123004 (2013)
R31. D.L.Shepelyansky, *Delocalization of Quantum Chaos by Weak Nonlinearity*, Phys. Rev. Lett. **70**, 1787 (1993)
R32. A.S.Pikovsky and D.L.Shepelyansky, *Destruction of Anderson localization by a weak nonlinearity*, Phys. Rev. Lett. **100**, 094101 (2008)
R33. L.Ermann and D.L.Shepelyansky, *Destruction of Anderson localization by nonlinearity in kicked rotator at different effective dimensions*, J. Phys. A: Math. Theor. **47**, 335101 (2014)
R34. D.L.Shepelyansky, *Kolmogorov turbulence, Anderson localization and KAM integrability*, Euro. Phys. J. B **85**, 199 (2012)
R35. T.V.Laptyeva, M.V.Ivanchenko, and S.Flach, *Nonlinear lattice waves in heterogeneous media*, J. Phys. A: Math. Theor. **47**, 493001 (2014)
R36. J. Bourgain and W.-M. Wang, *Quasi-periodic solutions of nonlinear random Schrodinger equations*, J. Eur. Math. Soc. **10**, 1 (2008)

Full References (in order of appearance - cont):

R37. K.B. Davis, M.-O. Mewes, M.R. Andrews, N.J. van Druten, D.S. Durfee, D.M. Kurn, and W. Ketterle, *Bose-Einstein Condensation in a Gas of Sodium Atoms*, Phys. Rev. Lett. **75**, 3969 (1995)

R38. Ya.G.Sinai, *On the foundations of the ergodic hypothesis for a dynamical system of statistical mechanics*, Doklady Akademii Nauk SSSR (in Russian) **153(6)**, 1261-1264 (1963) [in English, Sov. Math Dokl. **4**, 1818-1822 (1963)]

R39. Ya.G. Sinai, *Dynamical systems with elastic reflections. Ergodic properties of dispersing billiards*, Uspekhi Mat. Nauk **25(2)**, 141 (1970) [English trans.: Russian Math. Surveys **25(2)**, 137 (1970)]

R40. L.Ermann, E.Vergini and D.L.Shepelyansky, *Dynamical thermalization of Bose-Einstein condensate in Bunimovich stadium*, Europhys. Lett. **111**, 50009 (2015); *Dynamics and thermalization a Bose-Einstein condensate in a Sinai-oscillator trap*, Phys. Rev. A **94**, 013618 (2016);
Kolmogorov turbulence defeated by Anderson localization for a Bose-Einstein condensate in a Sinai-oscillator trap, Phys. Rev. Lett. **119**, 054103 (2017)

R41. E. Wigner, *Random matrices in physics*, SIAM Review **9(1)**, 1 (1967)

R42. O. Bohigas, M.J. Giannoni, and C. Schmit, *Characterization of chaotic quantum spectra and universality of level fluctuation laws*, Phys. Rev. Lett. **52**, 1 (1984)

R43. D.Ullmo, *Bohigas-Giannoni-Schmit conjecture*, Scholarpedia **11(9)**, 31721 (2016)

R44. F. Haake, *Quantum signatures of chaos*, Springer, Berlin (2010)

Full References (in order of appearance - cont):

R45. L. Pitaevskii, and S. Stringari, *Bose-Einstein condensation*, Oxford Univ. Press (2003)
R46. A.N. Kolmogorov, *The local structure of turbulence in an incompressible liquid for very large Reynolds numbers*, Dokl. Akad. Nauk SSSR **30**, 299 (1941); *Dissipation of energy in the locally isotropic turbulence*, **32**, 19 (1941) [in Russian] (English trans. Proc. R. Soc. Ser. A **434**, 19 (1991); **434**, 15 (1991))
R47. V.E. Zakharov and N.N. Filonenko, *Weak turbulence of capillary waves*, J. Appl. Mech. Tech. Phys. **8 (5)**, 37 (1967)
R48. V.E. Zhakharov, V. S. L'vov and G. Falkovich, *Kolmogorov spectra of turbulence*, Springer-Verlag, Berlin (1992)
R49. S. Nazarenko, *Wave turbulence*, Springer-Verlag, Berlin (2011)
R50. D.L.Shepelyansky, *Some statistical properties of simple classically stochastic quantum systems*, Physica D **8**, 208 (1983)
R51. J.Martin, B.Georgeot and D.L.Shepelyansky, *Time reversal of Bose-Einstein condensates*, Phys. Rev. Lett. **101**, 074102 (2008)
R52. A.Ullah and M.D. Hoogerland, *Experimental observation of Loschmidt time reversal of a quantum chaotic system*, Phys. Rev. E **83**, 046218 (2011)

Full References (in order of appearance - cont):

R53. B.V.Chirikov and V.V.Vecheslavov, *Arnold diffusion in large systems*, JETP 85(3): 616 (1997) (Zh. Eksp. Teor. Fiz. 112: 1132 (1997))
R54. M.Mulansky, K.Ahnert, A.Pikovsky, D.L.Shepelyansky, *Strong and weak chaos in weakly nonintegrable many-body Hamiltonian systems*, J. Stat. Phys. 145: 1256 (2011)
R55. S.Åberg, *Onset of chaos in rapidly rotating nuclei*, Phys. Rev. Lett. **64**, 3119 (1990)
R56. P.Jacquod and D.L.Shepelyansky, *Emergence of quantum chaos in finite interacting Fermi systems*, Phys. Rev. Lett. **79**, 1837 (1997)
R57. A.R.Kolovsky and D.L.Shepelyansky, *Dynamical thermalization in isolated quantum dots and black holes*, EPL **117**, 10003 (2017)