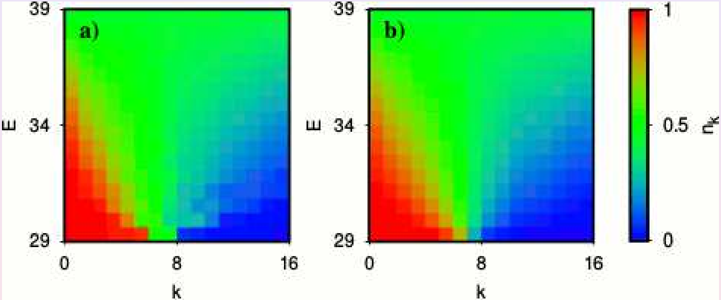
Dynamical thermalization and quantum small world

Dima Shepelyansky (CNRS, Toulouse)
www.quantware.ups-tlse.fr/dima

with L.Ermann (CNEA AR), K.Frahm (UPS Toulouse), A.Kolovsky (RAS RU)



Orbital occupation numbers n_k for interacting fermionic atoms in a Sinai-oscillator trap at Åberg parameter $A = 3.5$; numerical result from many-body eigenstates (left), theoretical Fermi-Dirac distribution (right); $M = 16$ orbitals and $L = 7$ fermions (Frahm, Ermann, DS (2019) [R1])

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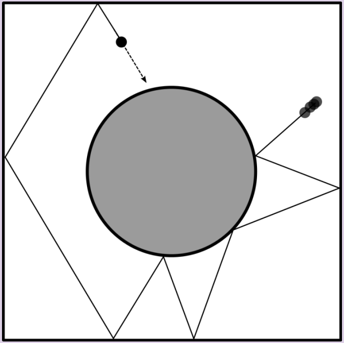
Loschmidt - Boltzmann dispute on time reversibility (1876-1877)

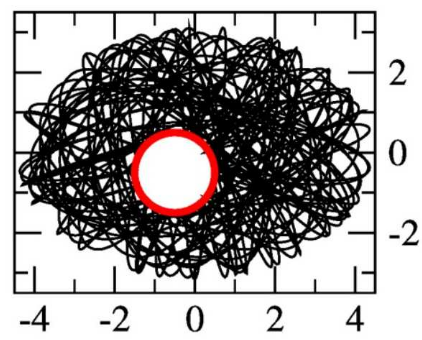
* irreversible statistical laws from reversible dynamical equations

Boltzmann (1872) [R2], Loschmidt (1876) [R3], Boltzmann (1877) [R4]

Dynamical Chaos as the origin of statistical laws

* dynamical systems with a few degrees of freedom





Poincaré (1893) [R5], Kolmogorov-Arnold-Moser theorem (1954-1963) [R6], Sinai (1963) [R7], Chirikov (1959-1979) [R8]

Time reversal breaking due to exponential instability of chaotic dynamics

Sinai billiard (left) and

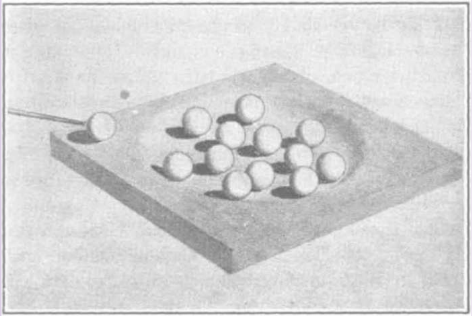
Sinai oscillator [R1,R9] (right) → Ketterle (1995-2002) [R10] (BEC 3d)

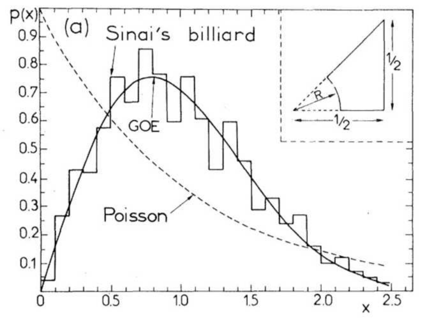
$H = ({p_x}^2 + {p_y}^2 + {\omega_x}^2 x^2 + {\omega_y}^2 y^2)/2 + V_d(x, y); \omega_x = 1, \omega_y = \sqrt{2}$,

elastic disk $r_d = 1$, $x_d = y_d = -0.5$

Quantum Chaos and many-body systems

* nuclear scattering and spectra of nuclei





Einstein (1917) [R11], Bohr (1936) [R12],
Wigner and Random Matrix theory (RMT) (1955-1967) [R13],
Bohigas-Giannoni-Schmit (1984) [R14]

Bohr billiard (left) [R12] and RMT validation for Sinai billiard (right)[R14]

Two-body Random Interaction Model (TBRIM)

* two-body interactions in nature but RMT statistics

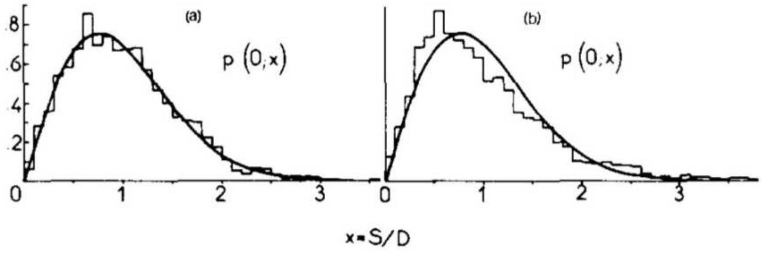


Fig.4. Nearest-neighbour spacing distributions in the ground state region (a) Gaussian orthogonal ensemble; (b) a two-body random hamiltonian ensemble. The curve is the same as in fig. 2 for $k = 0$.

Bohigas, Flores (1971) fig of [R15]; French, Wong (1971) [R16]

Same Hamiltonian as SYK black hole model (TBRIM=SYK)

Sachdev, Ye (1993) [R17], Kitaev (2015) [R18], Sachdev (2015) [R19], Garcia-Garcia, Verbaarschot (2016) [R20]

TBRIM with one-particle orbitals

* fermions of one-particle orbitals with two-body random interactions

TBRIM Hamiltonian of L spinless fermions on M energy orbitals ϵ_k:

$$\hat{H} = \hat{H}_0 + \hat{H}_{int} \ , \ \hat{H}_0 = \frac{1}{\sqrt{M}} \sum_{k=1}^{M} v_k \hat{c}_k^\dagger \hat{c}_k \ , \hat{H}_{int} = \frac{1}{\sqrt{2M^3}} \sum_{ijkl} J_{ij,kl} \hat{c}_i^\dagger \hat{c}_j^\dagger \hat{c}_k \hat{c}_l \ ,$$

$\hat{c}_i^\dagger, \hat{c}_i$ are fermion operators for the M orbitals; v_k ($J_{ij,kl}$) are real Gaussian random variables with zero mean and variance $\langle v_k^2 \rangle = V^2$ ($\langle J_{ij,kl}^2 \rangle = J^2(1+\delta_{ik}\delta_{jl})$) so that the non-interacting orbital one-particle energies are given by $\epsilon_k = v_k/\sqrt{M}$. The variance of the interaction matrix elements is chosen such they correspond to a GOE-matrix of size $M_2 \times M_2$ with $M_2 = M(M-1)/2$. The matrix size of H is $N = M!/L!(M-L)!$ with $K = 1 + L(M-L) + L(L-1)(M-L)(M-L-1)/4$ nonzero links.

One-particle energy spacing $\Delta = \Delta_1 \approx V/M^{3/2} \gg \Delta_L \propto 1/N$;
two-body coupling element $U \approx J/M^{3/2}$; conductance $g = \Delta/U \approx V/J \gg 1$

Åberg (1990,1992) [R21,R22] (random signs); Flambaum, Izrailev (1997) [R23]; Jacquod, DS (1997) [R24] (TBRIM quantum chaos border); DS (2001) [R25]; Kolovsky, DS (2017) [R26]; Frahm, DS (2018) [R27] (link to SYK)

Onset of quantum chaos in finite many-body systems

COMMON LORE IN NUCLEAR PHYSICS:
energy level spacing drops exponentially with a number of particles L (e.g. fermions) or energy excitation above Fermi energy (or density of states grows exponentially) and thus an exponentially small two-body interaction leads to emergence of quantum chaos, RMT, thermalization

see e.g.
A.Bohr, B.Mottelson (1969) [28];
Guhr, Muller-Groeling, Weidenmuller (1998) [R29];
Flambaum, Gribakin, Sushkov (1997) [R30]

Åberg criterion for many-body quantum chaos

Onset of quantum chaos in systems with two-body interactions:
spacing between adjacent energy levels drops exponentially
with number of particles L: $\Delta_L \propto \exp(-L)$.
Interaction induced coupling, two-body matrix element between directly coupled states $U_c = U$.

Spacing between directly coupled states: $\Delta_c \gg \Delta_L$.

Åberg criterion for onset of quantum chaos: $U_c \approx \Delta_c \gg \Delta_L$ $(A = U_c/\Delta_c > 1)$

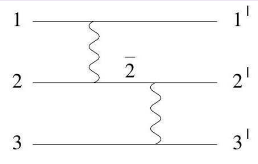
Åberg (1990,1992) [R21,R22] (random signs of interaction, argument refers on Weidenmuller works even if in (1998) [R29] there is no theory for that);
DS, Sushkov (1997) [R31] (argument of 3 interacting particles);
Jacquod, DS (1997) [R24]
$\rightarrow$ dynamical thermalization near Fermi energy in TBRIM:

$$\delta E > \delta E_{ch} \approx g^{2/3}\Delta \gg \Delta;$$

$\rightarrow$ dynamical thermalization border/conjecture (DTC)
(in metalic quantum dots $U/\Delta = J/V \approx 1/g \ll 1$,
$g = E_{Thouless}/\Delta \gg 1$ dot conductance)
Analytical confirmation of DTC for TBRIM:
Gornyi, Mirlin, Polyakov *et al.* (2016,2017) [R32,R33]

Three interacting particles in a metalic dot

M one-particle levels with average spacing $\Delta \sim B/M$
(e.g. 2d Anderson model of size much smaller than localization length)



Interaction coupling matrix element $U_2 \sim U_c$
Density of two-particle states
$\rho_2 \approx 1/\Delta_2 \sim M^2/B \sim M/\Delta$
Density of three-particle states
$\rho_3 \approx 1/\Delta_3 \sim M^3/B \sim \Delta/{\Delta_2}^2$

The matrix element between initial three-body state $|123>$ and final state $|1'2'3'>$ is given by diagram presented in Fig. with intermediate state $|1'\bar{2}3>$

$$U_3 = \sum_{\bar{2}} \frac{<12|U_{12}|1'\bar{2}><\bar{2}3|U_{23}|2'3'>}{(E_1+E_2+E_3-E_{1'}-E_{\bar{2}}-E_3)} \sim \frac{{U_c}^2}{\Delta} \;;$$

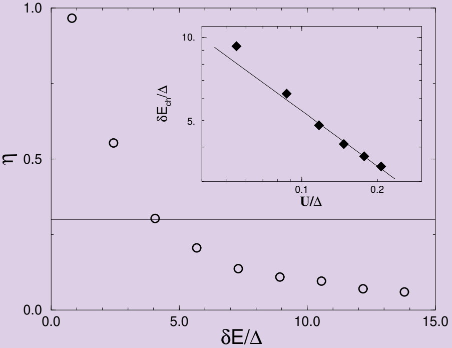
the summation is carried out only over single particle states $\bar{2}$ and hence the minimal detuning in the dominator is about Δ. Mixing of 3-particle levels takes place when $U_3 \sim {U_c}^2/\Delta \sim \Delta_3 \sim {\Delta_2}^2/\Delta$. Hence the transition from Poisson to Wigner-Dyson statistics for three interacting particles takes place at

$$U_c \sim \Delta_c \approx \Delta_2 \gg \Delta_3 \;\; (\text{Åberg parameter } A > 1)$$

DS, Sushkov (1997) [R31]

Quantum chaos and DTC in TBRIM

$U \approx J/M^{3/2}, \Delta \approx V/M^{3/2}; M = 2L = 2n = m$



exitation energy above ground state $\delta E \approx T\delta n$ with temperature T; density of effectively coupled TIP states $\rho_{2ef} \sim \epsilon/\Delta^2$, number of excited electrons $\delta n \sim Tn/\epsilon_F \sim T/\Delta$ with $\rho_{2ef} \sim T/\Delta^2$
$n = L = 6$, $U/\Delta = 0.15$;
inset: $\eta = 0.3$, line is theory at $C = 1.08$

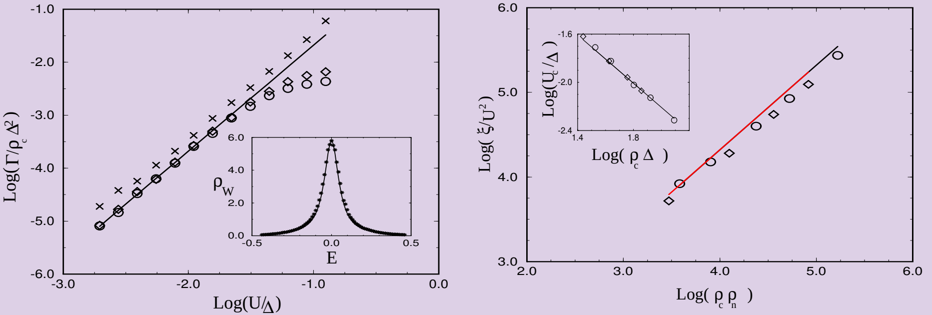
[$\eta = \int_0^{s_0}(P(s) - P_W(s))ds / \int_0^{s_0}(P_P(s) - P_{WD}(s))ds$, intersection point $s_0 = 0.4729...$; $\eta = 1$ for $P(s) = P_P(s)$, 0 for $P(s) = P_W(s)$]

$U > U_c \approx C/\rho_2 n^2 \approx C/(\rho_{2ef}(\delta n)^2)$ => $T_c = C\Delta(\Delta/U)^{1/3}$

$\delta E > \delta E_{ch} \approx T\delta n \approx C\Delta(\delta/U)^{2/3} \approx g^{2/3}\Delta$ => dynamical thermalization border (in metals $U \approx \Delta/g$, $g \gg 1$ conductance)

Jacquod, DS (1997) [R24]; [q-dot experiment Sivan *et al.* (1994) [R34]]

Breit-Wigner width and participation ratio in TBRIM



$n/m = 3/17; 3/130; 4/60$

$n = 3, 4; 30 \leq m \leq 130$

Breit-Wigner distribution in many-body quantum chaos regime

$\rho_W(E - E_n) = \sum_\lambda |\psi_\lambda(n)|^2 \delta(E - E_\lambda) = \Gamma/[2\pi((E - E_n)^2 + \Gamma^2/4)]$

Fermi golden rule: $\Gamma = 2\pi < U^2 > \rho_c = 2\pi U^2 \rho_c/3$

number of populated states IPR: $\xi \approx \Gamma \rho_n \approx 2U^2 \rho_c \rho_n$

Georgeot, DS (1997) [R35]

Dynamical thermalization ansatz for TBRIM

At $g \gg 1$ => Fermi-Dirac thermal distribution of M one-particle orbitals:

$$n_k = \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1} \; ; \;\; \beta = 1/T \; ,$$

with the chemical potential μ determined by the conservation of number of fermions $\sum_{k=1}^{M} n_k = L$.
At a given temperature T, the system energy E and von Neumann entropy S are

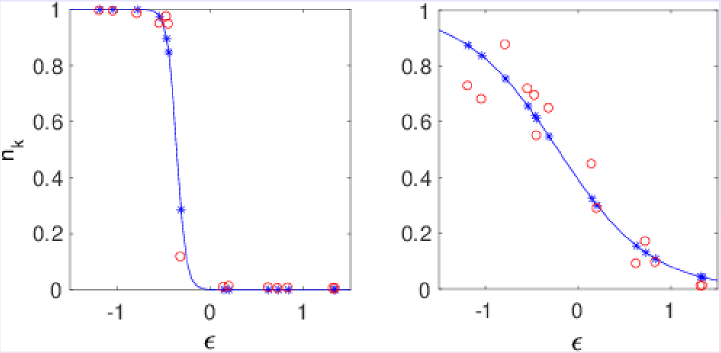
$$E(T) = \sum_{k=1}^{M} \epsilon_k n_k \; , \quad S(T) = -\sum_{k=1}^{M} n_k \ln n_k \; .$$

Fermi gas entropy is $S_F = -\sum_{k=1}^{M} (n_k \ln n_k + (1 - n_k) \ln(1 - n_k))$.
S and E are obtained from eigenstates ψ_m and eigenenergies E_m of H via $n_k(m) =< \psi_m | \hat{c}_k^{+} \hat{c}_k | \psi_m >$ $(n = L, m = M)$.

$S(T)$ and $E(T)$ are extensive and self-averaging.

This gives the implicit dependence S(E).

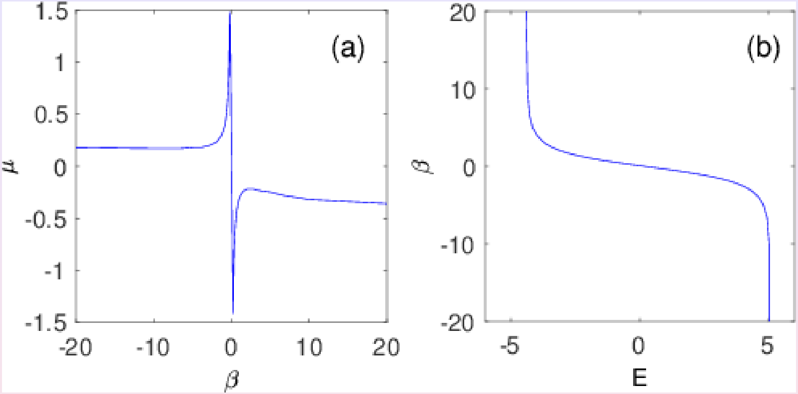
Single eigenfunction thermalized ($g \gg 1$)



Dependence of filling factors n_k on energy ϵ for individual eigenstates obtained from exact diagonatization of (red circles) and from Fermi-Dirac ansatz with one-particle energy ϵ (blue curve); blue stars are shown at one-particle energy positions $\epsilon = \epsilon_k$). Here $M = 14$, $L = 6$, $N = 3003$, $J = 1$, $V = \sqrt{14}$ and eigenenergies are $E = -4.4160$ (left), -3.0744 (right); the theory (blue) is drown for the temperatures corresponding to these energies $\beta = 1/T = 20$ (left), 2 (right).

Kolovsky, DS (2017) [R26]

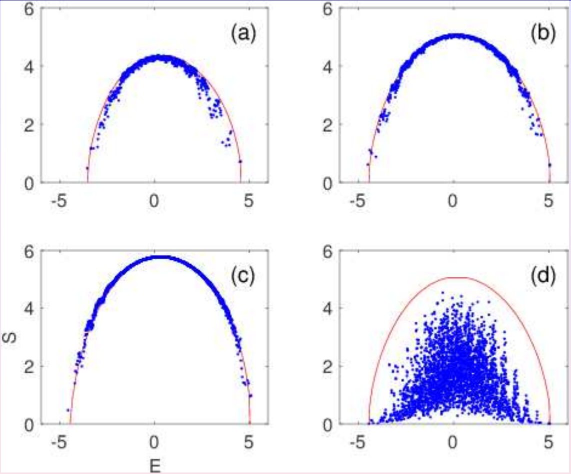
Quantum dot regime $\mu(T), E(T)$



Dependence of inverse temperature $\beta = 1/T$ on energy E (right) and chemical potential μ on β (left) given by the Fermi-Dirac ansatz for the set of one-particle energies ϵ_k as in above Fig.

Negative temperatures $T < 0$.

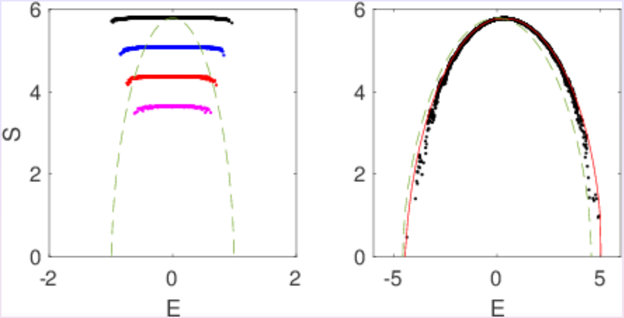
Quantum dot regime $S(E)$



(a) $M = 12$, $L = 5$, $N = 792$, $J = 1$; (b) $M = 16$, $L = 7$, $N = 3003$, $J = 1$; (c) $M = 14$, $L = 6$, $N = 11440$, $J = 1$; (d) $M = 16$, $L = 7$, $N = 3003$, $J = 0.1$. Blue points show the numerical data E_m, S_m for all eigenstates, red curves show the Fermi-Dirac thermal distribution; $V = \sqrt{14}$.

$S(E = 0) = -L \ln(L/M)$ (equipartition).

SYK black hole regime $S(E)$



$S(E)$ for SYK black hole at $V = 0$ (left) and quantum dot regime $V = \sqrt{14}$ (right); $M = 16, L = 7, N = 11440$ (black), $M = 14, L = 6, N = 3003$ (blue), $M = 12, L = 5, N = 792$ (red), $M = 10, L = 4, N = 210$ (magenta); here $J = 1$. Points show numerical data E_m, S_m for all eigenstates, the full red curve shows FD-distribution (right). Dashed gray curves in both panels show FD-distribution for a semi-empirical model of non-interacting quasi-particles for black points case. Here $S(E = 0) \approx L \ln 2; L \approx M/2$.

Semi-empirical model: non-interacting particles on orbital energies ϵ_k reproducing many-body density of states

Quantum chaos and quantum computers

The quantum computer hardware is modeled as a two-dimensional lattice of qubits (spin halves) with static fluctuations/imperfections in the individual qubit energies and residual short-range inter-qubit couplings. The model is described by the many-body Hamiltonian

$$H_S = \sum_i (\Delta_0 + \delta_i)\sigma_i^z + \sum_{i<j} J_{ij}\sigma_i^x\sigma_j^x ,$$

where the σ_i are the Pauli matrices for the qubit i, and Δ_0 is the average level spacing for one qubit. The second sum runs over nearest-neighbor qubit pairs, and δ_i, J_{ij} are randomly and uniformly distributed in the intervals $[-\delta/2, \delta/2]$ and $[-J, J]$, respectively. Eigenstate entropy $S_q = -\sum_i W_i \log_2 W_i$.

Quantum chaos border for quantum hardware:

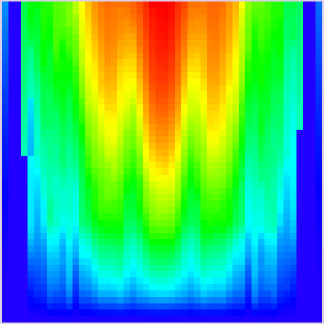
$$J > J_c \approx \Delta_c \approx 3\delta/n_q \gg \Delta_n \sim \delta 2^{-n_q}$$

Emergency rate of quantum chaos:

$$\Gamma \sim J^2/\Delta_c .$$

Georgeot, DS (2000) [R36] [R37]; DS (2001) [R25]

Quantum hardware melting induced by quantum chaos

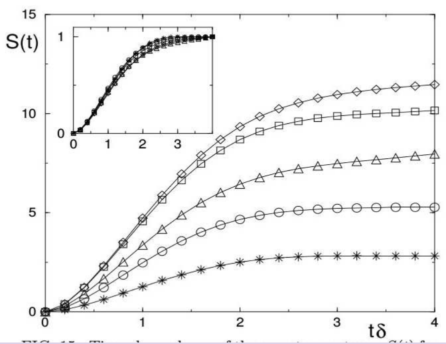


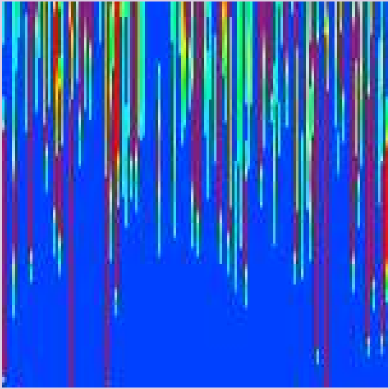
Quantum computer melting induced by inter-qubit couplings. Color represents the level of quantum eigenstate entropy S_q (red for maximum $S_q \approx 11$, blue for minimum $S_q = 0$). Horizontal axis is the energy of the computer eigenstates counted from the ground state to the maximal energy ($\approx 2n_q\Delta_0$). Vertical axis gives the value of J/Δ_0 (from 0 to 0.5). Here $n_q = 12$, $J_c/\Delta_0 = 0.273$, and one random realization of couplings is chosen.

What are effects of quantum many-body chaos on the accuracy of quantum computations?
Static imperfections vs. random errors in quantum gates of a quantum algorithm.
Åberg criterion confirmed up to $n_q = 18$ qubits

Georgeot, DS (2000) [R36]

Quantum hardware explosion in time

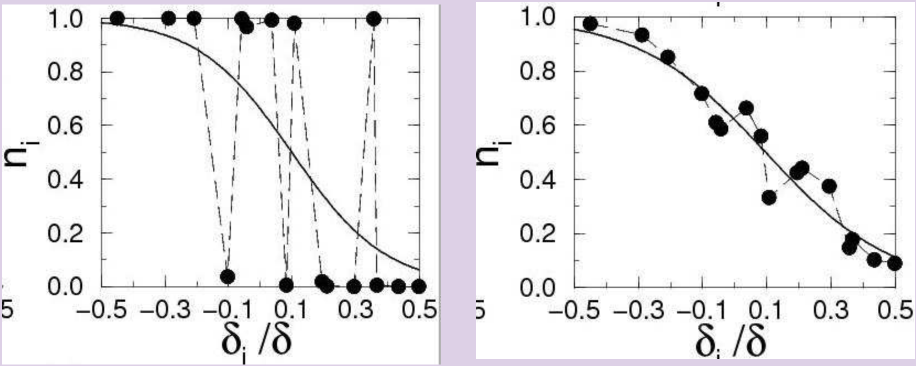




Entropy growth in time $S_q \sim \Gamma t$; $A = J/J_c \approx 2$, $n_q = 6$ to 16

Right: time explosion of quantum chaos in the quantum register: color represents the value of the projection probability of an initial state on the quantum register states ordered in energy (150 states in x-axis); time is in y-axis from $0 \le t\delta \le 2$; the initial state is the superposition of two quantum register states; $n_q = 16$, $J/\delta = 0.4$ ($J/\delta > J_c/\delta = 0.22$): $t > 1/\Gamma$

Dynamical thermalization of quantum computer

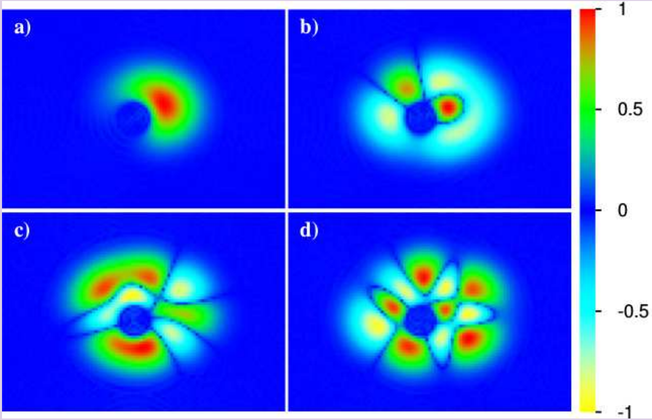


First example of DTC (ETH): Quantum computer with $n_q = 16$ qubits, central band (spin up/down as fermions). One quantum eigenstate: Occupation numbers n_i vs. rescaled exitation energies $\epsilon_i = \delta_i$.
Left: 5th state, $J/J_c \approx 0.15$, $T_{FD} = 0.15\delta$, $\delta E = 0.97\delta$, entropy $S = 0.49$.
Right: 100th state, $J/J_c \approx 1.5$, $T_{FD} = 0.20\delta$, $\delta E = 1.19\delta$, $S = 8.41$.
Full curves: Fermi-Dirac thermal distribution with given temperature T.

Benenti *et al.* (2001) [R38]

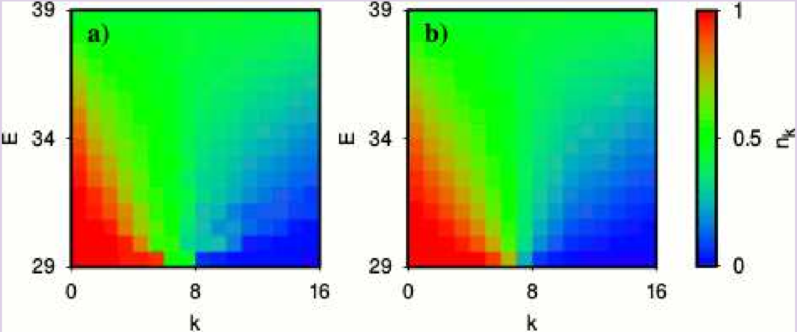
Quantum chaos in Sinai-oscillator trap

* experimentally realized by Kettrele (1995-2002) [R10]



Examples of one-particle eigenstates for orbital numbers $k = 1$ (ground state) (a), $k = 6$ (b), $k = 11$ (c) and $k = 16$ (d). There are only very small stability islands for classical dynamics embeded in a chaotic sea; energy level statistics is RMT [R1],R9]; color bar shows wavefuncton amplitude (at max/min scale). Frahm, Ermann, DS (2019) [R1]

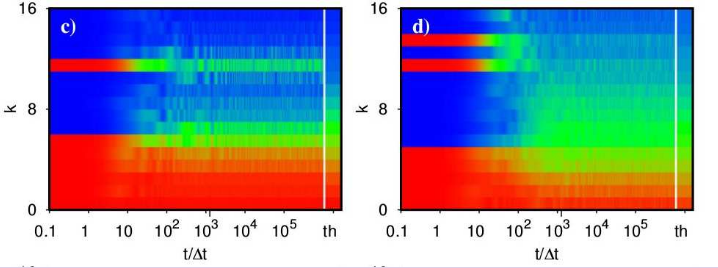
Dynamical thermalization in Sinai-oscillator trap



Orbital occupation numbers n_k for interacting fermionic atoms in a Sinai-oscillator trap at Åberg parameter $A = 3.5$; numerical result from many-body eigenstates (left), theoretical Fermi-Dirac distribution (right); $M = 16$ orbitals and $L = 7$ fermions, $N = 11440$.
Interaction vetween fermionic atoms $v(r)$ has a form of disk of certain small radius $r_c = 0.2$ and amplitude U.

Frahm, Ermann, DS (2019) [R1]

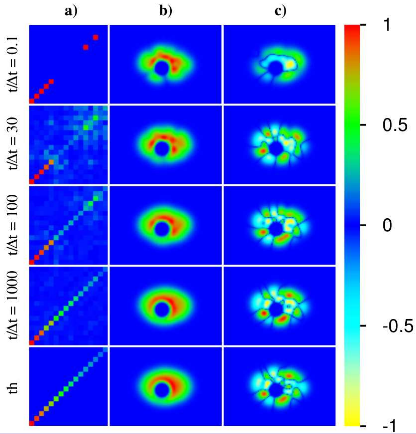
Dynamical thermalization in Sinai-oscillator trap



Color density of occupation numbers n_k in the plane of orbital index k and time t for time evolution $|\psi(t)\rangle = \exp(-iHt)|\psi(0)\rangle$. The bar behind the vertical white line labeled “th” shows the theoretical thermalized Fermi-Dirac distribution; panels are for the initial state
$|\psi(0)\rangle = |\phi_1\rangle = |0000100000011111\rangle$ (c) and
$|\psi(0)\rangle = |\phi_2\rangle = |0010100000011111\rangle$ (d) for Åberg parameter $A = 3.5$.

Frahm, Ermann, DS (2019) [R1]

Dynamical thermalization in Sinai-oscillator trap



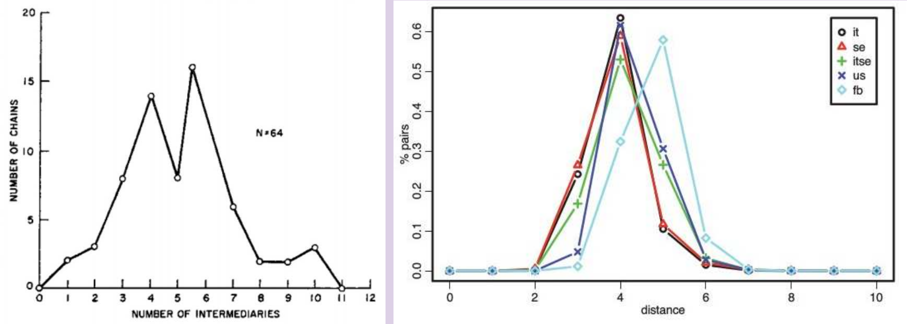
Time dependent density matrix $|n_{kl}(t)|$ (a), spatial density $\rho(x, y, t)$ (b), spatial density difference with respect to the initial condition (c) all computed from $|\psi(t)>$ for the initial state $|\psi(0)>= |\phi_2>= |0010100000011111>$ at Åberg parameter $A = 3.5$. Color bar shows $|n_{kl}|$ (a), $(\rho/\rho_{\max})^{1/2}$ (b), $\mathrm{sgn}(\Delta\rho)(|\Delta\rho|/\Delta\rho_{\max})^{1/2}$ (c) where $\rho_{\max}$ or $\Delta\rho_{\max}$ for max/min values.

Small-world network and 6-degrees of separation

Erdös-Rényi random graphs, Erdös number (1959) [R39]

Milgram small-world experiment (1967) [R40]
(left $d \approx 6$ from Nebraska to Boston)

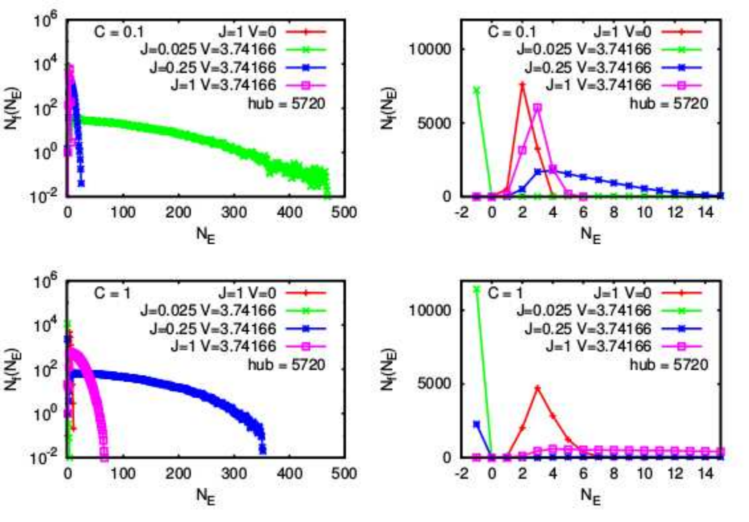
Backstrom *et al.* entire Facebook of 721M users (2012) [R41]
(right $d = 4.74 \pm 0.02$; on average only about 190 nonzero links)



more details: Dorogovtsev (2010) [R42]

TBRIM as a quantum small-world network

Erdös numbers N_E distributions for TBRIM;
links satisfy the Åberg criterion: $|H_{ij}| > C|H_{ii} - H_{jj}|$; $J = 0.25$, $V = 3.74$ in DTC ($L = 7$, $M = 16$, $N = 11440$) Frahm, DS (2018) [R27]



Open problems

* Thermalization and quantum chaos in SYK near ground state (Garcia-Garcia, Verbaarschot *et al.*)

* more analytical, numerical results for DTC and Åberg criterion; DTC for repulsive bosons in 1d Anderson model with first results Schlageck, DS PRE 93, 012126 (2016)

* Ground state delocalization of electrons in disordered 2D potential: 2D experiments: Kravchenko et al. RMP 73,251 (2001); Baenninger *et al.* PRL 100, 016805 (2008); Melnikov *et al.* PRB 101, 045302 (2020)); Song, DS PRB 61, 15546 (2000) RMT near ground state; delocalization of pinned Wigner crystal in 1d periodic incommensurate potential Garcia-Mata, Zhirov, DS EPJD 41, 325 (2007)

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