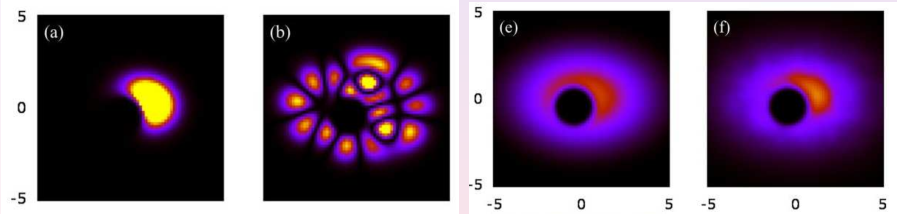
Dynamical thermalization in generic nonlinear systems

Dima Shepelyansky (CNRS Toulouse FR)
www.quantware.ups-tlse.fr/dima

with Leonardo Ermann, Eduardo Vergini (CNEA Buenos Aires)

* Dynamical Gibbs paradox (DGP):
classical or quantum dynamical thermalization ?
energy equipartition over modes or quantum Gibbs?



Sinai-oscillator trap: ground state $m = 1$ (a), linear eigenstate $m = 24$ (b), steady state of Bose-Einstein condensate $\beta = 4$, $m = 24$ from GPE (e), theoretical Bose-Einstein thermal distribution (f)

Ermann, Vergini, DS PRA (2016) [R1]

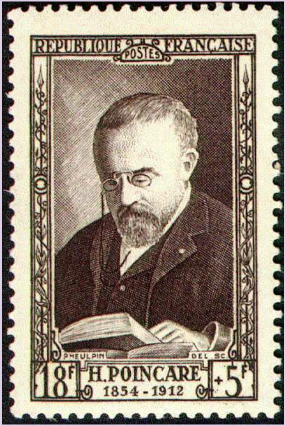
Support: ANR LABEX NANOX MTDINA project

Loschmidt - Boltzmann dispute on time reversibility (1876-1877)

* irreversible statistical laws from reversible dynamical equations
- 150 years ago

Boltzmann (1872) [R2], Loschmidt (1876) [R3], Boltzmann (1877) [R4]

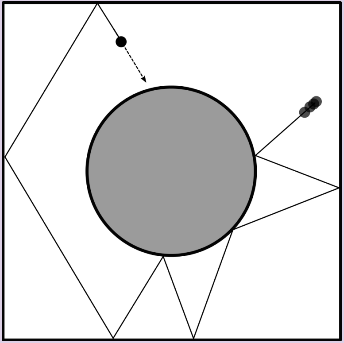
Dynamical chaos as the origin of statistical laws

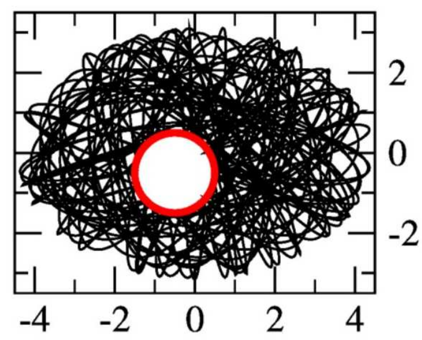


* Poincaré (1893) [R5] - mathematics;
* Chirikov resonance overlap criterion
for plasma experiments at Kurchatov Inst. (1959) [R6] - physics

Dynamical Chaos as the origin of statistical laws

* dynamical systems with a few degrees of freedom





Poincaré (1893) [R5], Kolmogorov-Arnold-Moser theorem (1954-1963) [R7], Sinai (1963) [R8], Chirikov (1959-1979) [R6]

Time reversal breaking due to exponential instability of chaotic dynamics

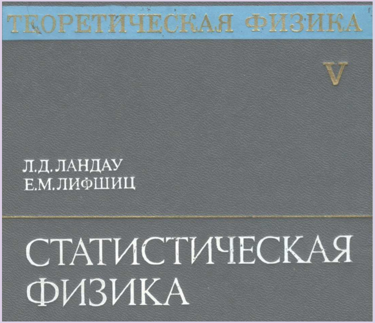
Sinai billiard (left) and

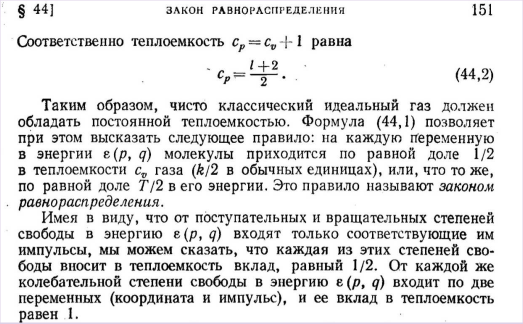
Sinai oscillator [R1,R9] (right) → Ketterle (1995-2002) [R10] (BEC 3d)

$H = ({p_x}^2 + {p_y}^2 + {\omega_x}^2 x^2 + {\omega_y}^2 y^2)/2 + V_d(x, y); \omega_x = 1, \omega_y = \sqrt{2}$,

elastic disk $r_d = 1$, $x_d = y_d = -0.5$

Energy equipartition over degrees of freedom





§ 44] ЗАКОН РАВНОРАСПРЕДЕЛЕНИЯ 151

Соответственно теплоемкость $c_p = c_v + 1$ равна

$$c_p = \frac{l+2}{2}. \qquad (44,2)$$

Таким образом, чисто классический идеальный газ должен обладать постоянной теплоемкостью. Формула (44,1) позволяет при этом высказать следующее правило: на каждую переменную в энергии $\varepsilon(p, q)$ молекулы приходится по равной доле 1/2 в теплоемкости c_v газа ($k/2$ в обычных единицах), или, что то же, по равной доле $T/2$ в его энергии. Это правило называют *законом равнораспределения*.

Имея в виду, что от поступательных и вращательных степеней свободы в энергию $\varepsilon(p, q)$ входят только соответствующие им импульсы, мы можем сказать, что каждая из этих степеней свободы вносит в теплоемкость вклад, равный 1/2. От каждой же колебательной степени свободы в энергию $\varepsilon(p, q)$ входит по две переменных (координата и импульс), и ее вклад в теплоемкость равен 1.

L.D.Landau, E.M.Lifshitz “Statistical Physics”, Nauka Moscow (1976)

Energy equipartition: Fermi-Pasta-Ulam problem

Fermi-Pasta-Ulam (FPU) problem (1955): “The results show very little, if any, tendency toward equipartition of energy among the degrees of freedom.”
$\rightarrow$ NON GENERIC MODEL

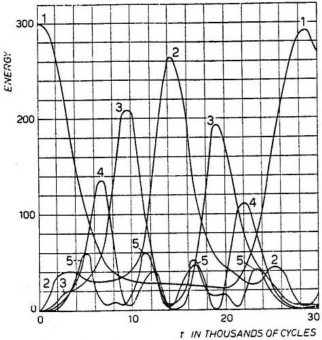


Fig. 1. – The quantity plotted is the energy (kinetic plus potential in each of the first five modes). The units for energy are arbitrary. N = 32 ; $\alpha = 1/4$; $\delta t^2 = 1/8$. The initial form of the string was a single sine wave. The higher modes never exceeded in energy 20 of our units. About 30,000 computation cycles were calculated.

Chirikov criterion for onset of chaos (1959)
Novosibirsk => FPU: Izrailev, Chirikov Dokl. Akad. Nauk SSSR 166: 57 (1966)
Integrable Toda lattice (1967) + Integrability of nonlinear Schrödinger equation
Zakharov, Shabat ZETF 61: 118 (1971), mKdV ... et al. Ruffo PRL (2022)

Generic nonlinear systems

* System of oscillators with (moderate) nonlinearity
* Random spectrum of linear oscillator frequencies
* Complex linear modes with complex nonlinear interactions
* Examples: disordered nonlinear lattices,
quantum chaos systems with nonlinearity
(e.g. Gross-Pitaevskii equation in Bunimovich stadium)

PHYSICAL REVIEW E **80**, 056212 (2009)

Dynamical thermalization of disordered nonlinear lattices

Mario Mulansky,[1] Karsten Ahnert,[1] Arkady Pikovsky,[1,2] and Dima L. Shepelyansky[2,3]
[1]*Department of Physics and Astronomy, Potsdam University, Karl-Liebknecht-Straße 24, D-14476 Potsdam-Golm, Germany*
[2]*Laboratoire de Physique Théorique (IRSAMC), Université de Toulouse–UPS, F-31062 Toulouse, France*
[3]*LPT (IRSAMC), CNRS, F-31062 Toulouse, France*
(Received 12 March 2009; revised manuscript received 2 October 2009; published 24 November 2009)

We study numerically how the energy spreads over a finite disordered nonlinear one-dimensional lattice, where all linear modes are exponentially localized by disorder. We establish emergence of dynamical thermalization characterized as an ergodic chaotic dynamical state with a Gibbs distribution over the modes. Our results show that the fraction of thermalizing modes is finite and grows with the nonlinearity strength.

Dynamical Thermalization Conjecture (DTC)

Nonlinearity and Anderson localization

Discrete Anderson nonlinear Schrödinger equation (DANSE) 1d-2d

$$i\hbar\frac{\partial\psi_n}{\partial t} = E_n\psi_n + \beta|\,\psi_n\,|^2\psi_n + V(\psi_{n+1} + \psi_{n-1})\;;[-W/2 < E_n < W/2]$$

localization length $\ell \approx 96(V/W)^2$ (1d); $\ln \ell \sim (V/W)^2$ (2d);

Hamiltonian: $H = \sum_n E_n|\psi_n|^2 + \psi_{n-1}\psi_n^* + \psi_{n-1}^*\psi_n + \frac{\beta}{2}|\psi_n|^4$

Amplitudes C in the linear eigenbasis are described by the equation

$$i\frac{\partial C_m}{\partial t} = \epsilon_m C_m + \beta \sum_{m_1 m_2 m_3} U_{m m_1 m_2 m_3} C_{m_1} C_{m_2}^* C_{m_3}$$

interaction induced transition matrix elements $U_{m m_1 m_2 m_3} \sim 1/l^{3d/2}$

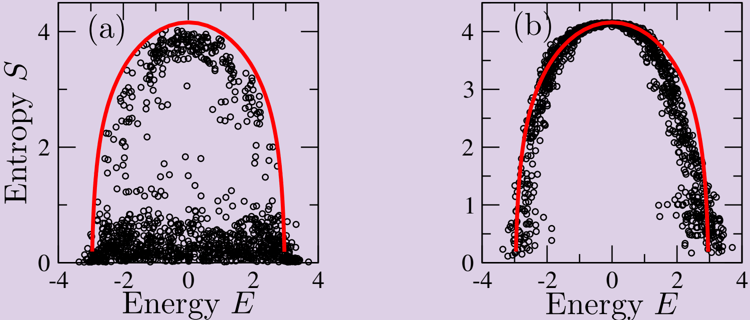
$$(\Delta n)^2 \propto t^\alpha;\; \alpha = 2/(3d+2);\;\; 1d \to \alpha = 0.3 - 0.4$$

Different DANSE type models with extremely large times $t = 10^7, ...10^{12}$:
DS PRL (1993 $d = 1, \alpha = 0.4$); Pikovsky, DS PRL (2008); García-Mata, DS PRE (2008) $d = 1, 2$); ... Flach et al. PRL (2009 $d = 1, \alpha = 1/3$), (2019 $t \sim 10^{12}$); Fishman et al. (2012) ...;
Ermann, DS CHAOS (2021 deconfinement of Yang-Mills fields in disorder)

But here we are interested not in spreading but in dynamical thermalization

Dynamical thermalization in DANSE (1d)

weaker and stronger nonlinearity $\beta \to$ *NO EQUIPARTITION*



$N = 64, N_d = 18, W = 4, \beta = 0.5(left), 2(right), t = 10^6$,
initial state: one linear eigenmode

QUANTUM Gibbs distribution with temperature T for localized linear modes,

$$\rho_m = |C_m|^2\text{: entropy } S = -\sum_m \rho_m \ln \rho_m, \;\; \rho_m = Z^{-1} \exp(-\epsilon_m/T),$$

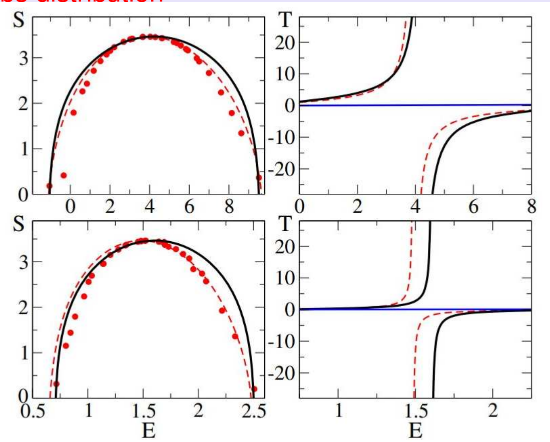
$$Z = \sum_m \exp(-\epsilon_m/T),\; E = T^2 \partial \ln Z/\partial T = \sum_m \epsilon_m \rho_m, \;\; S = E/T + \ln Z\;,$$

$$\langle \ln Z \rangle \approx \ln N + \ln \sinh(\Delta/T) - \ln(\Delta/T),\; \Delta \approx 3$$

Mulansky, Ahnert, Pikovsky, DS PRE (2009)

Dynamical thermalization in 1d nonlinear lattices

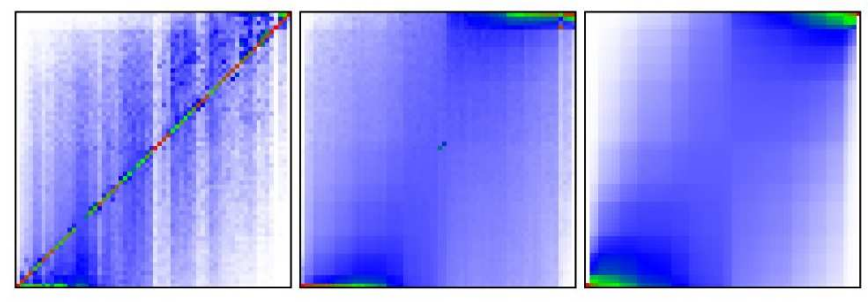
QUANTUM Gibbs distribution



Top: DANSE with $E_n = \delta E_n + f|n - n_0|$, $f = 0.5, \beta = 2, W = 2, N = 32, t = 10^7$;
Bottom: Klein-Gordon model (only energy integral)
$H = \sum_l [(p_l^2 + \tilde{\epsilon}_l u_l^2)/2 + \beta u_l^2/4 + (u_{l+1} - u_l)^2/(2W)]$,
$f = 0.125, \beta = 1, W = 2, t = 10^8$; blue line (right) shows energy equipartition
Ermann, DS NJP (2013)

Dynamical thermalization in 2d nonlinear lattices

QUANTUM Gibbs distribution in 2D DANSE



probabilities $\rho_m(m')$ in mode m (y-axis) for initial state in mode m' (x-axis), mode index is ordered by energy;
DANSE 2d $8 \times 8-$lattice, $f = 1$, $W = 2$, $t = 10^6$, $N_d = 10$ disorder realisations;
$\beta = 1$ - left, $\beta = 4$ - center, right - quantum Gibbs thermalization theory

Ermann, DS NJP (2013)

Bose-Einstein condensates (BEC) with cold atoms

BEC dynamics => Gross-Pitaevskii equation (GPE)

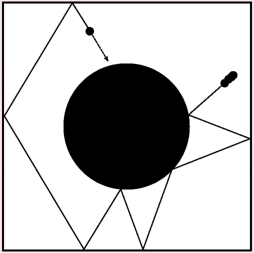
Sinai billiard → Sinai oscillator + GPE for BEC

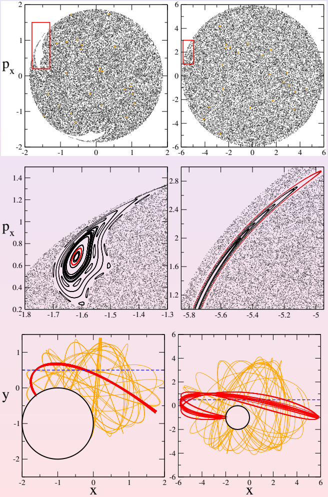
Sinai billiard (1963; 1970) (Abel prize 2014)

$H = ({p_x}^2 + {p_y}^2)/2 + x^2/2 + y^2$

$r_d = 1, x_d = y_d = -1;\ E = 2; 18 \quad \rightarrow$

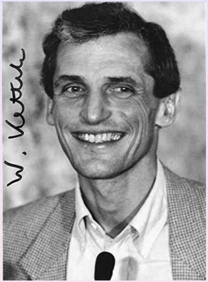
(also Bunimovich stadium + GPE)





BEC dynamics in Sinai oscillator with GPE

Ketterle BEC experiment (1995))



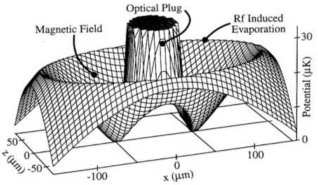


FIG. 1. Adiabatic potential due to the magnetic quadrupole field, the optical plug, and the rf. This cut of the three-dimensional potential is orthogonal to the propagation direction (y) of the blue-detuned laser. The symmetry axis of the quadrupole field is the z axis.

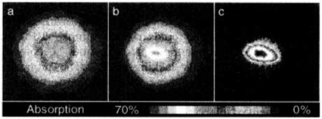


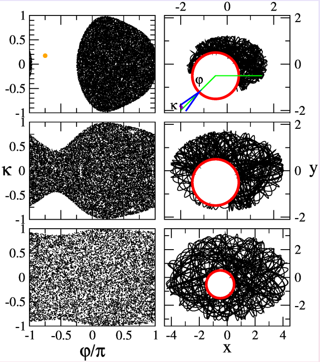
FIG. 2 (color). Two-dimensional probe absorption images, after 6 ms time of flight, showing evidence for BEC. (a) is the velocity distribution of a cloud cooled to just above the transition point, (b) just after the condensate appeared, and (c) after further evaporative cooling has left an almost pure condensate. (b) shows the difference between the isotropic thermal distribution and an elliptical core attributed to the expansion of a dense condensate. The width of the images is 870 μm. Gravitational acceleration during the probe delay displaces the cloud by only 0.2 mm along the z axis.

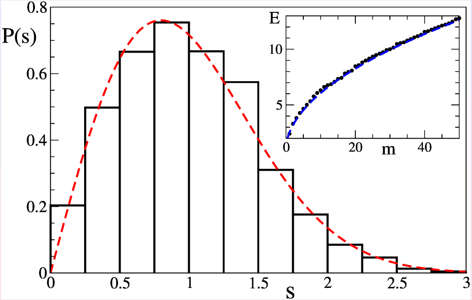
PRL (1995) (Nobel prize 2001)

Classical and quantum chaos in Sinai oscillator

Poincaré sections (left)

$H = (p_x{}^2 + p_y{}^2)/2 + x^2/2 + y^2;\ r_d = 1;\ \hbar = 1;\ x_d = y_d = -0.5;\ E = 1.5; 3; 10;$





Bohigas-Giannoni-Schmit conjecture (1984); Ullmo Scholarpedia (2016)

Wigner-Dyson statistics of lowest 2500 energy levels unfolded (right)

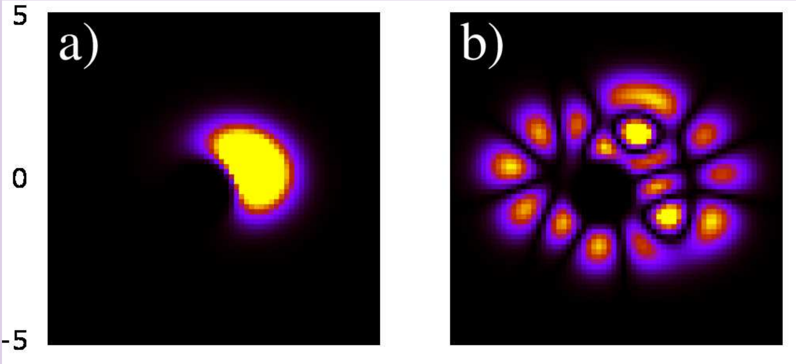
Random matrix theory (Wigner (1967)); quantum chaos (e.g. Haake (2010))

Ermann, Vergini, DS EPL-PRA (2015-2016)

Lazutkin type orbits under certain conditions

A.Lerman, V.Zharnitsky Physica D 425: 132960 (2021)

Quantum chaos eigenstates in Sinai oscillator



Eigenstates at $\beta = 0$; ground state $m = 1$ and $m = 24$

Bose-Einstein THERMALIZATION anzats:

$\rho_m = 1/[\exp((E_m - E_g - \mu)/T) - 1]$;

$\rho_m = < |\psi_m|^2 >$, energy $\sum_m E_m \rho_m = E$,
entropy $S = -\sum_m \rho_m \ln \rho_m \to S(E)$

BEC in Sinai oscillator with GPE

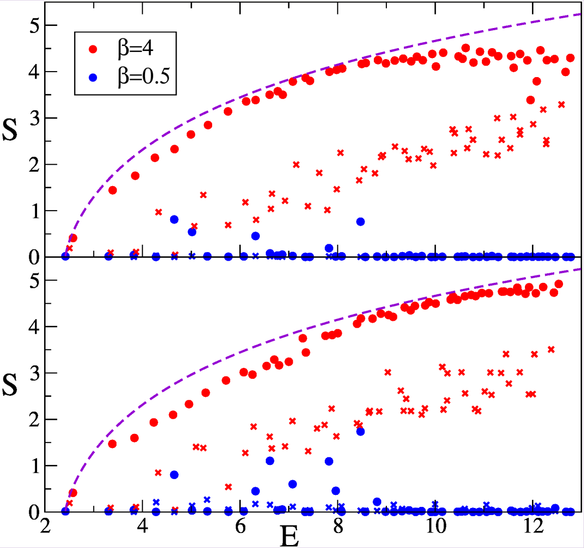
Gross-Pitaevskii equation (GPE or NSE) (Pitaevskii, Stringari (2003))

The BEC evolution in the Sinai oscillator trap is described by the GPE, which reads as

$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\vec{\nabla}^2\psi(\vec{r},t) + \left[\frac{m}{2}(\omega_x^2x^2+\omega_y^2y^2)+V_d(x,y)\right]\psi(\vec{r},t) + \beta|\psi(\vec{r},t)|^2\psi(\vec{r},t). \quad (2)$$

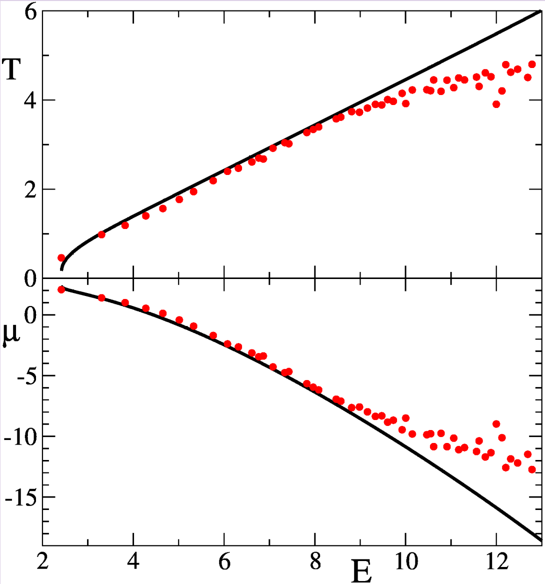
Here in (2), we use the same oscillator and disk parameters as in (1) and take $\hbar = 1$. The wave function is normalized to unity $W = \int|\psi(x,y)|^2 dx\,dy = 1$. Then, the parameter β describes the nonlinear interactions of atoms in BEC. All

Bose-Einstein anzats for dynamical thermalization



first 50 states; Sinai osc (dots), no disk (X); $500/1500 < t < 1500/2500$ (top/bottom); Bose-Einstein anzats (dashed) → no energy equipartition

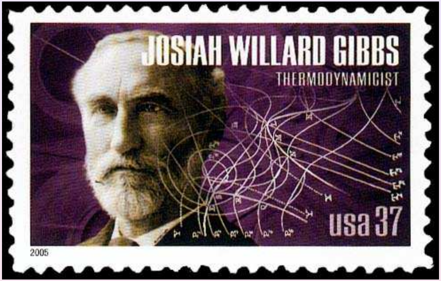
Bose-Einstein anzats



temperature and chemical potential dependence on energy ($\beta = 4$)

Discussion

* Dynamical Gibbs paradox → nonlinear thermostate for quantum system
* Behavior at very large times → no mathematical results: Arnold diffusion ...?



* Real quantum dynamical thermalization
in finite quantum many-body systems

see Frahm, Ermann, DS Condensed Matter 4: 76 (2019)
in memory of Shmuel Fishman

Many chaotic happy returns, Arkady!

at Einstein's summer house, Caputh 2007

special thanks to Maria Pikovsky

References:

R1. L.Ermann, E.Vergini and D.L.Shepelyansky, *Dynamics and thermalization of a Bose-Einstein condensate in a Sinai-oscillator trap*, Phys. Rev. A v.94, p.013618 (2016)
R2. L.Boltzmann, *Weitere Studien uber das Warmegleichgewicht unter Gasmolekulen*, Wiener Berichte v.66, p.275 (1872)
R3. J.Loschmidt, *Uber den Zustand des Warmegleichgewichts eines Systems von Korpern mit Rucksicht auf die Schwerkraft*; II-73; Sitzungsberichte der Akademie der Wissenschaften, Wien, Austria, p. 128 (1876)
R4. L.Boltzmann, *Uber die Beziehung eines Allgemeine Mechanischen Satzes zum Zweiten Haupsatze der Warmetheorie*; II-75; Sitzungsberichte der Akademie der Wissenschaften: Wien, Austria, p. 67 (1877)
R5. H.Poincaré, *Les methodes nouvelles de mecanique celeste* , Gauthier-Villars, Paris (1893) [New Methods of Celestial Mechanics, 3 vols. English trans., Ed. D.Goroff, AIP Press (1967); ISBN 1-56396-117-2]
R6. B.V.Chirikov, *A universal instability of many-dimensional oscillator systems*, Phys. Rep. v.52, p.263 (1979)
R7. V.Arnold, A.Avez, *Ergodic Problems in Classical Mechanics*, Benjamin: New York, NY, USA (1968)
R8. Y.G.Sinai, *Dynamical systems with elastic reflections. Ergodic properties of dispersing billiards*, Uspekhi Mat. Nauk v.25, p.141 (1970)

References (continued):

R9. L.Ermann, E.Vergini and D.L.Shepelyansky, *Dynamical thermalization of Bose-Einstein condensate in Bunimovich stadium*, EPL v.111, p.50009 (2015)
R10. W.Ketterle, *Nobel lecture: When atoms behave as waves: Bose-Einstein condensation and the atom laser*, Rev. Mod. Phys. v.74, p.1131 (2002)
R11. E. Fermi, J. Pasta, S. Ulam, and M. Tsingou, *Studies of nonlinear problems. I*, Los Alamos Report No. LA-1940, 1955 (unpublished); E. Fermi, Collected Papers, University of Chicago Press, Chicago, **2**, 978 (1965)
R12. F.M.Izrailev and B.V.Chirikov, *Statistical properties of a non-linear string*, Dokl. Akad. Nauk SSSR **166(1)**, 57 (1966) [in Russian] (English trans. Sov. Phys. - Doklady **11(1)**, 30 (1966))
R13. M.Toda, *Studies of a non-linear lattice*, Phys. Rep. **18**, 1 (1975)
R14. V.E.Zakharov, A.B.Shabat, *Exact theory of two-dimensional self-focusing and one-dinemsional self-modulation of waves in nonlinear media*, Zh. Eksp. Teor. Fiz. **61**, 118 (1971)