QUANTUM COMPUTERS GAMBLING CHAOS

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Content

2001 results

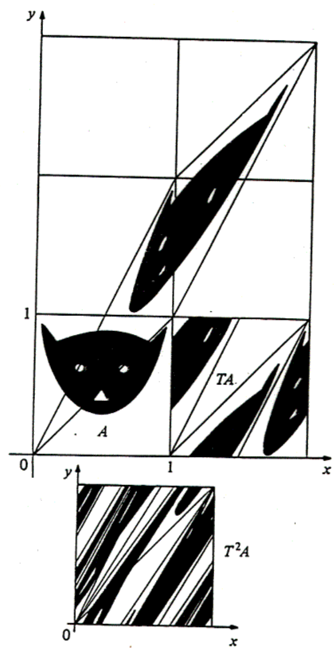
- **Stable quantum computation of unstable classical chaos:**
  Arnold - Schrödinger cat algorithm,
  Boltzmann - Loschmidt dispute

- **Schrödinger cat animated on a quantum computer:**
  kicked rotator model,
  double well map,
  chaos assisted tunneling,
  decoherence from noisy gates

- **Quantum computation of quantum chaos:**
  sawtooth map, *Anderson transition*
  static imperfections from inter-qubit interactions,
  eigenstates of operating quantum computer

Arnold cat map

$$\bar{x} = x + y \quad (mod\, 1)$$
$$\bar{y} = x + 2y$$
$$h = \ln\left(\frac{3+\sqrt{5}}{2}\right) \approx 1 > 0$$



Arnold's cat mapping, showing the cat A transformed to TA and to T^2A. This is a C-system (after Arnold and Avez, 1968).

$$\Delta x(t) \approx e^{ht} \Delta x(0)$$

$\Delta x(0) \sim 10^{-16} \rightarrow t \approx 38$

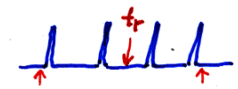
Pentium III

F1

Arnold - Schrödinger cat algorithm

$\bar{y} = y + x(mod1)$

$\bar{x} = x + \bar{y}(mod1)$ time inversion at $t = t_r$



periodic kicks

Discretization on a grid $N \times N$ with $N = 2^{n_q}$ where n_q is number of qubits for register $|x_i>$ or $|y_j>$: $x_i = i/N, \;\; y_j = j/N, 0 \le i,j \le N-1$ (i,j are integers)
Initial classical distribution in the phase space (x,y) is coded in the initial wave function:

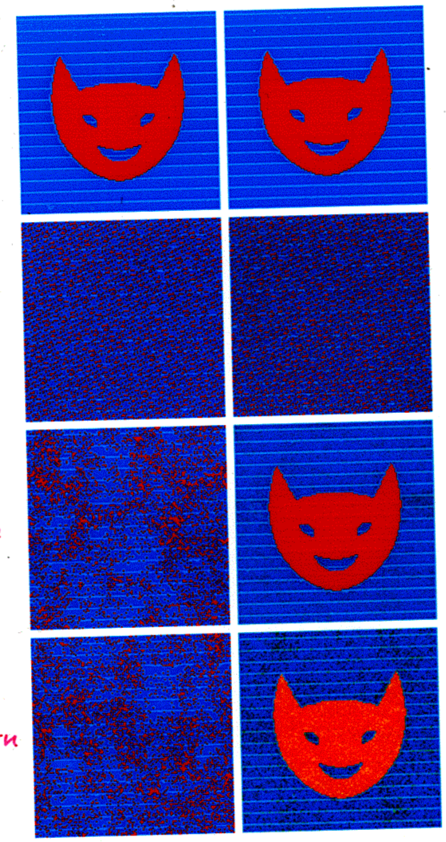
$$\boxed{\psi(t=0) = \sum_{i,j} a_{ij} |x_i > |y_j > |0 >}$$

with $a_{i,j} = 0$ or $1/\sqrt{N_d}$ where $N_d = O(N^2)$ is the number of classical orbits; workspace register $|0>$ has $n_q - 1$ qubits

* Quantum algorithm is based on modular additions (see e.g. V.Vegral, A.Barenco, A.E.Ekert Phys. Rev. A **54** (1996) 147), it uses $8n_q - 10$ C-NOT gates and $8n_q - 12$ C-C-NOT (Toffoli) gates.
In total one map iteration requires:
$O(n_q)$ quantum gates
versus $O(2^{2n_q})$ classical operations.
The Hilbert space has $N_H = 2^{3n_q - 1}$ states.

t Classical Quantum

128×128



$\varepsilon = 10^{-2}$

Figure 1: Dynamics of Arnold-Schrödinger cat simulated on a classical (left) and quantum computer (right), on a 128×128 lattice. Upper row: initial distribution; second row: distributions after 10 iterations; third row: distributions at $t_{2r} = 20$, with time inversion made at $t_r = 10$; bottom row: distributions at $t_{2r} = 400$, with time inversion made at $t_r = 200$. Left: inversion is done with classical error of one cell size ($\epsilon = 1/128$) at $t = t_r$ only; right: all quantum gates operate with quantum errors of amplitude $\epsilon = 0.01$; color from blue to red gives the probability $|a_{ij}|^2$; $n_q = 7$.

in total 20 qubits

High harmonics of Liouville density distribution

→ efficient simulation

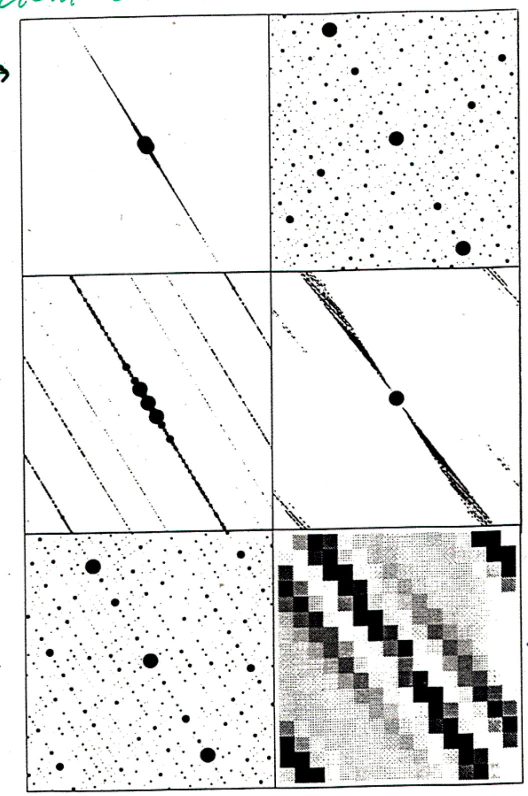


FIG. 1. Fourier coefficients $|\tilde{P}(k_x, k_y)|^2$ of Liouville distribution for $-N/2 \leq k_{x,y} \leq N/2$, initial state as in Fig.1 of [1]. Left column: cat map at $t = 3, 5, 7$ from top to bottom for $n_q = 10$. Top right: same at $t = 5$, $n_q = 7$. Middle right: $|\tilde{P}(k_x, k_y)|^2$ for perturbed cat map (see text) at $t = 5$, $n_q = 10$. Peaks are shown by circles; maximal circle size marks peaks with $1 > |\tilde{P}(k_x, k_y)|^2 > 0.1$, circles twice smaller those with $0.1 > |\tilde{P}(k_x, k_y)|^2 > 0.01$, etc... Bottom right: coarse-grained image of $|\tilde{P}(k_x, k_y)|^2$ (proportional to grayness) for the data of middle right panel, $n_f = 4$.

Quantum computer turning the arrow of time

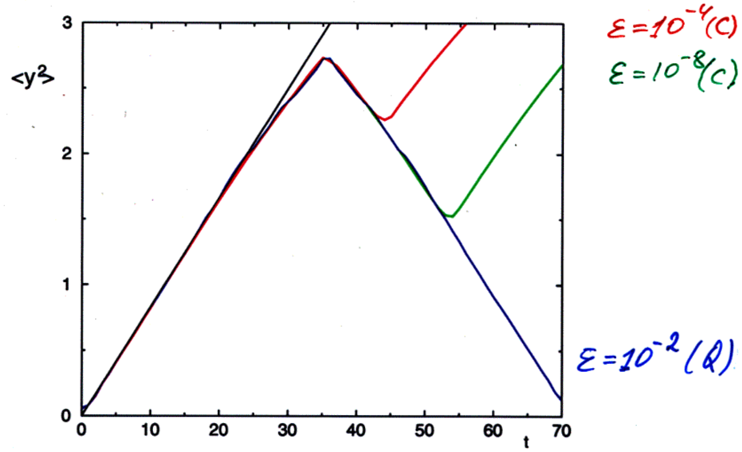
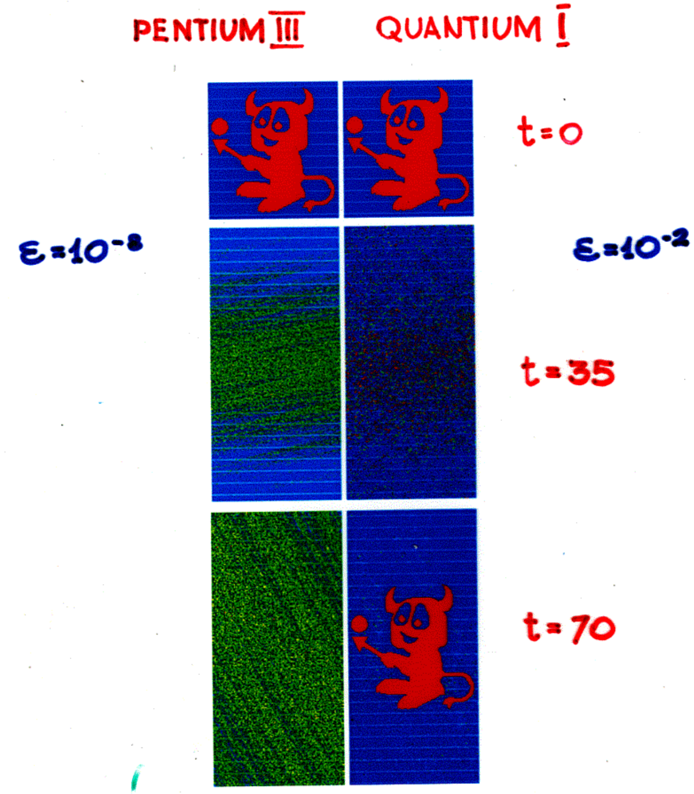


Figure 1: Diffusive growth of the second moment $< y^2 >$ of the distribution $w(y,t)$ generated by the Arnold cat map with $L = 8$, simulated on a classical (Pentium III) and quantum ("Quantium I") computers. At $t = t_r = 35$ Maxwell's demon inverts all velocities. For Pentium III inversion is done with precision $\varepsilon = 10^{-4}$ (red line) and $\varepsilon = 10^{-8}$ (green line); 10^6 orbits are simulated, initially distributed inside initial distribution. For Quantium I, the computation is done with 26 qubits ($n_q = 7, n_{q'} = 10$)(blue line); each quantum gate operates with imperfections of amplitude $\varepsilon = 0.01$ (unitary rotation on a random angle of this amplitude). The black straight line shows the theoretical macroscopic diffusion with $D = 1/12$.

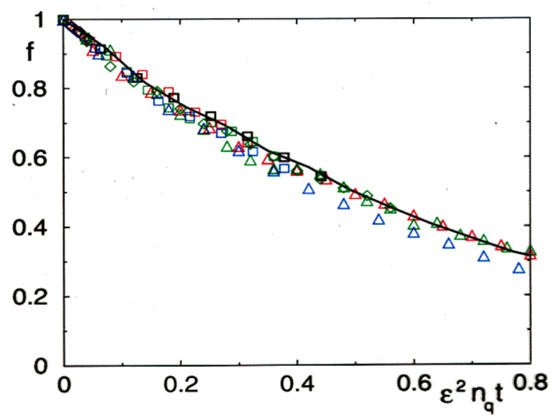
$\bar{y} = y + x \pmod{L}, \ \bar{x} = x + \bar{y} \pmod{1}$

1

Classical and quantum errors



Scaling of quantum errors



Universality of fidelity $f = |<\psi_\epsilon|\psi_0>|^2$ as a function of $tn_q\epsilon^2$ for Quantium I

$4 \leq n_q \leq 7; \quad 10^{-2} \leq \epsilon \leq 10^{-1}$

$$\boxed{t_f \approx 0.5/(n_q\epsilon^2)}$$

n_q number of qubits, ϵ quantum errors

t_f defined by $f(t_f) = 0.5$

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Boltzmann - Loschmidt dispute

A legend tells that once Loschmidt asked Boltzmann on what happens to his statistical theory if one reverses the velocity of all particles so that, due to the reversibility of Newton's equations, they return from the equilibrium to a nonequilibrium initial state. Boltzmann only replied "then go and invert them".

(from Mayer and Goeppert-Mayer, *Statistical mechanics*, Wiley & Sons, N.Y. 1976)

A quantum computer with 125 qubits can perform Boltzmann's demand for Avogadro's number of classical chaotic orbits.

Kicked rotator model

The Hamiltonian is time periodic

$$\hat{H} = \frac{\hat{n}^2}{2} + kV(\hat{\theta}) \sum_m \delta(t - mT)$$

The evolution operator is

$$\hat{U} = e^{-iT\frac{\hat{n}^2}{2}} e^{-ikV(\hat{\theta})}; \quad \bar{\psi} = \hat{U}\psi$$

Here $\hbar = 1, \hat{n} = -id/d\theta$ and the classical limit corresponds to $k \gg 1, T \ll 1, K = kT = const.$

The classical map in action/angle variables:

$$\begin{aligned} \bar{n} &= n - kV'(\theta) \\ \bar{\theta} &= \theta + T\bar{n} \ (mod 2\pi) \end{aligned}$$

Rescaled classical map ($y = Tn,\ x = \theta$):

$$\begin{aligned} \bar{y} &= y - KV'(x) \\ \bar{x} &= x + \bar{y} \ (mod 2\pi) \end{aligned}$$

Examples of classical and quantum maps:

$V(\theta) = cos\theta$ gives the Chirikov standard map and the quantum kicked rotator

$\bar{y} = y + K \sin x, \quad \bar{x} = x + \bar{y}$ mod (2π)

$V(\theta) = (\theta - \pi)^2/2$ gives the sawtooth map
$V(x) = (x^2 - a^2)^2$ gives the double well map
Arnold cat map
Baker map (quantum computing of quantum baker map Schack (1998))

Classical properties:
from Kolmogorov-Arnold-Moser integrability to chaos and diffusion

Quantum properties: dynamical localization, quantum ergodicity, chaos assisted tunneling
Anderson transition

Numerical simulations on

classical computer

1. $e^{-\frac{iTn^2}{2}} \psi_n = \tilde{\psi}_n$

$O(2^{n_q})$ multiplications

$N = 2^{n_q}$ level

2. $\tilde{\psi}_\theta = FFT\, \tilde{\psi}_n$

$O(2^{n_q} n_q)$ operations

3. $\tilde{\tilde{\psi}}_\theta = e^{-ik\cos\theta} \tilde{\psi}_\theta$

$O(2^{n_q})$ multiplications

4. $\bar{\psi}_n = FFT^{-1}\, \tilde{\tilde{\psi}}_\theta$

$O(2^{n_q} n_q)$ operations

In total

$O(2^{n_q} n_q)$ operations

quantum computer

1. $n = \sum_{j=0}^{n_q-1} \alpha_j 2^j$

$e^{-i\frac{Tn^2}{2}} = \prod_{j_1 j_2} e^{-i\frac{T}{2}\alpha_{j_1}\alpha_{j_2} 2^{j_1+j_2}}$

$O(n_q^2)$ gate operations

2. $\tilde{\psi}_\theta = QFT\, \tilde{\psi}_n$

$O(n_q^2)$ gate operations

3. Construction of a supplementary register holding

$|\theta_i\rangle\, |\cos\theta_i\rangle$

$\theta_i = \sum_{j=1}^{n_q} \beta_{ij}\, 2\pi/2^j$

$e^{i\theta_i} = \prod_{j=1}^{n_q} e^{i\beta_{ij} 2\pi/2^j} =$

$= \prod_{j=1}^{n_q} (\cos \beta_{ij}\frac{2\pi}{2^j} + i \sin \beta_{ij}\frac{2\pi}{2^j})$

$O(n_q^3)$ gate operations

4. $\bar{\psi}_n = QFT^{-1}\, \tilde{\tilde{\psi}}_\theta$

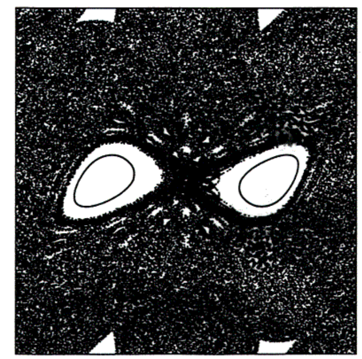
In total

$O(n_q^3)$ gate operations

Similar algorithm for Chirikov standard map

Poincaré section for double well map

$K = 0.04, a = 1.6$

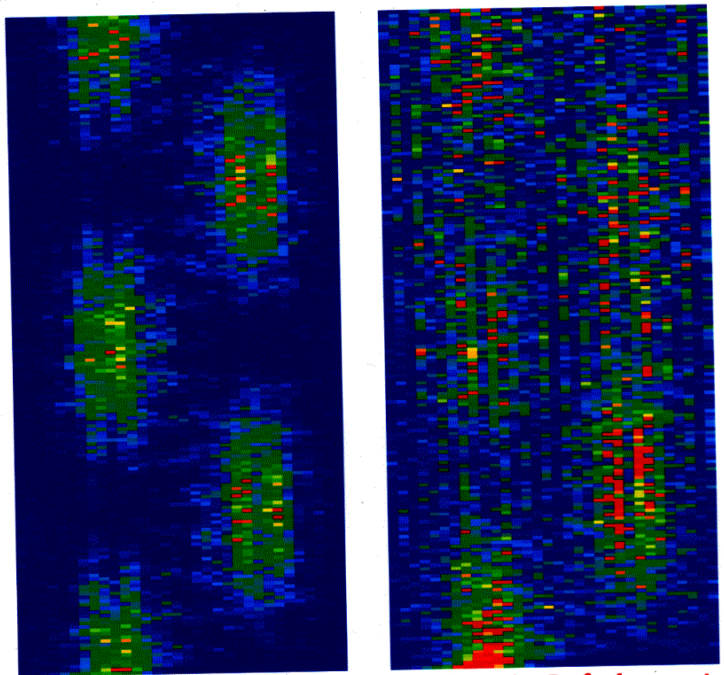


$$\bar{y} = y - KV'(x)$$
$$\bar{x} = x + \bar{y}$$

Here the double well potential is $V(x) = (x^2 - a^2)^2$ and the dynamics is taken by $mod(2\pi)$ in the square $(-\pi, \pi; -\pi, \pi)$. The frequency of small oscillations is $\omega_0 = 2\sqrt{2K}$. The classical dynamics depends on two parameters: K and a. The map becomes integrable in the continuous limit $K \to 0$.

Schrödinger cat animated on a quantum computer

$\epsilon = 0$ $\epsilon = 0.04$



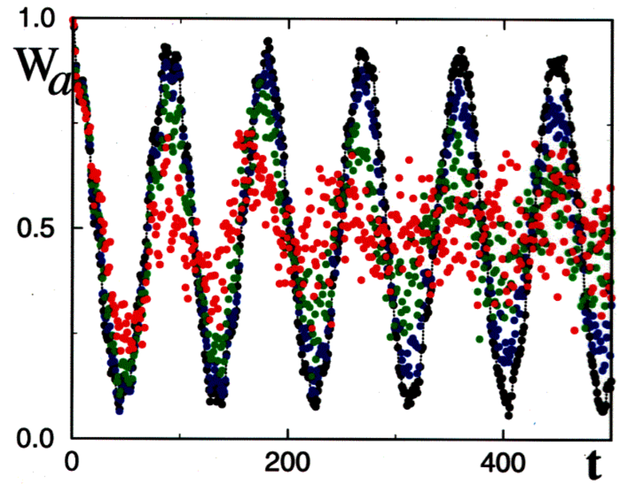
A. Chepelianskii, D.S. (coming up)

Time evolution of the Schrödinger cat: probability distribution in x–axis (blue for zero, red for maximum) is shown for 180 kicks (from bottom to top in y–axis). Here $K = 0.04, a = 1.6$ and simulations are done with $n_q = 5$, Hilbert space size $N = 2^{n_q} = 32$. The algorithm uses the gates of QFT, controlled phase shift $C^{(1)}(\varphi)$ and $C^{(2)}(\varphi)$, $C^{(3)}(\varphi)$ gates; it takes $O(n_q^4)$ gates. Amplitude of random unitary gate rotations is ϵ

Decoherence induced by noisy gates

$K = 0.04, a = 1.6, \quad n_q = 5$

amplitude of noise: $\epsilon = 0$, 0.01, 0.02, 0.04



Probability for **Schrödinger cat to be alive** W_a as a function of number of kicks t (W_a =total probability for $x < 0$). The time dependence allows to determine the period T_u of chaos assisted tunneling oscillations (here $T_u \approx 90$) and their decoherence decay rate Γ. [Raizen *et al.* experiment (2001)]

Effects of static inter-qubit interactions for operating quantum computer

The quantum computer hardware is modeled as a two-dimensional lattice of qubits (spin halves) with static fluctuations/imperfections in the individual qubit energies and residual short-range inter-qubit couplings. The model is described by the many-body Hamiltonian (B.Georgeot, D.S. (1999)):

$$H_S = \sum_i (\Delta_0 + \delta_i)\sigma_i^z + \sum_{i<j} J_{ij}\sigma_i^x\sigma_j^x,$$

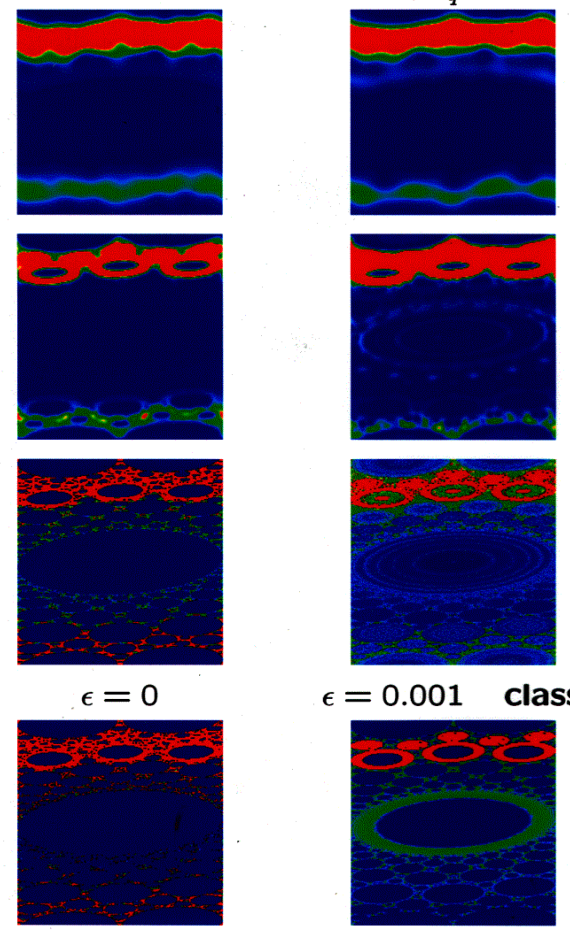
where the σ_i are the Pauli matrices for the qubit i, and Δ_0 is the average level spacing for one qubit. The second sum runs over nearest-neighbor qubit pairs, and δ_i, J_{ij} are randomly and uniformly distributed in the intervals $[-\delta/2, \delta/2]$ and $[-J, J]$, respectively.

Quantum chaos border for quantum hardware:

$$J > J_c \approx 3\delta/n_q \gg \Delta_n \sim \delta 2^{-n_q}$$

What happens for operating quantum computer ? (model: gates are perfect but between gates a propagator with H_S is applied during time τ_g, Δ_0 rotation is compensated, hence effective imperfection strength is $\epsilon = \tau_g\delta$).

Effects of static imperfections: Husimi distribution for the sawtooth map



cover page PRL 26 nov 01

$\overline{n} = n + k(\theta - \pi)$, $\overline{\theta} = \theta + T\overline{n}, mod(2\pi)$

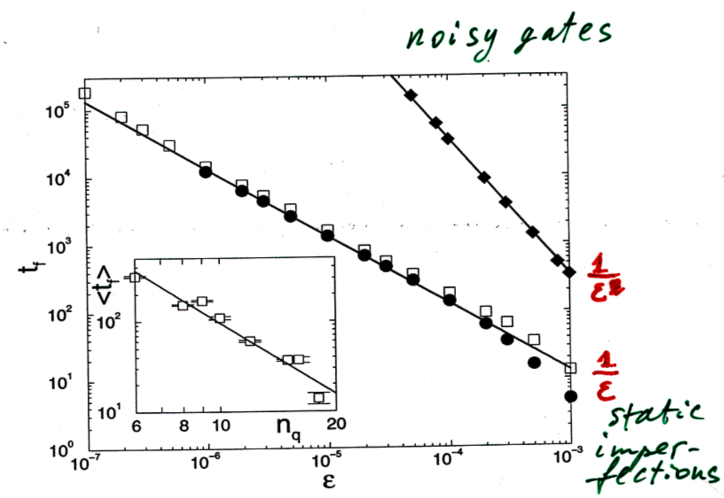
$K = kT = -0.1$, $T = 2\pi/2^{n_q}$, $t \approx 1000, J = 0$

Measurements:

* Wigner function
  at given point (p,q)
  Miquel, Paz, Saraceno,
  Knill, Laflamme, Negrevergne
  quant-ph/0109072
  efficient procedure

* Localized regime
  with localization length ℓ:
  measure W_n probabilities
  in $O(\ell)$ measurements
  to determine
  the localization length
  or transition to
  Anderson delocalized
  regime

Time scale for fidelity: simulation of sawtooth map

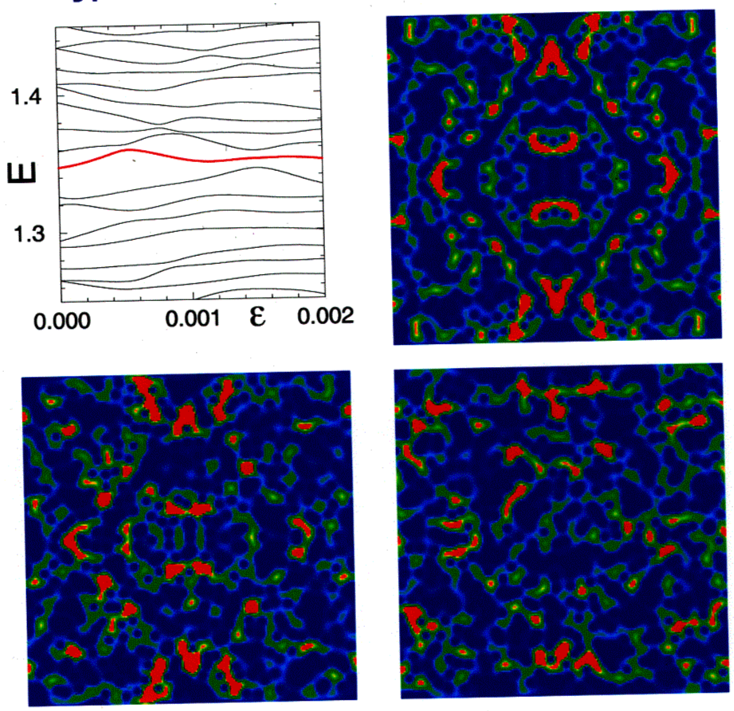


Fidelity time scale t_f as a function of ϵ, for $n_q = 9$, in the case of **static imperfections** ($J = \delta$ (circles) and $J = 0$ (squares)) and **noisy gates** (diamonds). The straight lines have slopes -1 and -2.

The **inset** shows the dependence of t_f on the **number of qubits**, for $\epsilon = 10^{-4}$, $J = 0$; the power-law fit (straight line) gives $t_f \propto n_q^{-2.6}$.

G. Benenti et al. (coming up)

Eigenstates of operating quantum computer: hypersensitivity to static imperfections



Variation of quasienergy (red curve) and corresponding eigenstate (shown by Husimi function) of unitary evolution operator of quantum sawtooth map with strength of static imperfections ϵ:

$$\overline{\psi} = \hat{U}\psi = e^{-iT\hat{n}^2/4} e^{ik(\hat{\theta}-\pi)^2/2} e^{-iT\hat{n}^2/4}\psi = e^{-iE}\psi$$

Here $\epsilon = 0$, 4×10^{-4}, 10^{-3} (right top, left/right bottom); and $K = kT = \sqrt{2}, J = 0, n_q = 9$.

Mixing of eigenstates induced by static imperfections

Analogy with parity breaking in the scattering of polarized neutrons on heavy nuclei (Sushkov - Flambaum enhancement (1982)):

In the regime of quantum chaos the quasienergy eigenstates are ergodic

$$\phi_\alpha^{(0)} = \sum_{m=1}^{N} c_\alpha^{(m)} u_m$$

with u_m quantum register states and $|c_\alpha^{(m)}| \sim 1/\sqrt{N}$

Imperfection induced matrix elements are

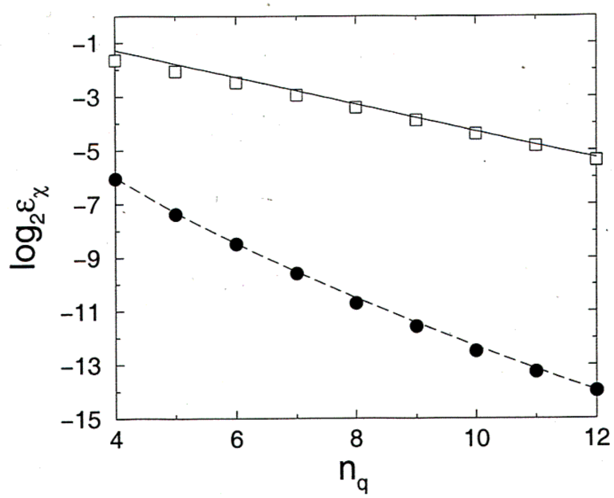
$$V_{\text{typ}} \sim |\langle \phi_\beta^{(0)} | \delta\hat{\sigma}_i^z \tau_g | \phi_\alpha^{(0)} \rangle| = \epsilon | \sum_{m=1}^{N} c_\alpha^{(m)} c_\beta^{(m)\star} |$$

and the mixing of levels takes place at

$$V_{\text{typ}}/\Delta E \sim \epsilon_\chi \sqrt{N} \sim 1$$

Critical interaction strength:

$$\epsilon_\chi \sim 1/\sqrt{N}$$



Dependence of ϵ_χ, at which perfect quasienergy eigenstates become mixed by imperfections, on number of qubits n_q. Here *circles* are for $J = 0$ and *squares* are for the single impurity model, The curves give the theoretical dependences $\epsilon_\chi = A^{-1/2}N^{-1/2}$ (above) and $\epsilon_\chi = B^{-1/2}N^{-1/2}n_q^{-5/2}$ (below), with the constants $A = 0.37$ and $B = 0.25$.

For $\epsilon > \epsilon_\chi$ the entropy of eigenstates ($S_\alpha = -\sum_{\beta=1}^{N} p_{\alpha\beta} \log_2$ where $p_{\alpha\beta} = |\langle\phi_\beta^{(0)}|\phi_\alpha^{(\epsilon)}\rangle|^2$) is exponentially large but fidelity remains close to unity for time scales $t < t_f$.

Conclusions

* Rich physics of classical and quantum maps can be studied on quantum computers with 6-11 qubits

* New information about classical and quantum chaos from efficient quantum computation

* and also ....

THOMAS A. BASS

THE NEWTONIAN CASINO

PENGUIN BOOKS (1991)

UC students in free time... (D. Farmer et. al. ~1980)

The program — a set of mathematical equations similar to those used by NASA for landing spaceships on the moon — tracks ball in orbit around a spinning disk of numbers . . . . and then announces where in this heavenly cosmos a roulette ball will likely come to rest on a still-spinning rotor. Its predictive power lies in the fact that the computer in our shoes can play out in microseconds a game that in real life takes a million times longer.

A 44 percent advantage is significantly larger than any other gambling system extant. The payout in roulette is thirty-five to one. For every hundred dollars invested — compounded fifty times an hour — one can expect a tidy hourly return of $2200. The money is sweet, but so too is the glory in beating roulette.

NEXT?

THE SCHRÖDINGER CASINO