Lecture series in Classical and Quantum Chaos

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Place: room 52, Physics Dept., Yale University.

Time : 1:00 - 2:30 p.m each Thursday.

First (last) lecture: 6 October (1 December) 1994

The course of 7 lectures is devoted to the modern problems of classical Hamiltonian chaos and its manifestations in quantum mechanics.

Classical chaos (pp. 1 - 49)

1) Exponential local instability and Kolmogorov definition of randomness. Ergodicity, mixing, correlations decay. Billiards and area-preserving maps. (p. 1 - 17)

2) Kolmogorov-Arnold-Moser theory, Chirikov criterion of overlapping resonances, transition to global chaos, Kolmogorov-Sinai entropy, Lyapunov exponents, diffusive excitation. (pp. 18 - 30

3) Standard map, golden curve and its destruction. Renormalization group picture. Frenkel-Kontorova model and Aubry transition. Poincaré recurrences and correlations decay. (31 - 49

Quantum chaos (pp. 50 - 141)

4) Different time scales of quantum dynamics. Kicked rotator model. Quantum suppression of classical chaos. Dynamical and Anderson localization. Localization length, Lyapunov exponents and transfer matrix technique. (pp. 50 - 65)

5-6) Diffusive photoelectric effect for hydrogen atom in a microwave field. Rydberg atoms, experiment of Bayfield and Koch, classical and quantum theories. Stabilization of atoms in strong field. Autoionization of molecular Rydberg states, Halley's comet. (pp. 66 - 96) (pp. 97 - 114)

7) Multifractal spectrum and kicked Harper model. Triangular well model. Dynamical localization in higher dimensions, Anderson transition. Effects of noise. Many degrees of freedoms: Fermi-Pasta-Ulam problem. (pp. 115 - 141)

<u>Problems</u>: pp. 142 - 146

References

1. A. J. Lichtenberg and M. A. Lieberman. Regular and Chaotic Dynamics. Springer-Verlag, Berlin, 1992. C

2. B. V. Chirikov. Phys. Rep. <u>52</u> (1979) 263. C

3. L. E. Reichl. The transition to chaos: In conservative classical systems: Quantum manifestations. Springer-Verlag, Berlin, 1992. Q

4. Chaos and Quantum Physics, eds. M.-J. Giannoni, A. Voros and J. Zinn-Justin Les Houches Lectures 1989 (North-Holland, Amsterdam, 1991). Q

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5. F. Haake. Quantum signatures of Chaos, Springer-Verlag, Berlin, 1991. Q

6. M. C. Gutzwiller. Chaos in Classical and Quantum Mechanics, Springer-Verlag, Berlin, 1990 C-Q

General

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LECTURE 1

[1] V.I.Arnold and A.Avez, Ergodic Problems of Classical Mechanics, Benjiamin (1968).

[2] I.P.Kornfeld, Ya.Sinai, S.V.Fomin, Ergodic Theory, Nauka, 1980 (English translation is also available).

[3] V.M.Alekseev and M.V.Yakobson, Phys. Reports, v.75, p.287 (1981). [Kolmogorov randomness].

[4] Levin, Zvonkin, Uspekhi Mat. Nauk v.25, N6, p.85 (1970). (translated in English). [Kolmogorov randomness]

[5] Chaitin, Information, Randomness and Incompleteness, World Scient., (1987). [Kolmogorov randomness]

Classical and Quantum Chaos

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Boris Chirikov — Siberian view: $-40^\circ C = -40^\circ$

God does not play dice.
A. Einstein

classical & quantum aspects

* Newton's equations (dynamics)
* Laplace determinism
* unpredictability and randomness

XAOC

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Dynamical system

$$\bar{X} = 2X \pmod 1$$

$$X = 0.\overbrace{1\,1\,0\,1\,0\,1\,1\,1\,0}^{N}0\ldots\ldots$$

Uncomputable numbers $[0, 1]$

complexity (Kolmogorov)

```
DO  I = 1, M
  . . . . . . . . .
  . . . . . . . . .
  . . . . . . . . .
  - - - - - - - -
END DO
```

$L_c(X) + C_c \geq K_N(X)$ (Alekseev - Brudno)

$K_N \to \infty$

$$h = \lim_{N \to \infty} \frac{K_N}{N} = \text{const} > 0$$

any randomness test

Measure of uncomputable numbers is one.

Uncomputable number $\equiv$ random number

Exponential local instability

unstable dynamics

$$\Delta X_n = e^{hn} \Delta X_0$$

$$h = \ln 2$$

stable dynamics

$$\bar{X} = X + \alpha \pmod 1$$

$$\Delta X_n = \Delta X_0$$

Entropy for a sequence of length N

partition

$$H_T = -N \sum_{i=1}^{2} p_i \ln p_i \quad ; \quad p_i = 1/2 \quad (\max)$$

$$h = \frac{H_T}{N} > 0 \ (N \to \infty) \qquad h = \ln 2$$

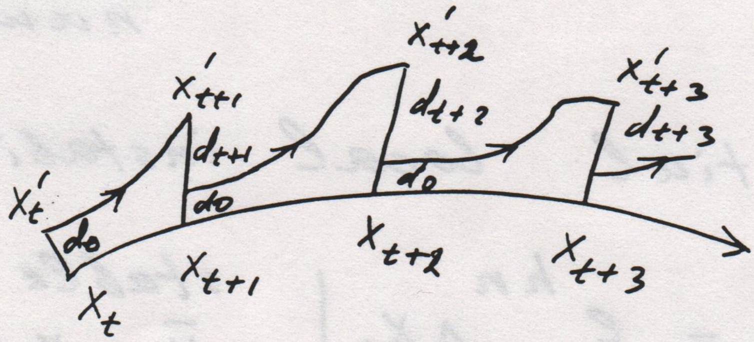
$$N = t \ (\text{time}) \qquad \bar{X} = 2X \ (\text{mod } 1)$$

KS-entropy (Kolmogorov-Sinai entropy)

$$h = \int_{\mu} \sum_{\lambda_i > 0} \lambda_i(x)\, d\mu \quad \left(\begin{array}{l}\text{Lyapunov} \\ \text{exponents}\end{array}\right)$$

$$h = \sum_{\lambda > 0} \lambda_i \qquad (\text{Pesin}) \ 1977$$

Numerical method



Toda lattice

(J. Ford (1973))

All Lyapunov exponents.

Cat map

Arnold

$$\bar{x} = x + y$$
$$\bar{y} = x + 2y \qquad (\text{mod } 1)$$

1

$$\frac{\partial(\bar{x}\ \bar{y})}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$$

Eigenvalues $\begin{vmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$$\lambda_{1,2} = \frac{3 \pm \sqrt{5}}{2}$$

Eigenvectors

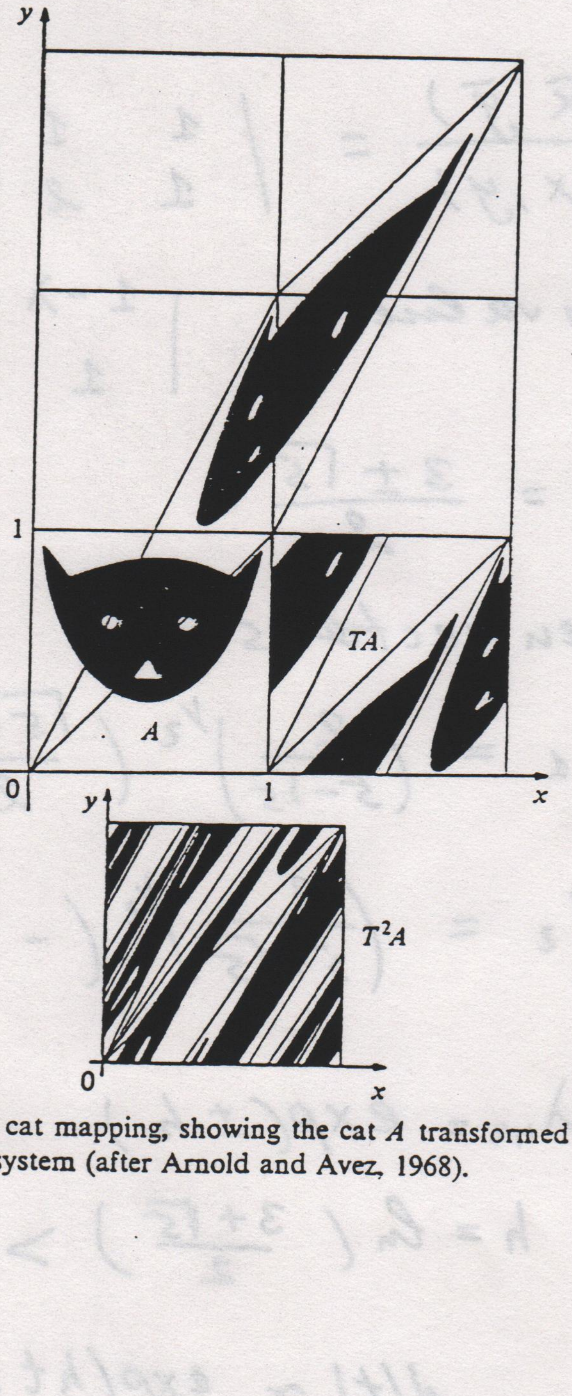
$$\underline{\xi}_1 = \left(\frac{2}{5-\sqrt{5}}\right)^{1/2} \left(\frac{\sqrt{5}-1}{2}\,\underline{x} + \underline{y}\right) \quad \Big|\ \lambda_1 > 1$$

$$\underline{\xi}_2 = \left(\frac{2}{5+\sqrt{5}}\right)^{1/2} \left(-\frac{\sqrt{5}+1}{2}\,\underline{x} + \underline{y}\right) \quad \Big|\ \lambda_2 < 1$$

$$\lambda_{1,2} = \exp(\pm h)$$

$$h = \ln\left(\frac{3+\sqrt{5}}{2}\right) > 0$$

$$d(t) \sim \exp(ht)\, d(0)$$



Arnold's cat mapping, showing the cat A transformed to TA and to T^2A. This is a C-system (after Arnold and Avez, 1968).

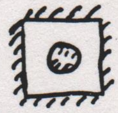
F1

Dynamical systems

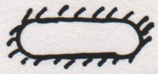
Ergodic systems — Time average = space average — $\bar{x} = x + \alpha \pmod 1$

Mixing systems — decay of correlations, equilibrium; coarse-graining — shaker 20% vodka 80% orange

K-systems — average exp. divergence; $h > 0$; positive Lyapunov exp. and KS-entropy — Sinai billiards



Bunimovich billiards (stadium)



Anosov systems — exponential instability (always stretching) — cat map

structural stability → strange attractor

exp. correlations decay

Bernouilli systems — zero correlations — Baker's transformation; negative curvature flows

5)

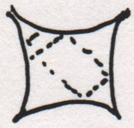
$h > 0$

unpredictability of motion
exponential growth of errors
(two neutron on opposite sides of universe)

Time reversing

5

Billiards and maps



Sinai

$C(\tau) \sim \exp(-\tau^{\alpha})$

$\alpha > 1/3$

Bunimovich

$C(\tau) \sim 1/\tau$

$h > 0$

$$\bar{I} = I + \varepsilon f(\bar{I}, \theta)$$

$$\bar{\theta} = \theta + \alpha(\bar{I}) + \varepsilon g(\bar{I}, \theta)$$

generating function

$$F_2 = \bar{I}\,\theta + A(\bar{I}) + \varepsilon\, G(\bar{I}, \theta)$$

$$\alpha = \frac{dA}{d\bar{I}}\ ; \quad f = -\frac{\partial G}{\partial \theta}\ ; \quad g = \frac{\partial G}{\partial \bar{I}}$$

Area preservation $\dfrac{\partial(\bar{I}, \bar{\theta})}{\partial(I, \theta)} = 1$

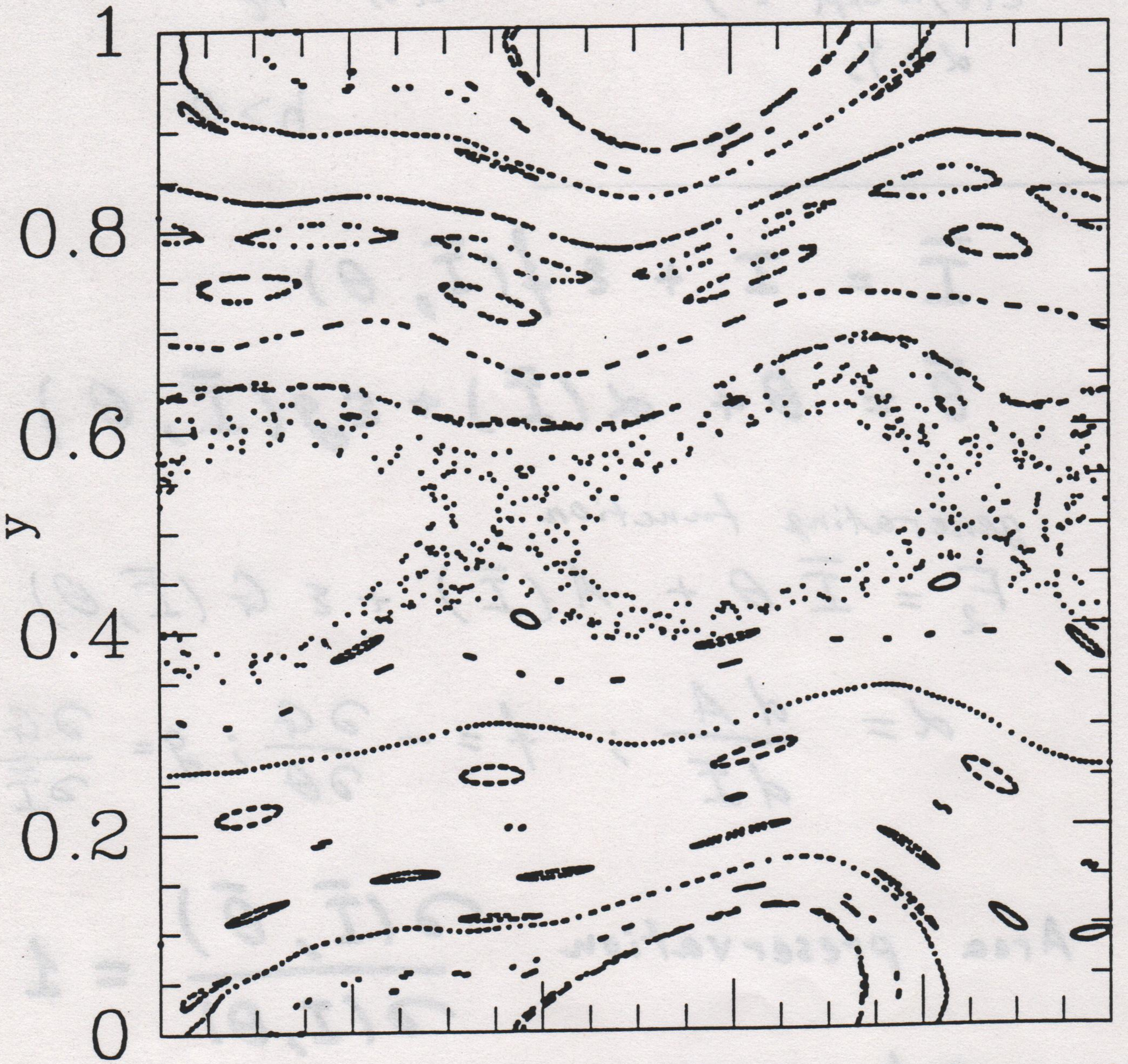
F2 $$\frac{\partial f}{\partial \bar{I}} + \frac{\partial g}{\partial \theta} = 0$$

$$\bar{p} = p - \frac{\partial V}{\partial x}$$

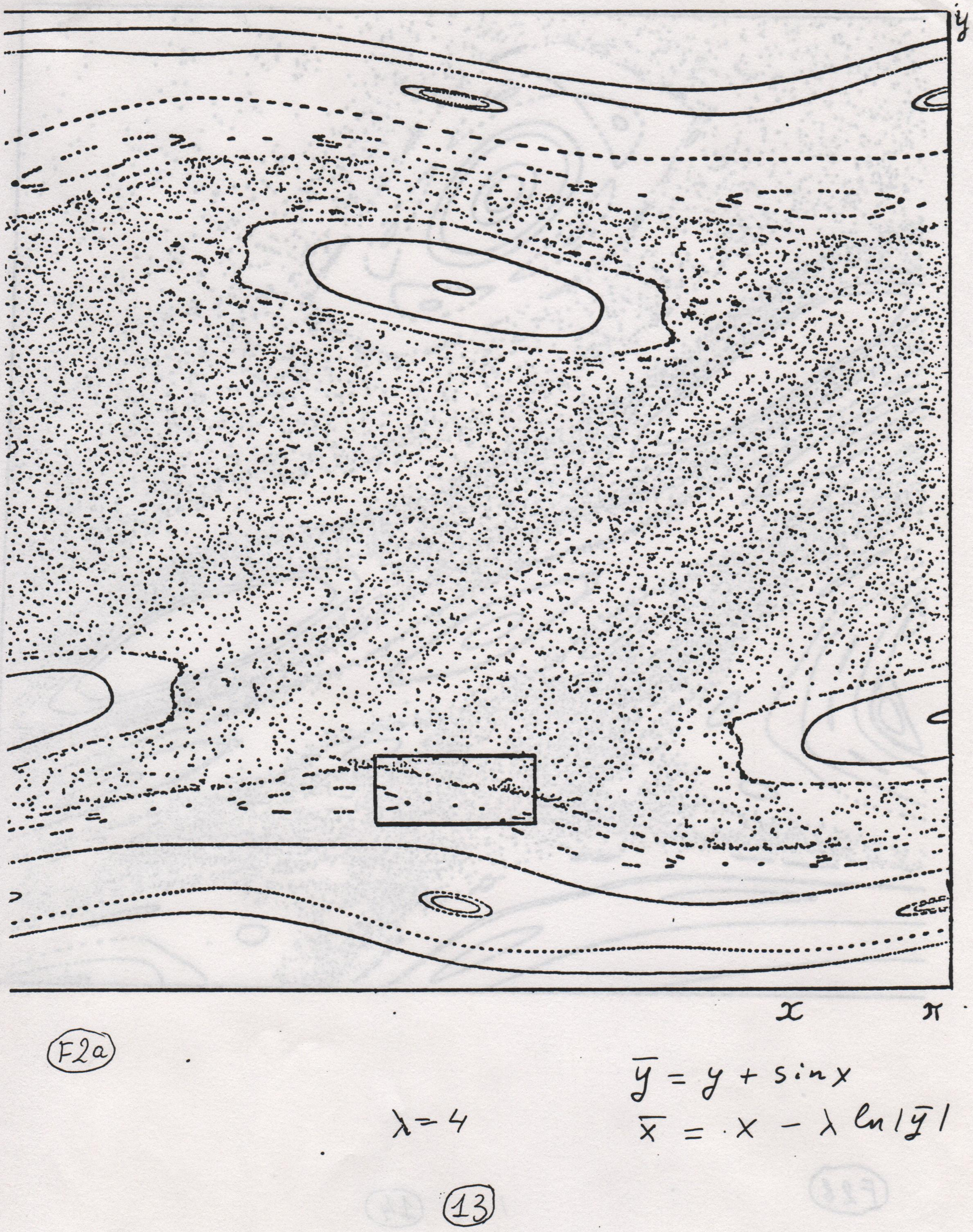
$$\bar{x} = x + \bar{p} \;; \qquad V = K(\cos x - \tfrac{1}{2}\sin 2x)$$

$$K = 0.25$$

F2



a



F2a

$$\bar{y} = y + \sin x$$
$$\bar{x} = x - \lambda \ln|\bar{y}|$$

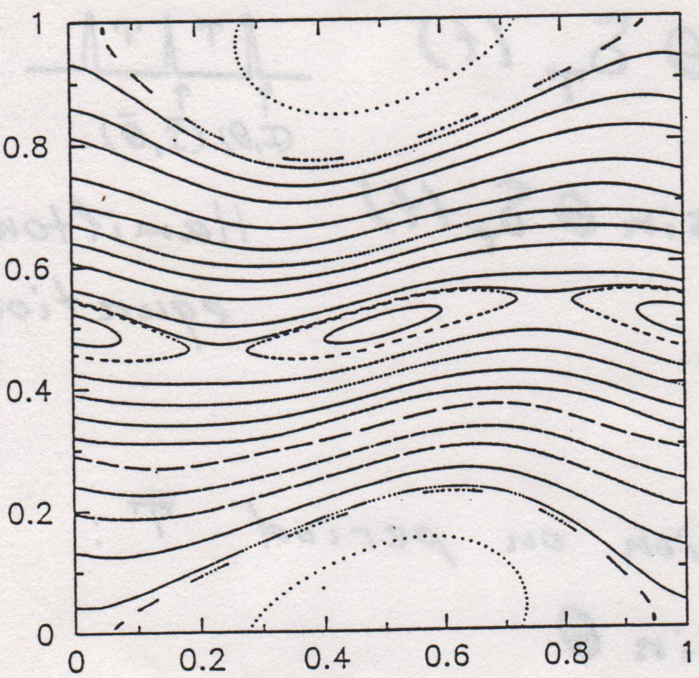
$\lambda = 4$

B

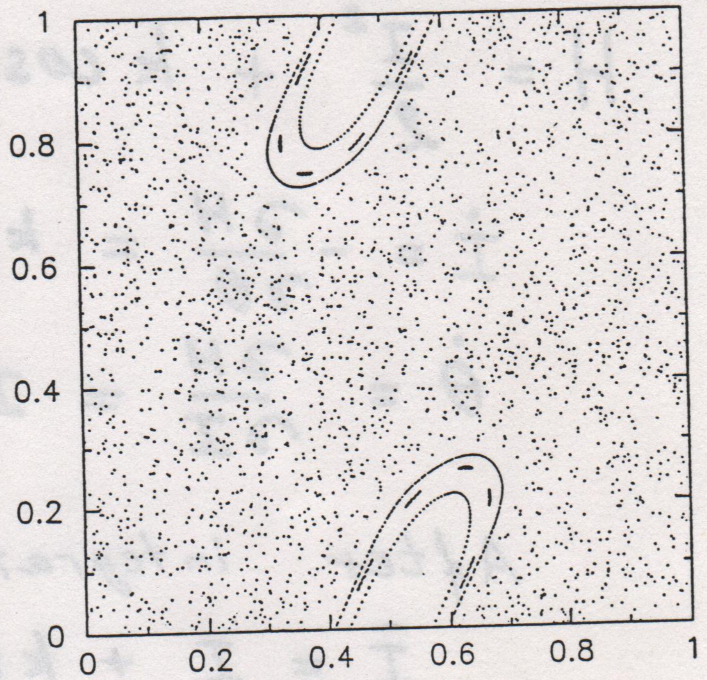
F2b

Chirikov standard map

$$\bar{y} = y + K \sin x , \quad \bar{x} = x + \bar{y}$$



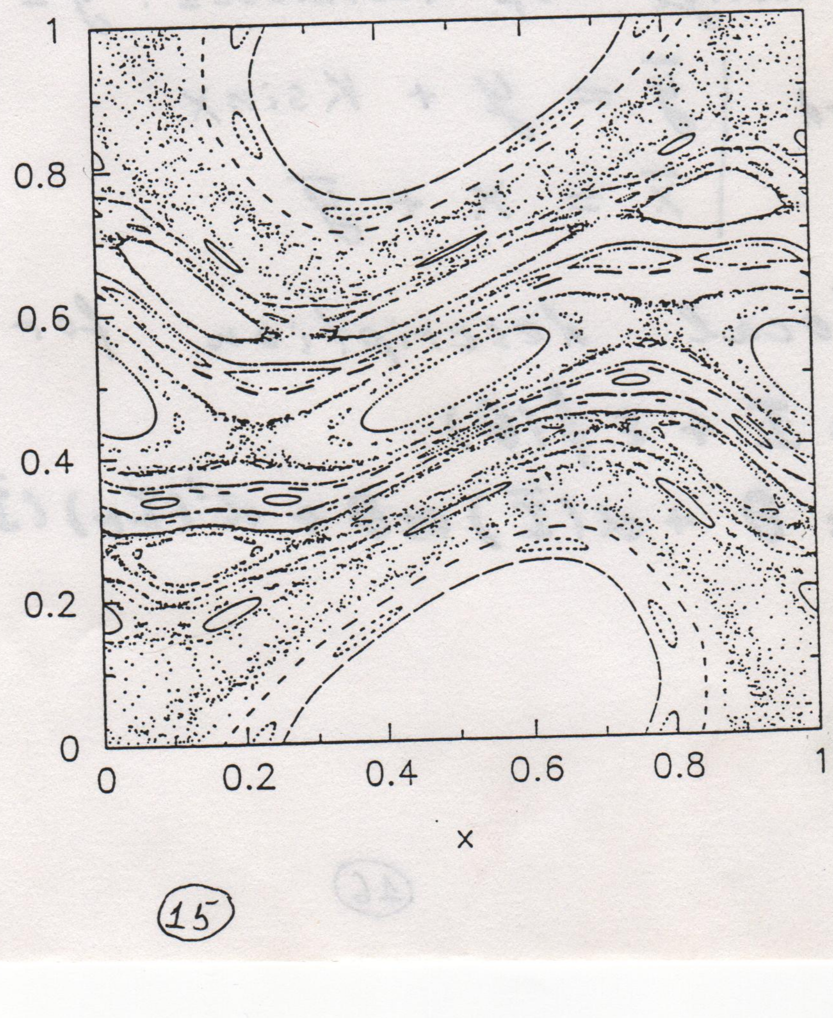
$K = 0.5$



$K = 3$

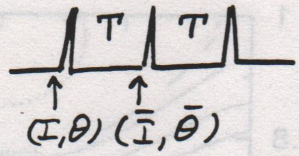
Surface of Section for $K=0.971635406$

F3



Chirikov standard map

$$H = \frac{I^2}{2} + k\cos\theta\,\delta_T(t)$$



$$\dot{I} = -\frac{\partial H}{\partial \theta} = k\sin\theta\,\delta_T(t)$$

$$\dot{\theta} = \frac{\partial H}{\partial I} = I$$

Hamiltonian equations

After integration on period T:

$$\bar{I} = I + k\sin\theta$$
$$\bar{\theta} = \theta + T\bar{I}$$

Change of variables: $y = TI$, $x = \theta$

Chirikov standard map

$$\begin{cases} \bar{y} = y + K\sin x \\ \bar{x} = x + \bar{y} \end{cases}$$

$K = kT$

$x = x \pmod{2\pi}$

Local description for other maps

$$\bar{I} = I + \varepsilon f(\theta)$$
$$\bar{\theta} = \theta + \alpha(\bar{I}) \approx \theta + \alpha'(I_r)(\bar{I} - I_r)\Big(+\underset{2\pi n}{\underset{\parallel}{\alpha(I_r)}}\Big)$$

Kolmogorov - Arnold - Moser theorem (KAM)

conjecture, analytic, finite number (1954; 1961; 1962)
derivatives

Invariant curves for zero perturbation

$$\bar{y} = y, \quad \bar{x} = x + \bar{y} \Rightarrow x_t = x_0 + 2\pi r t; \; r = y/2\pi$$

r - rotation number

are only weakly perturbed by small perturbation if:

$$\sum_i \underline{m}_i \underline{\omega}_i (\underline{I}) \neq 0$$ in some domain of action I

$$\underline{\omega} = \partial H / \partial \underline{I}$$

perturbation has sufficient number of derivatives

$$|\underline{m}\,\underline{\omega}| \geq \frac{c}{m^{\gamma}} \qquad \left(|qr - p| > \frac{1}{q^{\gamma}} \right)$$

$$1 \leq \gamma < 2$$

Divergence of denominators:

$$\dot{I} = \varepsilon \cos(\underline{m}\,\underline{\theta}) = \varepsilon \cos(\underline{m}\,\underline{\omega} t)$$

$$I \sim \frac{\varepsilon}{\underline{m} \cdot \underline{\omega}} \sin(\underline{m}\,\underline{\omega} \cdot t)$$

Small perturbation $\Longrightarrow$ small changes