LECTURE 2

[6] B.V.Chirikov, Atomnaya Energiya v.6, p.630 (1959) (translation: J. Nucl. Energy Part C Plasma Physics v.1 pp.253-260 (1960), Pergamon Press Ltd.) [Chirikov criteria of overlapping resonances].

[7] D.F.Escande, Phys. Reports, v.121, p.165 (1985). [renormalization of two resonances].

[8] A.B.Rechester, R.B.White, Phys. Rev. Lett. v.44, p.1586 (1980); A.B.Rechester, M.N.Rosenbluth and R.B.White, Phys. Rev. A, v.23, p.2664 (1981). [diffusion in Chirikov standard map].

[9] B.V.Chirikov, D.L.Shepelyansky, "Stochastic Oscillations of Classical Yang-Mills Fields", Pis'ma Zh. Eksp. Teor. Fiz. v.34, p.171-175 (1981) [JETP Lett., v.34, p.163-166 (1981)] [first numerical demonstration of chaos for Yang-Mills fields].

[10] B.V.Chirikov, D.L.Shepelyansky, "Dynamics of Some Homogeneous Models of Classical Yang-Mills Fields", Yader. Fiz. v.36, p.1563-1576 (1982) [Sov. J.Nucl. Fiz. v.36, p.908 (1982)] [introduction of Higgs mass, chaos at small nonlinearity and energy].

F4

J. Nucl. Energy. Part C Plasma Physics. 1960. Vol. 1. pp. 253 to 260. Pergamon Press Ltd. Printed in Northern Ireland

RESONANCE PROCESSES IN MAGNETIC TRAPS*

B. V. CHIRIKOV*

Abstract—Consideration is given to resonances between the Larmor rotation of charged particles and their slow oscillations along the lines of force. Under certain conditions these resonances can result in a complete exchange of energy among the degrees of freedom of the particle, so that the particle escapes from the trap. The influence of resonances on adiabatic processes associated with a time variation of the magnetic field is also examined.

1. INTRODUCTION

ONE of the methods for thermally insulating a plasma in order to realize a controlled thermonuclear reaction is the use of so-called adiabatic traps, or traps with magnetic mirrors, proposed and calculated by BUDKER.[1] Similar systems have been proposed by YORK[2] and calculated by JUDD, MCDONALD and ROSENBLUTH.[3] Recently, considerable developments in this direction have occurred and therefore it is of interest to study further similar systems.

The action of an adiabatic trap is based[1] on the conservation of orbital magnetic moment of a charged particle in a magnetic field ($\mu = Mv_\perp^2/(2H)$ where $v_\perp$ is the component of the particle velocity in a direction perpendicular to the magnetic field H). It is a necessary, but of course not a sufficient condition for the usefulness of a trap that it can entrap a single charged particle. Generally speaking the lifetime of such a particle in the trap is not infinite because the magnetic moment is only an adiabatic invariant, i.e. it can change slowly and so allow a redistribution of energy among the longitudinal and transverse degrees of freedom of the particle and consequent escape from the trap.

The question of the time variation of an adiabatic invariant has been considered in a number of papers.[6–8] However, only KULSRUD[7] takes his calculations as far as concrete results for a harmonic oscillator, obtaining

$$\frac{\Delta I}{I} = \frac{2\Delta^{(q)}}{(2\omega_0)^{q+1}} \cdot \cos\left(2\theta_0 - \frac{\pi q}{2}\right) \qquad (1.1)$$

Here I is the adiabatic invariant. $\Delta^{(q)}$ is the discontinuity in the qth derivative of $\omega(t)$. θ_0 and ω_0 are the phase and the frequency of the oscillator at the time of the discontinuity in the derivative. The basically unsatisfactory feature of the above expression is its asymptotic nature. This means that it is correct only if $1/(\omega T) \to 0$ (T being the characteristic time for the variation $\omega(t)$. For finite values of the adiabaticity parameter $1/(\omega T)$ equation (1.1) is not always correct. (The conditions for its applicability are given in the Appendix.) In the particular case where $\omega(t)$ is an analytic function, equation (1.1) gives $\Delta I/I = 0$. This means that when $1/(\omega T) \to 0$ the quantity $\Delta I/I$ tends to zero faster than any power of the parameter $1/(\omega T)$ (for instance as $\exp(-\omega T)$), but it remains unknown how exactly it behaves. For this reason the normally used methods of asymptotic expansion in powers of a small parameter such as $(1/\omega T)$ are not applicable in this case.

In the present paper we consider a different approach to this problem. It is based on the simple physical model of resonances between the Larmor rotation of the charged particle and slow oscillations of the particle along the magnetic lines of force.† Such resonances are possible in spite of the differences in frequency if the slow oscillations of the particle are anharmonic and contain high harmonics of their basic frequency. The action of the resonances leads in particular to a change in the magnetic moment of an individual particle (ignoring collisions).

2. BASIC EQUATIONS

The present paper does not aim to produce formulae for computation. The main attention is directed to the physical processes taking place when a charged particle moves in a magnetic trap. We therefore confine ourselves to the study of the simple Hamiltonian used by FIRSOV[6] ($M = 1$)

$$\mathcal{H} = \frac{p_x^2 + p_y^2 + \omega^2(x)y^2}{2}; \quad p_x = \dot{x}; \quad p_y = \dot{y}. \qquad (2.1)$$

Here x and y are the co-ordinates along and across the magnetic line of force respectively and ω is the Larmor frequency. The equations of motion have the form

$$\ddot{y} = -\omega^2 y; \quad \ddot{x} = -\omega \frac{d\omega}{dx} y^2. \qquad (2.2)$$

* Translated by N. KEMMER from *Atomnaya Energiya* 6, 630 (1959).

!

† The importance of resonances for the change of adiabatic invariants has been pointed out by ANDRONOV, LEONTOVICH and MANDEL'SHTAM.[9]

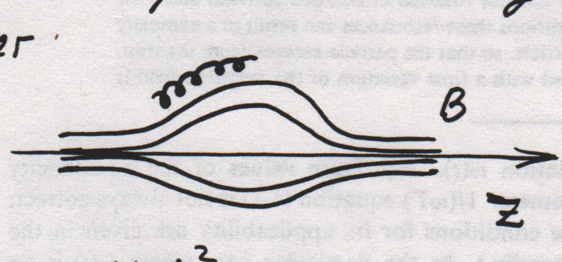
19

(3) # Chirikov criterion of overlapping resonances (1959) !

even 1957 !

(F4) one-particle dynamics in a magnetic trap

Budker

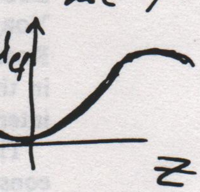


explanation of Rodionov's experiment

Kurchatov Inst.

$\mu = \frac{Mv_\perp^2}{2H}$ — adiabatic invariant ($E_\perp = \omega I$, $\omega = \frac{eH}{mc}$)

$E = \frac{Mv_\parallel^2}{2} + \frac{Mv_\perp^2}{2} \approx \frac{Mv_\parallel^2}{2} + \mu H(z)$



KAM theorem, KS entropy,

Arnold diffusin, standard map...

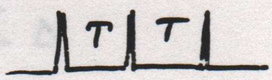
kicked rotator

Resonances overlap in Chirikov standard map

(CSM) $\begin{cases} \bar{I} = I + K\sin\theta \\ \bar{\theta} = \theta + \bar{I} \end{cases}$

$$H = \gamma \frac{I^2}{2} + \frac{k}{T}\cos\theta \sum_{s=1}^{\infty}\left(1 + 2\cos(\omega s t)\right)$$

$(\gamma = T = 1 \to k \to K)$ $\qquad \boxed{\tilde{k} = \frac{k}{T} \qquad \omega = \frac{2\pi}{T}}$



sT

$$H = \gamma \frac{I^2}{2} + \tilde{k}\sum_{s=-\infty}^{\infty}\cos(\theta - s\omega t)$$

$\cos(\theta - \omega s t) \to$ resonance condition

$$\dot{\psi} = \dot{\theta} - s\omega = 0 \quad ; \quad \dot{\theta} = \frac{\partial H}{\partial I} = \Omega_s = \gamma I_s = s\omega$$

Distance between resonances $\delta\Omega$

$$\delta\Omega = \Omega_{s+1} - \Omega_s = \omega = \frac{2\pi}{T}$$

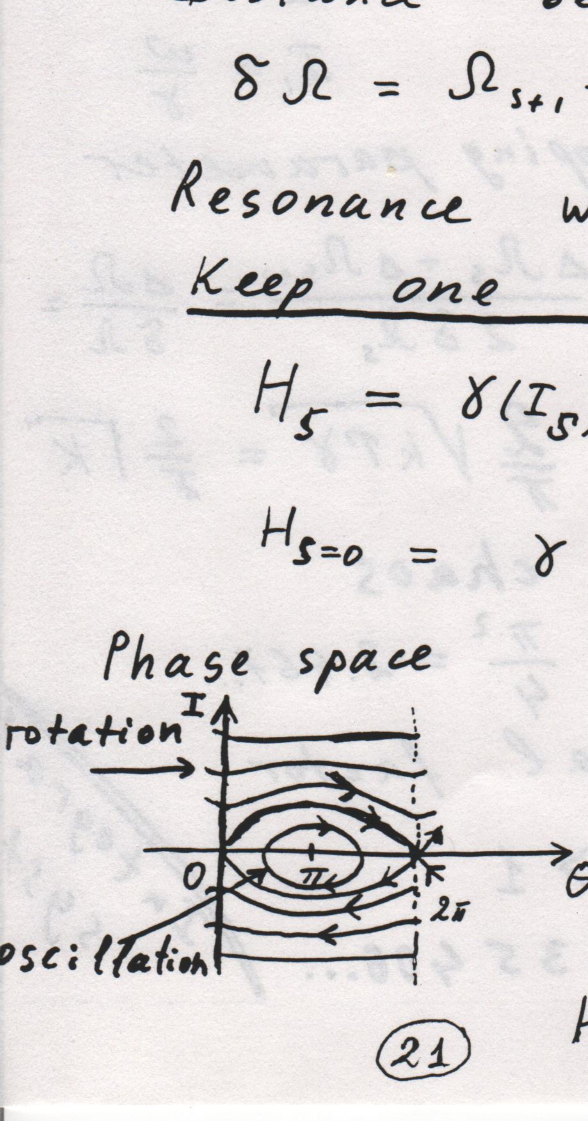
Resonance width $\Delta\Omega$

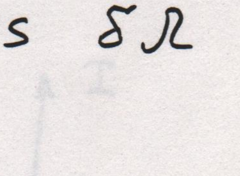
Keep one resonance

$$H_s = \gamma(I_s)\frac{(I - I_s)^2}{2} + \tilde{k}(I_s)\cos\theta$$

$$H_{s=0} = \gamma \frac{I^2}{2} + \tilde{k}\cos\theta \longrightarrow \text{pendulum}$$

Phase space





Fixed points

$\theta = \pi$ - stable

$\theta = 0$ - unstable

Frequency of small phase oscillations $\quad \Omega_\phi = \sqrt{\gamma\tilde{k}}$

$$H_{s=0} = \gamma\frac{I^2}{2} + \frac{\tilde{k}}{2}(\theta - \pi)^2$$

Separatrix

$$H_{s=0} = \frac{\gamma I^2}{2} + \tilde{k}\cos\theta = \tilde{k} \quad \left(\begin{array}{l}\text{energy at unstable} \\ \text{fixed point } \theta = 0\end{array}\right)$$

$$I^2 = \frac{\tilde{k}}{\gamma}\, 4 \sin^2\frac{\theta}{2} \rightarrow I = \pm 2\sqrt{\frac{\tilde{k}}{\gamma}}\,\sin\frac{\theta}{2}\,; \quad \tilde{k} = \frac{k}{T}$$

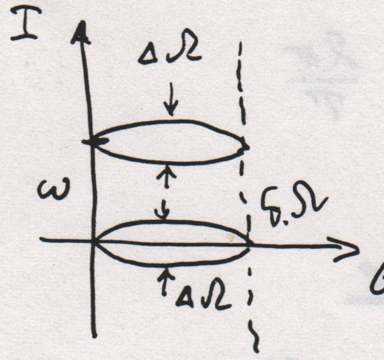
$$\Delta I = 4\sqrt{\frac{k}{\gamma T}}\,; \quad \Delta\Omega = \gamma\Delta I = 4\sqrt{\frac{k\gamma}{T}} = 4\,\Omega_\phi$$

Resonance width

$$H_{s=1} = \frac{\gamma I^2}{2} + \tilde{k}\cos(\underbrace{\theta - \omega t}_{=\varphi})$$

$$H = \frac{\gamma I^2}{2} - \omega I + k\cos\varphi = \frac{\gamma(I - I_1)^2}{2} + k\cos\varphi + \text{co}$$

$$I_1 = \frac{\omega}{\gamma}$$



Overlapping parameter

$$S = \frac{\Delta\Omega_s + \Delta\Omega_{s+1}}{2\,\delta\Omega_s} = \frac{\Delta\Omega}{\delta\Omega} =$$

$$= \frac{2}{\pi}\sqrt{kT\gamma} = \frac{2}{\pi}\sqrt{K} > 1$$

$$\Delta\Omega = 4\sqrt{\frac{k\gamma}{T}}$$

$$\delta\Omega = \frac{2\pi}{T}$$

Global chaos

$$K > \frac{\pi^2}{4} = 2.467\ldots$$

Empirical numerical factor

Second order resonances	$K = 2.5\, S^2 > 1$
Chaotic separatrix layer	$K_{cr} = 0.971635406\ldots$

Integrable system

11a

KS-entropy in Chirikov standard map

$$\delta \underline{r}(t+1) = \begin{pmatrix} 1 & K\cdot\cos x \\ 1 & 1+K\cos x \end{pmatrix} \delta \underline{r}(t)$$

$$\lambda_+ \approx K\cos x \qquad (K \gg 1)$$

$$h = \lambda_1 = \frac{1}{2\pi}\int_0^{2\pi} \ln(|K\cos x|)\,dx = \ln\frac{K}{2}$$

for $K=6$ → 2% accuracy

Rigorous mathematical results

⟹ HET

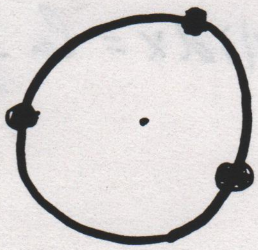
V. Lazutkin (1994 – ...) preliminar results	measure of chaotic component is positive; $h>0$
S. Aubry	anti-integrable limit $K\to\infty$

8

Integrable systems:

Toda (Hamiltonian) Lattice (1970)

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \exp[-(\phi_1 - \phi_3)] + \exp[-(\phi_2 - \phi_1)] + \exp[-(\phi_3 - \phi_2)] - 3$$



Three particles on a ring

Cubic terms → Hénon-Heiles system

Zero Lyapunov exponent (Ford (1973))

Complete integrability Hénon (1974)

Failure of overlapping criterion for system with hidden symmetries

---

Escande, Doveil (1981)
renormalization of two resonances
($K_{cr} = 1.2$)

Diffusion rate and Fokker – Plank equation

$$\bar{y} = y + K \sin x$$
$$\bar{x} = x + \bar{y}$$

random phase-approximation

$$\langle \Delta y \rangle = 0 \; ; \; \langle \Delta y^2 \rangle = K^2 \langle \sin^2 x \rangle \overset{\Downarrow}{=} \frac{K^2}{2} = D_{ql}$$

quasi-linear diffusion rate

$$(\Delta y)^2 = D t \; ; \quad D \approx \frac{K^2}{2}$$

$f(y)$ – distribution function

$$\frac{\partial f}{\partial t} = \frac{D}{2} \frac{\partial^2 f}{\partial y^2}$$

Correlations $\quad y_t = y_0 + K \sum_{m=1}^{t} \sin x_m$

$$C(\tau) = \langle \sin x_\tau \sin x_0 \rangle$$

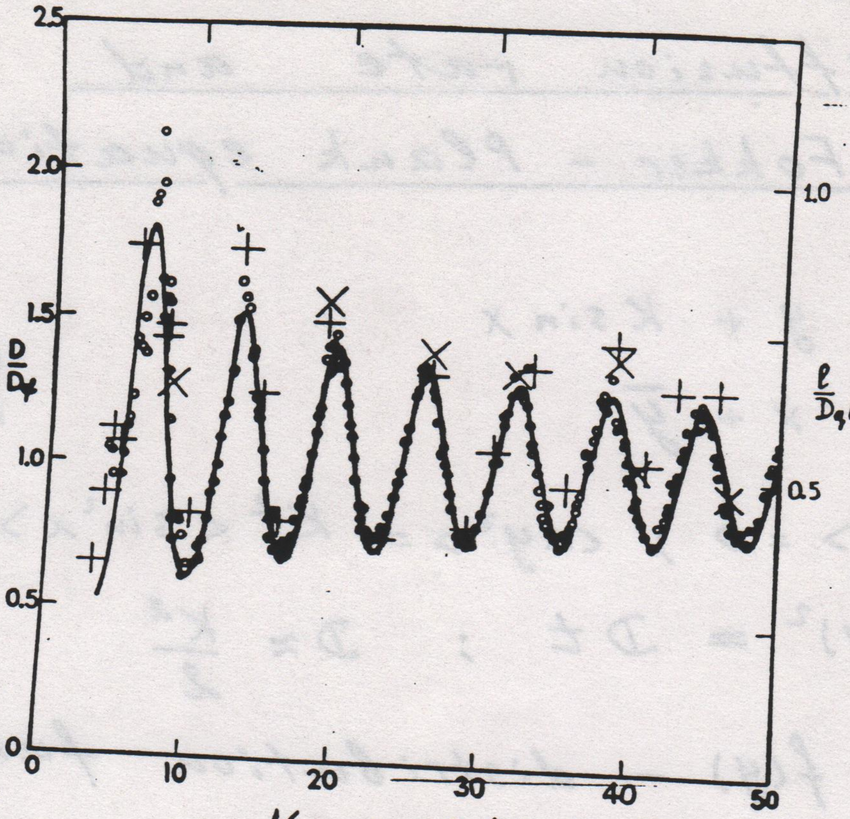
$$D = D_{ql} \left[1 + 4 \sum_{\tau=1}^{\infty} C(\tau) \right] \rightarrow D_{ql} \left[1 - \left(\frac{8}{\pi K} \right)^{1/2} \cos\left(K - \frac{5\pi}{4} \right) \right]$$

$$C(1) = 0 \; ; \quad C(2) = -J_2(K)/2 \sim 1/\sqrt{K}$$
$$C(3) = \frac{1}{2} \left(J_3^2(K) - J_1^2(K) \right) \sim 1/K^2$$
$$C(4) = J_2^2(K)/2 \sim 1/K$$

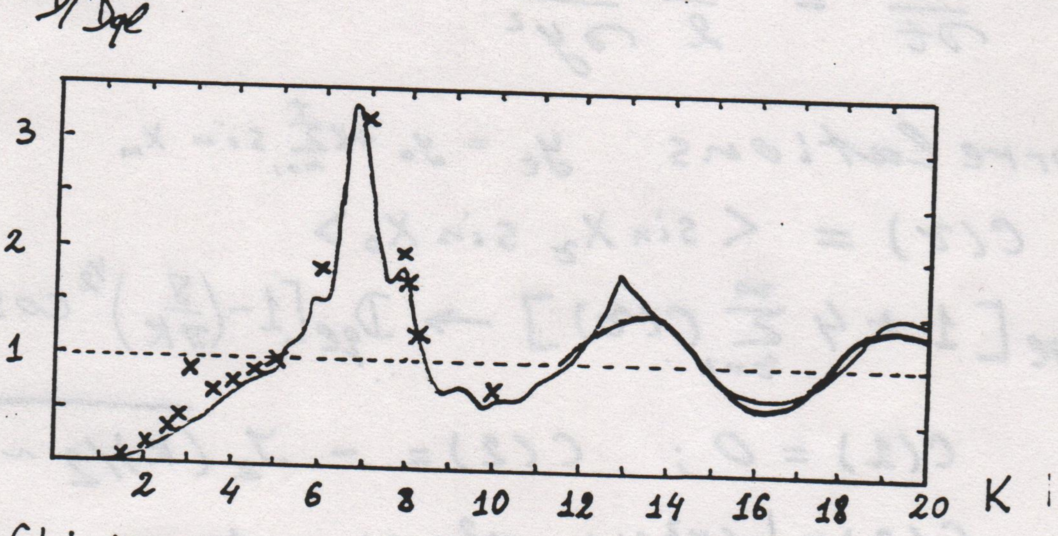
$$D \approx \begin{cases} \frac{K^2}{2} \left(1 - 2 J_2(K) + 2 J_2^2(K) \right), & K \geq 4.5 \\ 0.3 \, (K - K_{cr})^3, & K < 4.5 \end{cases}$$

F5

$k = 30$



$K \rightarrow 2k\sin\left(\frac{T}{2}\right)$



× Chirikov, Shepelyansky (1986) numerical data

Rechester, Rosenbluth, White (1981) theory

accelerating modes

$$(\Delta y)^2 \sim t^{\alpha}$$

$\alpha > 1$

$K \approx 2\pi m$

$\bar{y} = y + K\sin x \approx$
$\approx y + 2\pi m$

F5

26

Diffusion rate for continuous time

$$\frac{dI}{dt} = \sum_n F_n \cos\left[\int \omega_n dt + \theta_n(t)\right]$$

$\omega_n(t)$, $\theta_n(t)$ – slow variables

$$\Delta I \approx \sum_n^N \frac{F_n}{\omega_n}\left[\sin(\omega_n t + \theta_n) - \sin\theta_n\right]$$

$$(\Delta I)^2 \approx \sum_n^N \left(\frac{2F_n}{\omega_n}\right)^2 \sin^2\frac{\omega_n t}{2}\, \overbrace{\cos^2\left(\frac{\omega_n t}{2} + \theta_n\right)}^{\Rightarrow 1/2} +$$

$$+ \underbrace{\sum_{m \neq n} \frac{4F_m F_n}{\omega_m \omega_n} \sin\frac{\omega_m t}{2} \sin\frac{\omega_n t}{2} \cos\left(\frac{\omega_m t}{2} + \theta_m\right)\cos\left(\frac{\omega_n t}{2} + \theta_n\right)}_{\Rightarrow 0}$$

random phases θ_n

$$(\Delta I)^2 \simeq 2\int_{-\infty}^{\infty} F_n^2 \frac{\sin^2 \omega_n t}{\omega_n^2}\, dn = \pi \frac{F_0^2}{\Delta_0} t$$

$$\omega_n = \Delta_0 n$$

$$D = \pi J(0)\,; \quad J(\omega) = \frac{F_\omega^2}{\Delta_0}$$ – spectral density

Chirikov SM

$$\dot{I} = K \sum_n \sin(\theta - 2\pi n t)$$

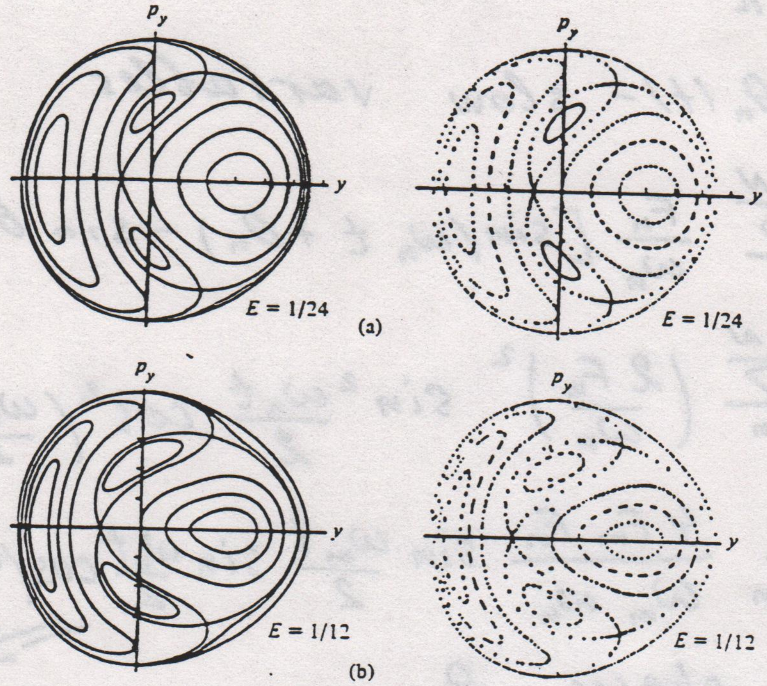
$$D = \pi \cdot \frac{K^2}{2\pi} = \frac{K^2}{2}$$

$t_{diffusive} \gg \tau_{correlations}$

1)

Hénon - Heiles model (1964)

$$H = \frac{p_x^2 + p_y^2}{2} + \frac{(x^2+y^2)}{2} + x^2 y - \frac{1}{3} y^3$$



Comparison of surfaces of section computed from adiabatic theory with those computed numerically (after Gustavson, 1966).

Poincaré surface of section

$(x = 0, \; p_x > 0)$

Homogeneous classical Yang-Mills fields

$$H = \frac{1}{2}\left[E_1^2 + E_2^2 + E_3^2 + m(A_1^2 + A_2^2 + A_3^2) + (A_1A_2)^2 + (A_2^2A_3^2) + (A_1A_3)^2\right]$$

(Matinyan, Savvidy 1981-82)

m – mass from Higgs field ($m=1$)

(E_i, A_i) – canonical pair (Chirikov, D.S. 1981 1982)

Arbitrarily small nonlinearity at $H \to 0$

KAM-theorem cannot be applied

$\Rightarrow$ CHAOS at $H \to 0$ $\quad \omega_1 = \omega_2 = \omega_3$

$$A_k = \sqrt{2I_k}\cos\theta_k, \quad E_k = -\sqrt{2I_k}\sin\theta_k$$

Averaged, resonant Hamiltonian

$$H_r = I_1 + I_2 + I_3 + \frac{1}{2}V$$

$$V = I_1I_2\left[1 + \frac{1}{2}\cos 2(\theta_1-\theta_2)\right] + I_1I_3\left[1 + \frac{1}{2}\cos 2(\theta_1-\theta_3)\right] + I_2I_3\left[1 + \frac{1}{2}\cos 2(\theta_2-\theta_3)\right]$$

$$\underbrace{+\cos 2(\theta_1+\theta_2) + \cos 2(\theta_1+\theta_3) + \cos 2(\theta_2+\theta_3)}_{\to 0}$$

rescaling: $H^0 = I_1 + I_2 + I_3$; $\quad H^0 t \to t$

$\varphi_1 = 2(\theta_1 - \theta_2)$; $\quad \varphi_2 = 2(\theta_2 - \theta_3)$; $\quad J_1 = I_1/H^0$, $J_2 = I_2/H^0$

Renormalized Hamiltonian:

$$H_R = J_1(1 - J_1 - J_2)(1 + \tfrac{1}{2}\cos\varphi_1) + J_2(1 - J_1 - J_2)(1 + \tfrac{1}{2}\cos\varphi_2) + J_1J_2(1 + \tfrac{1}{2}\cos(\varphi_1 - \varphi_2))$$

$H = \frac{1}{2}(p_1^2+p_2^2+p_3^2+x_1^2+x_2^2+x_3^2+x_1^2x_2^2+x_2^2x_3^2+x_1^2x_3^2)$

Further, the motion along each of the axes will now be periodic: $A_1 = A_m \cos t$. The stability of this solution in the linear approximation is determined by the Mathieu equation ($A \equiv A_2$)

$$\ddot{A}+(1+H+H\cos 2t)A=0. \tag{4.6}$$

For $H \gg 1$ the stable and unstable intervals of H have approximately the same width (see, for example, Ref. 22). The centers of the intervals are given by the approximate relations

$$H_{\text{stab}} \approx {}^1/_4\pi^2(n+{}^1/_2)^2-1, \quad H_{\text{unstab}} \approx {}^1/_4\pi^2 n^2-1,$$

where $n > 1$ is an integer.

In this way, the mass terms in the Hamiltonian (1.11) actually stabilize the motion, so that for $H - 0$ the chaotic component is preserved only in an exponentially narrow layer around the separatrix. However, the situation changes fundamentally with an increase in the number of degrees of freedom. Let us consider, for example, the model (1.9) with the mass addition (1.11) but for $N = 3$.

Passing to the action-angle variable as for the case $N = 2$, we arrive at the averaged Hamiltonian

$$\begin{gathered} H_r = I_1+I_2+I_3+{}^1/_3V, \\ V = I_1I_2[1+{}^1/_2\cos 2(\theta_1-\theta_2)]+I_1I_3[1+{}^1/_2\cos 2(\theta_1-\theta_3)] \\ +I_2I_3[1+{}^1/_2\cos 2(\theta_2-\theta_3)]. \end{gathered} \tag{4.7}$$

The principal peculiarity of this model is the presence of, not one [as in (4.2)], but three resonances which are preserved for $H - 0$. For complete integrability of the system two additional integrals are now needed. Nevertheless, (4.7) contains two linearly independent combinations of phases, so that there is only one cyclic combination of phases and correspondingly only one additional integral $H^0 = I_1 + I_2 + I_3$. In these conditions one can expect a sizeable chaotic component of motion for any $H - 0$. Moreover, as for $N = 2$, the structure of the phase space generally does not depend on the quantity H, which determines only the time scale. Actually, thanks to the integral $H^0 = \text{const}$ the system can be reduced to two degrees of freedom. Then if we carry the scale transformation of time $H^0 t - t$ and pass to the canonical variables

$$\begin{gathered} \varphi_1 = 2(\theta_1-\theta_3), \quad \varphi_2 = 2(\theta_2-\theta_3), \\ J_1 = I_1/H^0, \quad J_2 = I_2/H^0, \end{gathered} \tag{4.8}$$

the Hamiltonian of the reduced system assumes the form

$$\begin{gathered} H_R = J_1(1-J_1-J_2)(1+{}^1/_2\cos\varphi_1) \\ +J_2(1-J_1-J_2)(1+{}^1/_2\cos\varphi_2)+J_1J_2(1+{}^1/_2\cos(\varphi_1-\varphi_2)) \end{gathered} \tag{4.9}$$

and does not depend on the energy of the initial system $H \approx H^0$. If the motion of this system is chaotic, then universal chaos in the initial system will be preserved for any weak nonlinear perturbation. This beautiful phenomenon was discovered and investigated in Ref. 23 in a similar model. We remark that the KAM (Kolomogorov-Arnold-Moser) theory is inapplicable in this case, since the unperturbed system (linear oscillator) is isochronous.[24]

The investigation of the dynamics of the system (4.9) was carried out by means of numerical modeling. The accuracy of the conservation of the integral H_R is in the interval from 10^{-3} to 10^{-8} and does not influence the

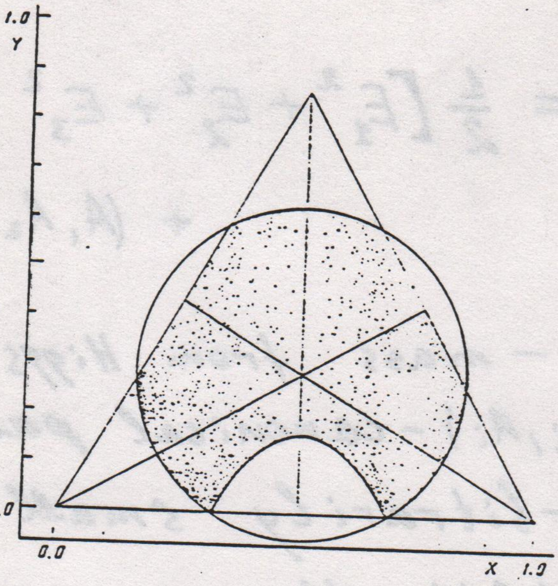


FIG. 3. Same as in Fig. 2; $H_R = 0.324$, $h_R = 0.14$.

F6

$$X = {}^1/_2(1+J_2-J_1), \quad Y = {}^1/_2\sqrt{3}(1-J_1-J_2). \tag{4.10}$$

The energetically accessible region of motion is represented by the intersection of the region inside the circle

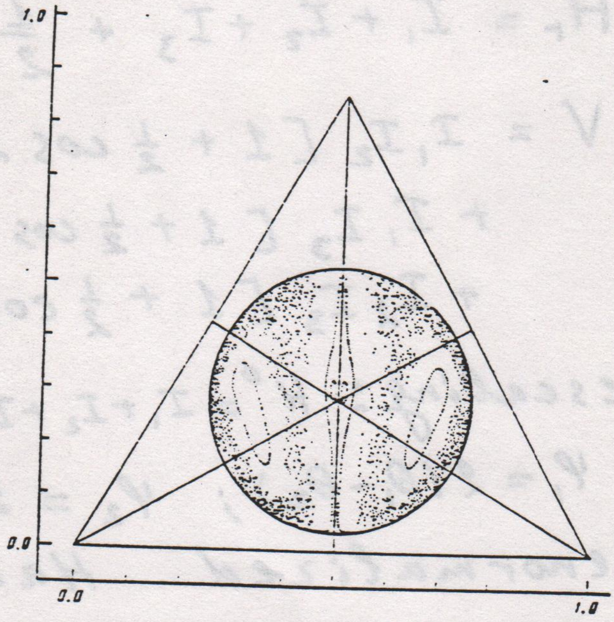


FIG. 2. Surface of the Poincare cross section for the system (4.9); $H_R = 0.404$. The picture of motion is symmetric relative to a vertical line. The center of the triangle coincides with the center of the circle, which bounds the energetically allowable region of motion. The irregularly distributed points belong to one chaotic trajectory; $h_R = 0.026$.

$h = H^0 h_R$