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6

## Destruction of invariant curves

$$\bar{y} = y + K\sin x \qquad x_t = 2\pi r t + \sum_{j=0}^{q_m - 1} A_j \cos\left(\frac{2\pi j t}{q_m}\right)$$
$$\bar{x} = x + \bar{y}$$

Winding number (rotation number)

$$r = \frac{\omega}{2\pi} = \frac{1}{2\pi} \lim_{t\to\infty} \frac{X(t) - X(0)}{t}$$

Continuous fraction expansion

$$r_n = \frac{p_n}{q_n} = [a_1, a_2, \ldots\ldots, a_n] \quad \left(\begin{array}{l} p = p_n \\ q = q_n \end{array}\right)$$

best rational approximates

Minimal detuning (n - renormtime)

$$\nu_q = q r \,(\mathrm{mod}\ 1) = q r - p; \quad \left| r - \frac{p_n}{q_n} \right| \sim \frac{C}{q_n^2}$$

$$\nu_q \to \frac{c}{q}; \quad \text{Maximal } \; C = \frac{1}{\sqrt{5}}$$

⇓

Most irrational number

$$r_G = [1, 1, 1, \ldots] = \frac{\sqrt{5} - 1}{2} \quad \text{(golden mean)}$$

$$q_{n+1} = q_n + q_{n-1}$$

$$\lambda^2 = \lambda + 1 \;\to\; \lambda = \frac{\sqrt{5} + 1}{2} \simeq 1.618\ldots$$

$$\frac{q_{n+1}}{q_n} \to \lambda = S_0$$

7

## Stability of periodic orbits with period $q_n$ ( Greene's approach (1979),

Matrix of linearized motion (monodromy matrix)

$$M = \prod_{i=1}^{q} \begin{bmatrix} 1 & K\cos x_i \\ 1 & 1+K\cos x_i \end{bmatrix}; \quad R \propto \exp(q(K-K_{cr}$$

Periodic orbit is stable if

Residua $0 < R = \dfrac{2 - SpM}{4} < 1$

$( \det(M-\lambda) = \lambda^2 - \lambda SpM + 1 = 0 \rightarrow -2 < SpM < 2 )$

$R = 1$ ( const) for $r_n = p_n/q_n$

$K \rightarrow K_{cr} = 0.971635406\ldots$ (golden mean in st. map)

$K = K_{cr}$

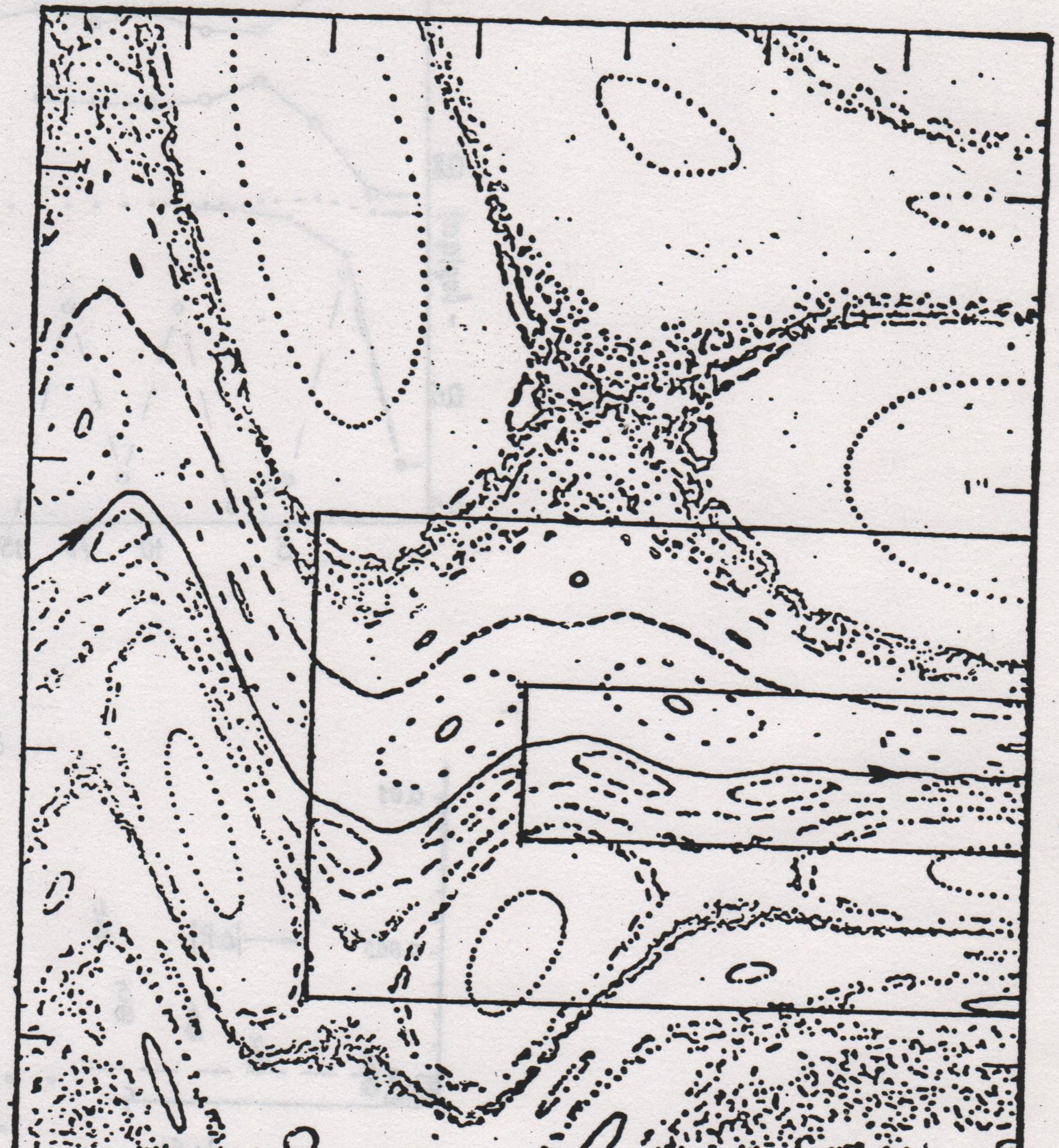
$R \rightarrow R_{cr} = 0.2500888\ldots$ for $r_G$ (universal value for any map)

( numerical tric to find highly periodic orbits — Greene (1979

⇓

symmetri line on which there is one periodic point $(x = \pi)$ )

R. S. MacKay (1982)

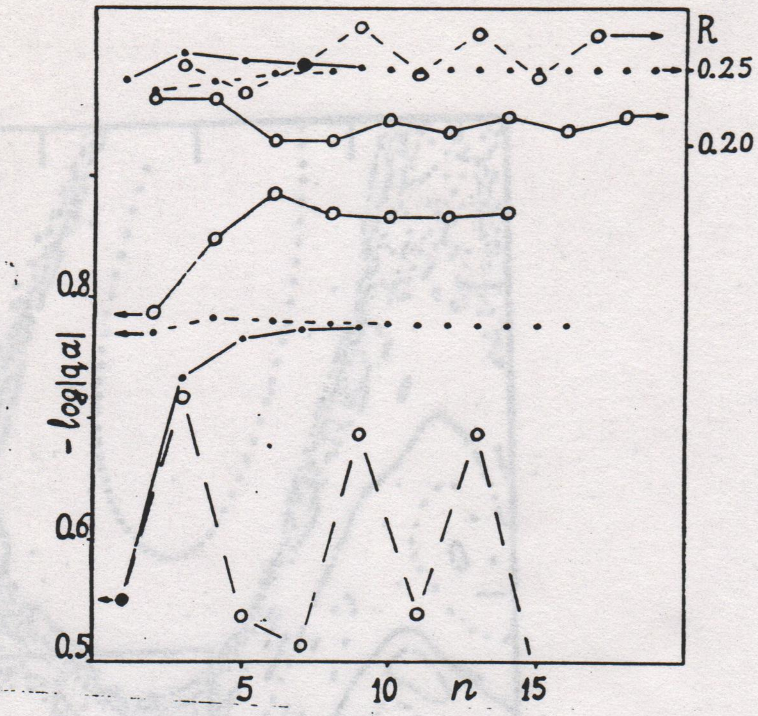


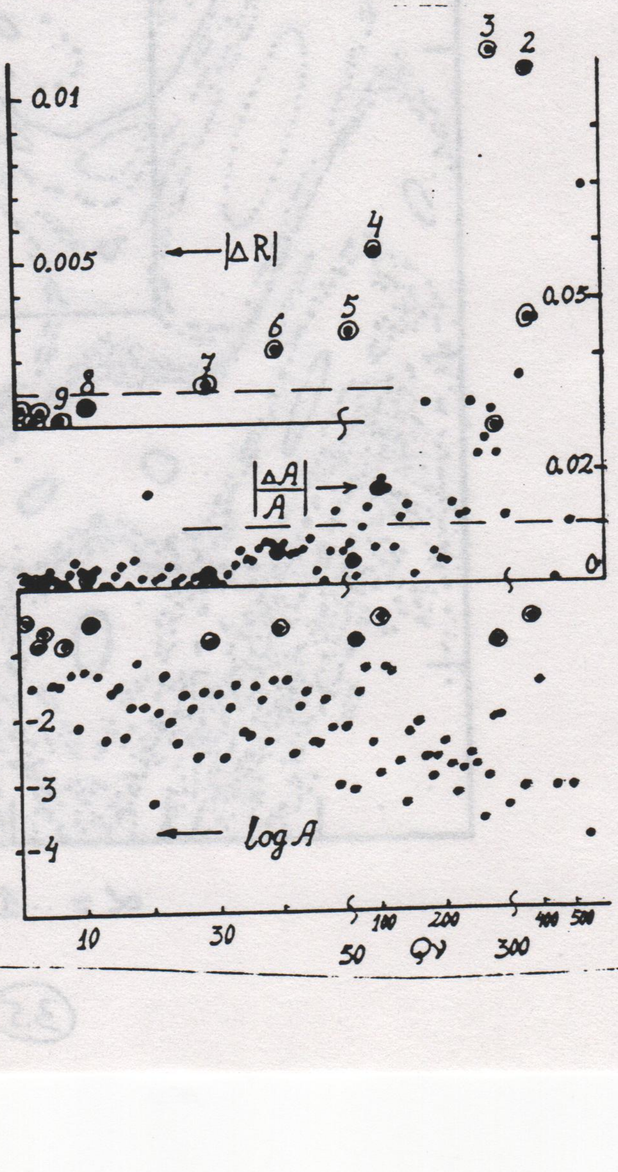
β = −3.0668882...

F7

α = 1.4148360...

Chirikov, D.S. (1988)




F8

# Estimates for Scaling Exponents

Period of resonant structure (along curv

$$\Delta x \sim 1/q_n \rightarrow S_0 = \lim_{n\to\infty} \frac{q_{n+1}}{q_n} \quad \left( \frac{1}{r_G} = 1.61803\ldots \right.$$

In ortogonal direction

$$\Delta y \sim \delta \omega_q = \left| r_G - \frac{p_n}{q_n} \right| \sim \frac{C}{q_n^2}$$ — distance between resonances

$$\Delta \omega_q \sim \Omega_{ph}/q_n$$ — resonance width

$$\left( H_{eff} = \frac{p^2}{2} + V_q \cos qx \rightarrow \Omega_{ph}^2 = V_q q^2 \right.$$

$$\Delta \omega_q \sim V_q^{1/2} \sim \Omega_{ph}/q$$

Overlapping of resonances

$$\frac{\Delta \omega_q}{\delta \omega_q} \sim q^2 \Delta \omega_q \sim q\,\Omega_{ph} \sim 1 \ (\text{const})$$

$\Downarrow$

$$s_y = s_0^2 \quad (\Delta y \sim q^{-2})$$

$$s_t = s_0^{-1} \quad (t \sim \Omega_{ph}^{-1} \sim q)$$

$$s_x = s_0 \quad (\Delta x \sim q^{-1})$$

(9)

# Analiticity of perturbation

$$V_q \propto \exp(\sigma q), \quad \sigma \propto \varepsilon = K - K_{cr}$$

$$s_\varepsilon = s_0 \quad (\varepsilon \sim q^{-1}; \quad K_n = K_{cr} + C\delta^{-n}, \ \delta = s_\varepsilon$$

## Numerical data vs. estimates

$$s_x / s_0 = 0.874 \qquad s_\mu = s_x s_y$$

$$s_y^{1/2} / s_0 = 1.082 \qquad s_\mu^{1/3} / s_0 = 1.0081$$

$$s_\varepsilon / s_0 = 1.0061$$

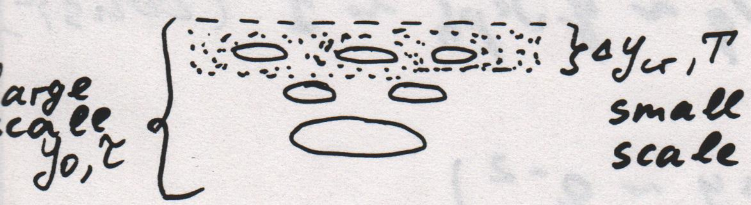
(for golden mean)

Destroyed invariant curve

$\Rightarrow$ cantorus (invariant Cantor set)

## Diffusion across cantorus

$$\varepsilon = K - K_{cr} > 0 \qquad \varepsilon \sim 1/q_{cr}$$




$$\Delta y_{cr} \sim q_{cr}^{-2}$$

$\tau$ – time of crossing $y_0 \sim 2\pi$

$T$ – time scale near cantorus

$$T \sim 1/\Omega_{ph} \sim q_{cr} \sim 1/\varepsilon$$

From ergodicity:

$$\frac{T}{\tau} \sim \frac{\Delta y_{cr}}{y_0}$$

MacKay Percival Meiss (1984)

$\tau \sim \varepsilon^{-2}$, $\eta = 3$; $\eta = 3.011722\ldots = \log s_\mu / \log s_\varepsilon$

Fractal diagram (Schmidt, Bialek (1981))

Given rotation number $r$

Critical parameter value

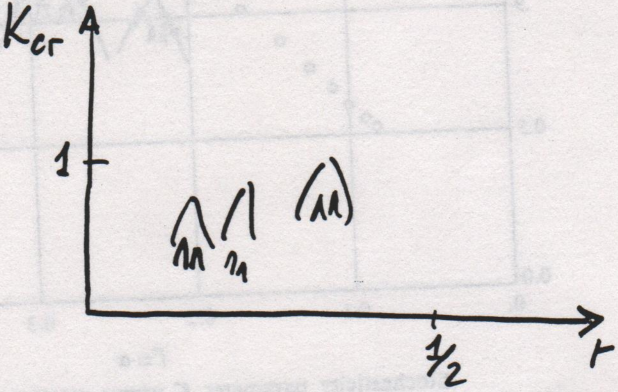
$K_{cr}(r)$

Where are local maxima in fractal diagram?

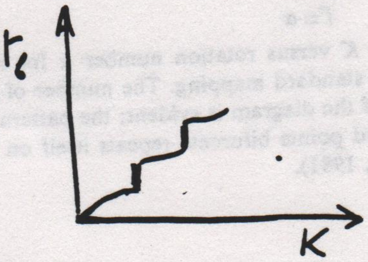
Greene → numbers with golden tailes

$r = [\ldots\ a_n, 1, 1, 1, \ldots]$

F9




$r_b(K)$




$r_b(K)$ - boundary rotation number

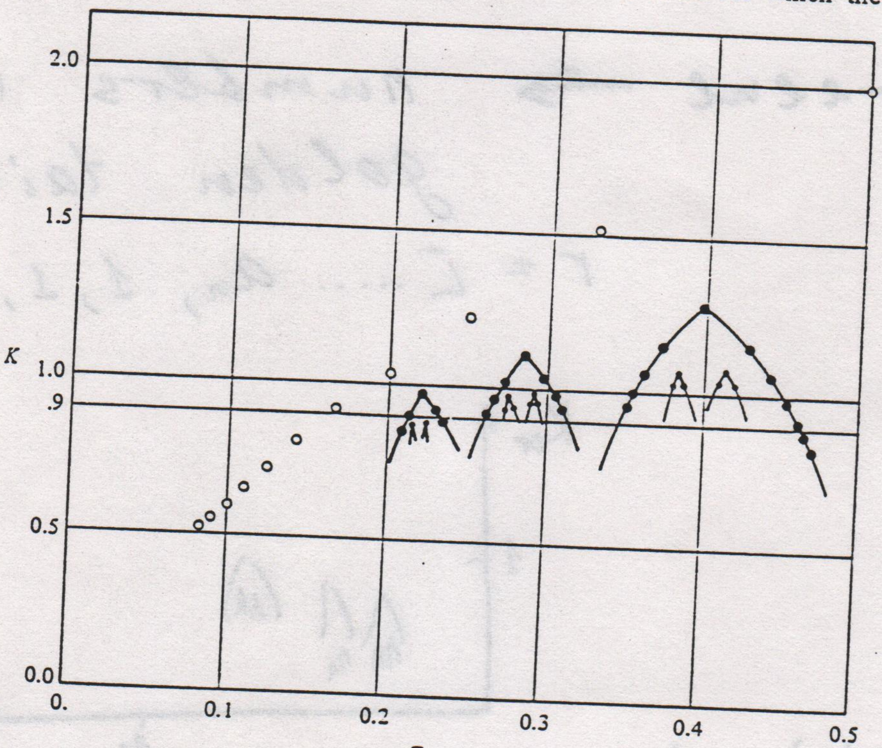
Markov tailes $qr - p \to \frac{c}{q}$ with $c > 1/3$

$c_M = \frac{M}{[(3M)^2 - 4]^{1/2}}$ ; $a_n = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$ ⇓ discrete spectrum of $C$

Another viewpoint that connects the destabilization of fixed points with the disappearance of KAM curves orders the marginally stable fixed points in a *fractal* diagram (Schmidt and Bialek, 1981). The basic idea is to order the fixed points by rotation number $\alpha$, with the first two orders given by

$$\Gamma_1 = \alpha_1 = \frac{1}{n}, \quad \Gamma_2 = \alpha_2 = \frac{1}{n \pm 1/m}, \tag{4.4.13}$$

respectively, where the $n$ and $m$ values are the positive integers. For $m$ large the fixed points approach the island separatrix of the associated $n$, while for $m = 1$, $\alpha_2 = \alpha_1$ of the neighboring island chains. For the first three such orderings, the value of $K$ at which the fixed points become unstable is plotted, for the standard mapping, in Fig. 4.10. Excluding the main island ($n = 1$), the values of $K$ for which the $n = 2, 3, 4, \ldots$ fixed points (open circles) become unstable are seen to fall on a smooth curve of descending values of $K$. (The curve is symmetric about $\alpha = 0.5$, with the other half not shown.) Between each pair of $n$ values, the values of $K$ for which the




Stochasticity parameter $K$ versus rotation number $\alpha$ for marginally stable families of fixed points for the standard mapping. The number of islands in the chain is $1/\alpha$. The fractal nature of the diagram is evident; the pattern of values of critical $K$ at which families of fixed points bifurcate repeats itself on finer and finer scales (after Schmidt and Bialek, 1981).

# External Renormchaos

$$r_B = r_{RANDOM} = [2, 1, 1, 1, 2, 1, 2, 1, 1, \dots]$$

Universality for all maps

Chaos - Chaos transition ( $r_G$ )

Chaos - Order transition ( $a_n < 5$ )

---

Destruction of 2-frequency torus :

Breakdown of universality

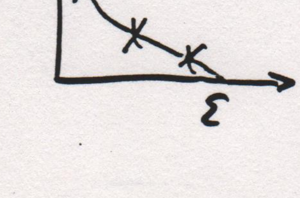
$$\bar{y} = y - (K + \varepsilon \cos z) \sin x$$
$$\bar{x} = x + \bar{y}$$
$$\bar{z} = z + 2\pi r_2$$

spiral mean

$$r_1 = \lim_{t \to \infty} \frac{x_t - x_0}{2\pi t}, \quad r_2 = \lim_{t \to \infty} \frac{z_t - z_0}{t}$$

$$r_1 = 1/\vartheta^2, \quad r_2 = 1/\vartheta \ ; \quad \vartheta = 1.324718\dots$$

$$\theta^3 - \theta - 1 = 0 \qquad (q_n = q_{n-2} + q_{n-3})$$

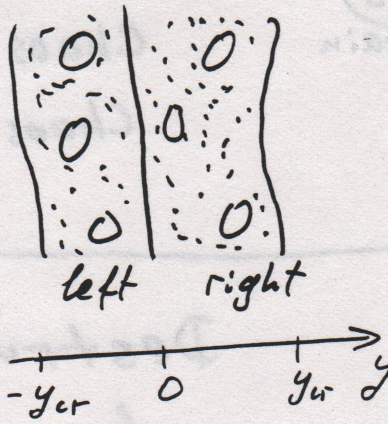



Different behaviour of residua $R_n$

2)

# Statistics of Poincare Recurrences

Probability to stay in a left half of chaotic layer during time $t > \tau$




$$P(\tau) = \frac{N(t>\tau)}{N_{total}} > 0$$

N - number of crossings.

$$\bar{y} = y + \sin x \qquad \lambda \gg 1$$
$$\bar{x} = x - \lambda \, | \ln \bar{y} |$$

$$|y| < y_{cr} = y_{cr}(\lambda) \gg 1$$

Diffusion time $t_D \approx y_{cr}^2 \gg 1$

$$P(\tau) \sim 1/\sqrt{\tau} \quad \text{for} \quad \tau < t_D$$

$$P(\tau) \sim \frac{1}{\tau^p} \quad \text{for} \quad \tau > t_D$$

Average numerical value

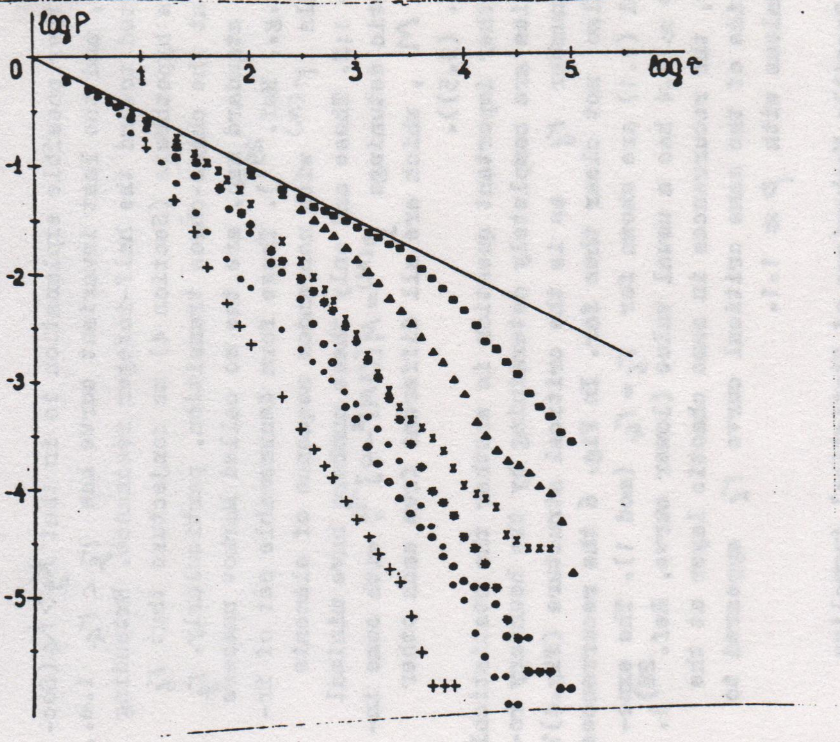
$$p \approx 1.45$$

(Chirikov, D.S. )
(1981)

F10

F11

*Fig. 1.* Distribution of Poincare's recurrences in the chaotic layer of map (1.1): $10^7$ iterations for each $\lambda = 1$ (+); 3 (•); 5 (○); 7 (∗); 10 (×); 30 (▲); and 100 (■); the straight line is $P(\tau) = \tau^{-1/2}$.




$\bar{y} = y + \sin x$

$\bar{x} = x - \lambda \ln|\bar{y}|$

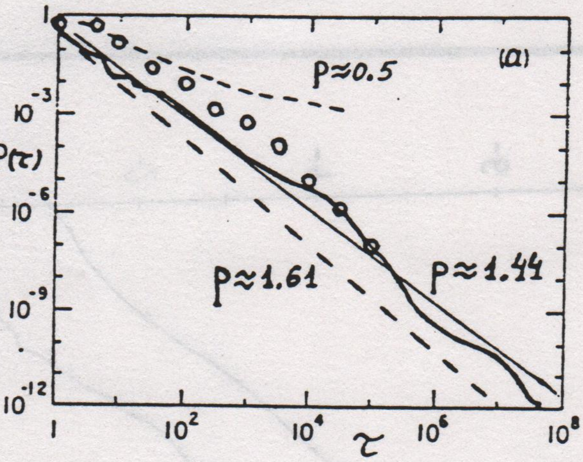
"wisker" map

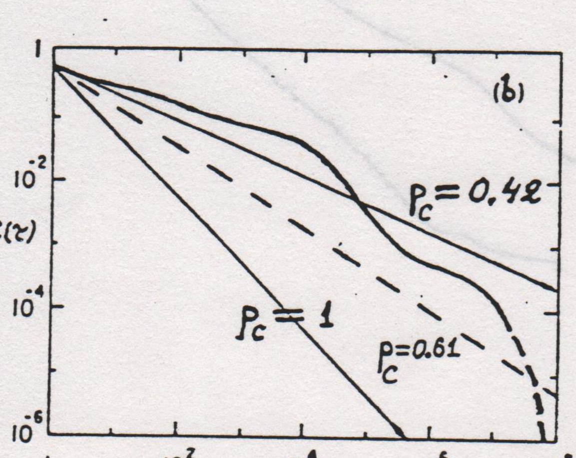
Chirikov, Shepelyansky (1981)

$P(\tau) \sim \frac{1}{\tau^{p}}$

$p \approx 1.45$

Karney (1983)




Chirikov, Shepelyansky (1988)

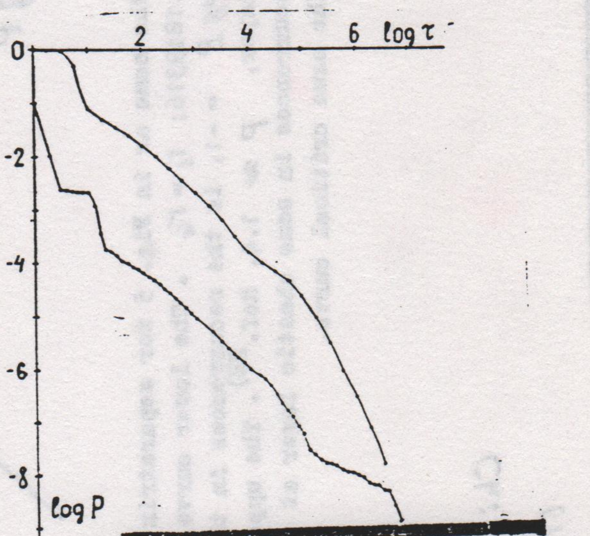



*Fig. 5.* Distribution of Poincare's recurrences in the two chaotic layers of standard map at $K = K_c$. Upper curve [28] relates to the layer with integer resonance while the lower one does so to that with half-integer resonance and is shifted by $\Delta \log P = -1$. $P(\tau)$ oscillation at small $\tau \lesssim 10$ is due to stable regions around resonance centers.

F10

(actually $\langle\tau\rangle \sim \ln T$ where $T$ is the total motion time).

Another possible explanation is in that $K_g > K_G$ (Section 3), and the last invariant curve has $r_g < r_G$ i.e., is shifted toward the half-integer resonance. Extending Greene's hypothesis (Section 4) we conjecture that $r_b$ values at the chaos-chaos transition, particularly, $r_g$ for the standard map, are the so called Markov numbers (see, e.g., Ref.[29]). Those form denumerable set of irrationals $r^{(M)}$ with nonrandom sequence of elements $m_n$ = 1;2. These and only these numbers have minimal asymptotic detunings $\rho^{(M)} = M[(3M)^2-4]^{-1/2}$, with some integers $M$, which are all different from each other (see eq. (3.5)).

Another important question is whether the statistical properties are completely determining by the boundary rotation number $r_b$ as is the critical structure (Fig.4)? It is also not clear thus far. In Fig. 6 the recurrences in model (1.1) are shown for $r_b = r_G$ (mod 1). The exponent $p \approx 1.4$ has a usual value (lower curve, Ref.[28]). However, the recurrences in some chaotic layer at the other side of the same critical curve $r_b$ appeared to be anomalous with $p \approx 1.1$.

## 6. Possible Mechanisms of Statistical Anomalies

The simplest quantitative conjecture for the trajectory "sticking" near a chaos border is in that the transition time $(\tau_n)$ from one scale to the next is of the order of a characteristic time for a given scale $(t_n)$:

$$\tau_n \sim t_n \sim q_n \qquad (6.1)$$

As $t_n$ rapidly drops with $n$ $(t_n/t_{n-2} \sim s^2)$ the total sticking time $\tau \sim \tau_n$. On the other hand, the measure

F11



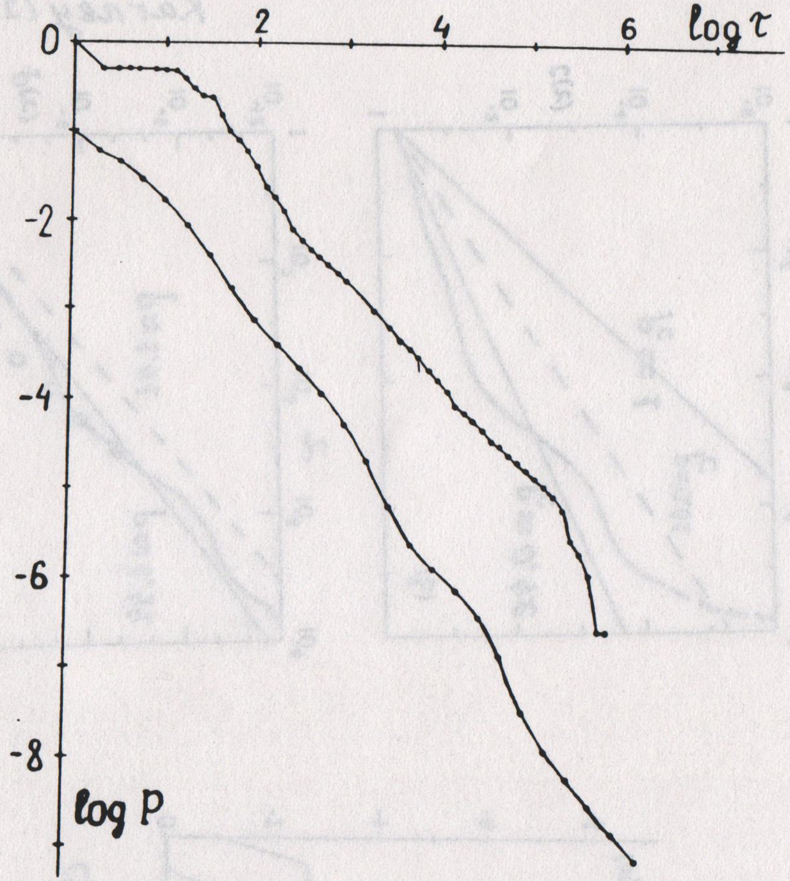



Fig. 6. The same as in Fig. 5 for separatrix map (1.1): $\lambda$ = 3.1819316; $r_b = r_G$ . The lower curve, shifted by $\Delta \log P$ = -1, is the recurrences in the main chaotic layer, $p \approx 1.4$, Ref.[28]. The upper curve is the recurrences in some chaotic layer at the other side of the same critical curve.

44

Chirikov, Shepelyan.
(1984, 1988)

23 Ergodicity relation and connection of $\rho(\tau)$ with correlation function:

$$\mu = \frac{\tau_\mu}{t} = \frac{\tau N_\tau}{N\langle\tau\rangle} \sim \frac{\tau \rho(\tau)}{\langle\tau\rangle} \sim \tau \rho(\tau)$$

↑ measure of sticking region

↑ ergodicity relation

$C(\tau)$ - correlation function

$$C(\tau) \sim \mu \sim \tau \rho(\tau) \sim \frac{1}{\tau^{p-1}} \sim \frac{1}{\tau^{p_c}}$$

$$\boxed{p_c = p - 1 \approx 0.45 < 1}$$

$$D \sim \int_0^t C(\tau)\, d\tau \sim t^{2-p} \sim t^{0.5}$$

$$(\Delta x)^2 \sim D t \sim t^{1.5}$$ — anomalous diffusion

(if $C(\tau)$ has one sign (not oscillat.

Scaling near the golden curve

$$\mu \sim \Delta y \sim \frac{1}{q^2} \sim \frac{1}{T_q^2} \sim \frac{1}{\tau^2} \rightarrow p = 3$$

$$q \sim T_q \sim \frac{1}{\Omega_{ph}}$$

4 

# Frenkel - Kontorova model

$$V = \sum_i \frac{1}{2}(x_i - x_{i-1})^2 - K\cos x_i \qquad (1938)$$

$$\frac{\partial V}{\partial x_i} = 2x_i - x_{i-1} - x_{i+1} + K\sin x_i = 0$$

$$p_{i+1} = x_{i+1} - x_i$$

$$\Longrightarrow \qquad p_{i+1} = p_i + K\sin x_i$$

$$x_{i+1} = x_i + p_{i+1}$$

Chirikov standard map

rotation number

$$r = \frac{x_N - x_0}{2\pi N} \Rightarrow \text{density of particles}$$

$$u_i = x_i \ (\mathrm{mod}\ 2\pi)$$

$$u_i = f(i \cdot 2\pi r + \alpha) \ (\mathrm{mod}\ 2\pi)$$ — hull functio

$K < K_{cr}(r) \rightarrow f$ — smooth function

$K > K_{cr}(r) \rightarrow f$ — devil's staircase (cantorus)

12 $K < K_{cr} \rightarrow$ phonon spectrum

13 $\omega(k) \sim ck$

$K > K_{cr} \rightarrow$ phonon gap

Aubry theorem (1978 – 1983) → transition of breaking analiticity

M Peyrard and S Aubry (1983)

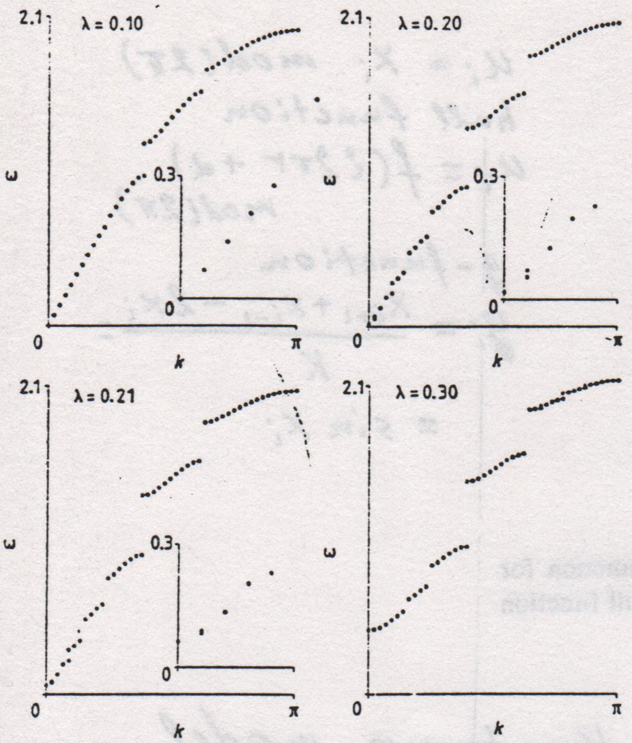



Figure 3. Phonon spectrum of the FK chain for different values of $\lambda$ (note that the lowest eigenvalue $\omega_G$ is non-zero for $\lambda \geqslant 0.2$). (Since the system has no real translational invariance the wavevector $k$ is not obviously defined. Eigenfrequencies are here plotted in increasing order so that in the limit $\lambda \to 0$ we obtain the usual phonon branch.)

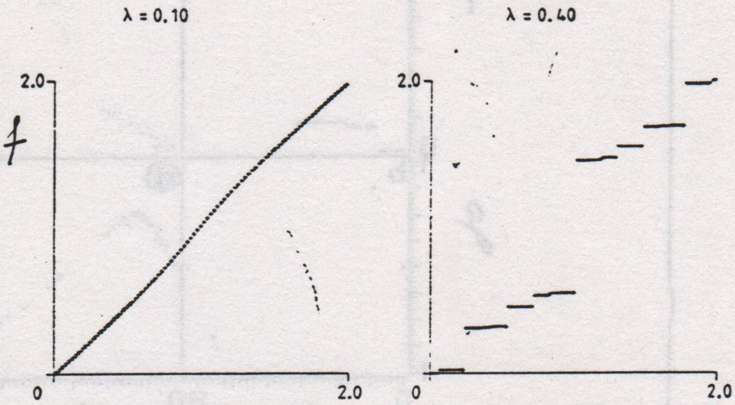



Figure 1: Two typical aspects of the hull function $f(x)$ describing the ground state for $\lambda < \lambda_c$ ($\lambda = 0.10$) and $\lambda > \lambda_c$ ($\lambda = 0.4$).

$$\lambda = \frac{2K}{\pi^2}$$

$$\lambda_c = \frac{2 \times 0.9716}{\pi^2} \simeq 0.1968\ldots$$

F12

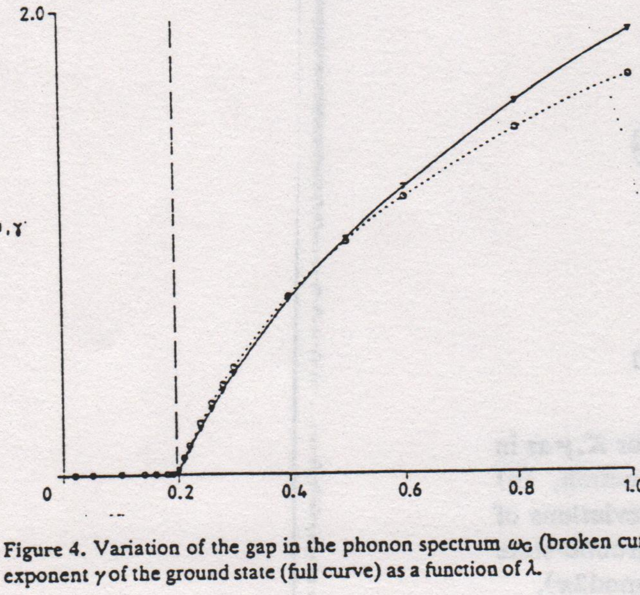
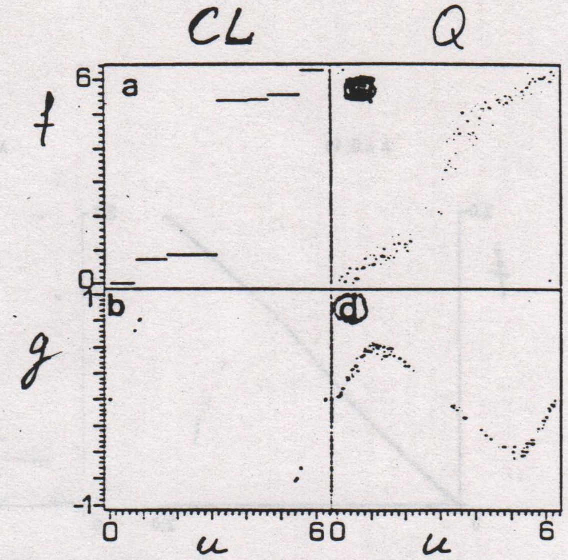



Figure 4. Variation of the gap in the phonon spectrum $\omega_G$ (broken curve) and Lyapunov exponent $\gamma$ of the ground state (full curve) as a function of $\lambda$.

F. Borgonovi, I. Guarneri, D. Shepelyansky
(1989)




$u_i = x_i \mod(2\pi)$

hull function

$u_i = f(i 2\pi r + \alpha) \mod(2\pi)$

g-function

$g_i = \dfrac{x_{i+1} + x_{i-1} - 2x_i}{K} = \sin x_i$

FIG. 1. Classical (a) hull function and (b) g function for winding number $\nu = 233/377$, $K = 5$; [redacted] hull function and [redacted] g function, for the same $\nu$ and $K$, and $\hbar = 3$.

# Quantum Frenkel - Kontorova model

F13

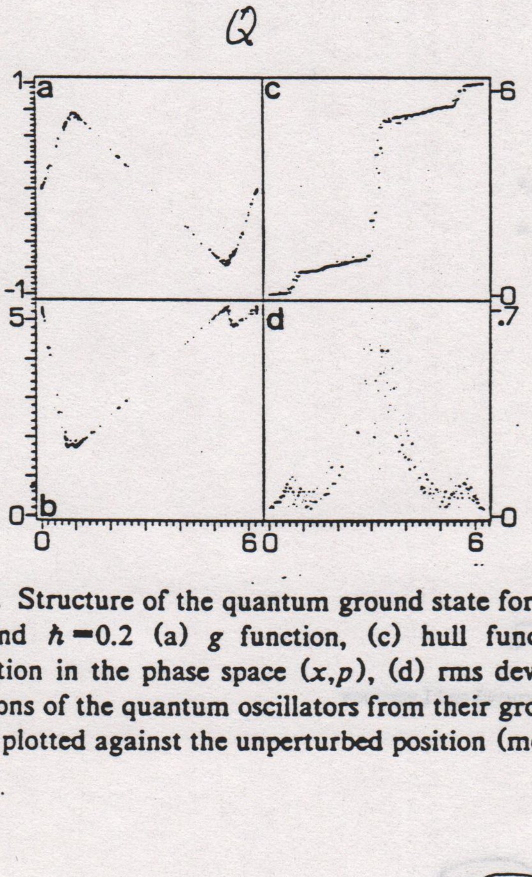



FIG. 2. Structure of the quantum ground state for $K, \nu$ as in Fig. 1 and $\hbar = 0.2$ (a) g function, (c) hull function, (b) configuration in the phase space $(x,p)$, (d) rms deviations of the positions of the quantum oscillators from their ground-state averages, plotted against the unperturbed position (mod$2\pi$).

$$x_i(t) = \bar{x}_i + \varepsilon_i(t)$$

$$\ddot{\varepsilon}_i(t) = - \sum_j \frac{\partial^2 V(x_i)}{\partial x_i \partial x_j} \varepsilon_i(t)$$

$$\omega^2 \varepsilon_i(\omega) = - \varepsilon_{i+1}(\omega) - \varepsilon_{i-1}(\omega) + 2\varepsilon_i(\omega) + K \cos x_i \, \varepsilon_i(\omega)$$

$\varepsilon_i = f'(i 2\pi r + \alpha)$ — for analytical $f$ $(K < K_c)$

$K > K_c$

$\omega_G(K) \propto (K - K_c)^{\chi}$ — gap size

M. Peyrard, S. Aubry (1983)

$1.00 < \chi < 1.03$

$\omega_G \sim \Omega_{ph} \sim \frac{1}{q} \sim \Delta K \rightarrow \chi = 1$

LECTURE 4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