LECTURE 4

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Quantum Chaos

1. Conservative systems:

discrete spectrum →

no exponential local instability,
regular time evolution of Ψ-function.

Manifestations of classical chaos in quantum systems.

a) ergodicity of eigenfunctions

A. Shnirelman theorem (I) (1974)

$$\int \Psi_n^* \hat{A} \Psi_n \, dx = \int_{\mu} A \, d\mu$$ (for chaotic billiards)

b) level spacing statistics

Wigner - Dyson $P_W = \frac{\pi}{2} s \exp(-\frac{\pi s^2}{4})$

(Bohigas, Giannoni, Schmidt) 1984 chaotic systems

Poisson $P_P = \exp(-s)$ $(\langle \Delta E \rangle = 1)$

(Berry, Tabor 1977) integrable systems

A. Shnirelman theorem (II) (1975)

Half of levels are degenerate

$P_S \sim 1/s$ (due to time reversability)

F14)

 (Chirikov, D.S. 1994)

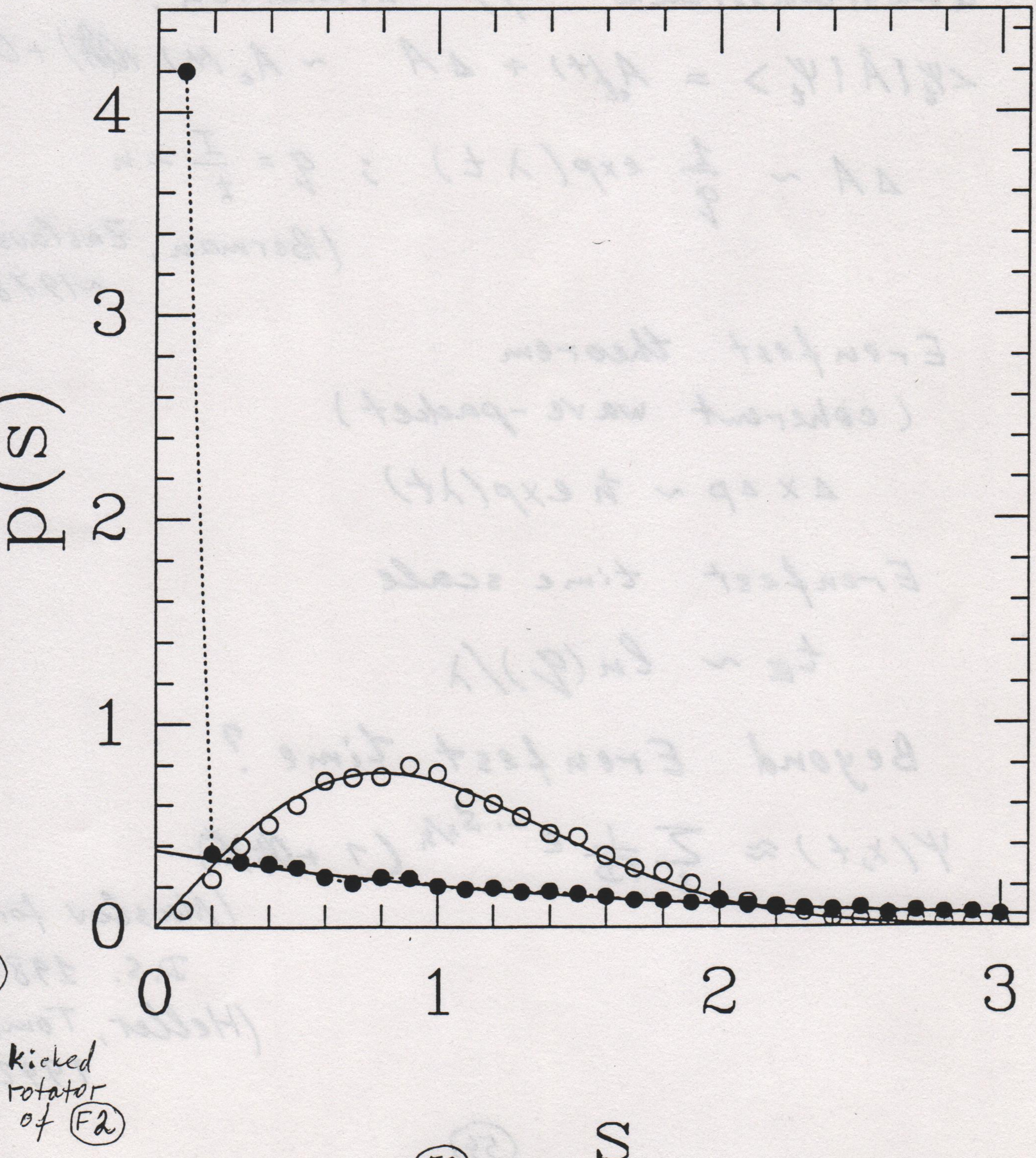
Fig. 1 $\alpha = 0, \ \gamma = 1/2$

● $k = 6 \div 10$ $r = 2$, $\sigma = 0.62$

$T = 4\pi/N \approx 0.025$ $N = 501$ Poisson
$p(s) = \sigma^2 \exp(-\sigma s)$

o $k = 25 \div 30$, $T = 40\pi/N$, $N = 50$, $D/N \approx 1.5$

statistics 10000



(14)

kicked rotator of (F2)

c) Gutzwiller quantization
unstable periodic orbits

$$N \sim \exp(T\lambda)$$

2. Quasiclassical approximation:

$$\langle \Psi_t | \hat{A} | \Psi_t \rangle = A_{cl}(t) + \Delta A \quad \sim A_c(t) + O(\hbar) + O(\hbar^2 \ldots)$$

$$\Delta A \sim \frac{1}{q} \exp(\lambda t) \quad ; \quad q = \frac{I}{\hbar} \sim n$$

(Berman, Zaslavsky ~1978)

Erenfest theorem
(coherent wave-packet)

$$\Delta x \Delta p \sim \hbar \exp(\lambda t)$$

Erenfest time scale

$$t_E \sim \ln(q)/\lambda$$

Beyond Erenfest time?

$$\Psi(x,t) \approx \sum_s \frac{1}{\sqrt{D}} e^{-iS_s/\hbar} (1 + O(\hbar)t^{*})$$

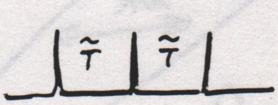
(Maslov formula
D.S. 1982
(Heller, Tomsovic,
1992

Periodically driven systems.

Kicked Rotator (Casati, Chirikov, Ford, Izrailev 1977-1979)

(quantized Chirikov standard map)

$$H = \frac{\hat{p}^2}{2} + \hat{k}\cos\hat{x}\ \delta_{\tilde{T}}(t)$$



$$[\hat{p},\hat{x}] = -i\hbar \qquad \psi(x+2\pi) = \psi(x)\ ; \quad \hat{p} = -i\hbar\frac{\partial}{\partial x}$$

Near the kick

$$i\hbar\frac{\partial\psi}{\partial t} = \tilde{k}\cos x\ \delta(t)\,\psi$$

$$\frac{d\psi}{\psi} = -i\frac{\tilde{k}}{\hbar}\cos x\ \delta(t)\,dt$$

$$\ln\psi|_{+0} - \ln\psi|_{0} = -i\frac{\tilde{k}}{\hbar}\cos x$$

$$\rightarrow \quad \psi_{+0} = \exp\left(-i\frac{\tilde{k}}{\hbar}\cos x\right)\psi_{-0}$$

rotation

$$\psi_{\tilde{T}} = \exp\left(-i\frac{H_0\tilde{T}}{\hbar}\right)\psi_0 = \exp\left(+i\frac{\hbar\tilde{T}}{2}\frac{\partial^2}{\partial x^2}\right)\psi_0$$

$$T = \hbar\tilde{T}; \quad k = \frac{\tilde{k}}{\hbar}; \quad K_{cl} = kT$$

$(T\to 0,\ k\to\infty,\ kT = K = \text{const})$

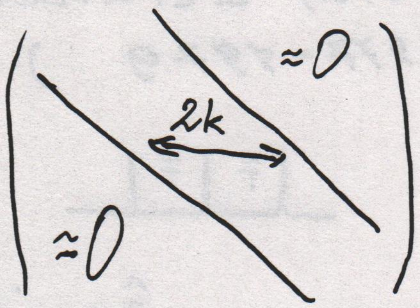
Evolution operator on a period

$$\bar{\psi} = U\psi; \quad U = \exp\left(-iT\frac{\hat{n}^2}{2}\right)\exp(-ik\cos x)$$

$$\hat{n} = -i\frac{\partial}{\partial x}\ ; \quad \psi_n = \frac{1}{\sqrt{2\pi}}\exp(inx) \quad U\psi_\nu = e^{-i\nu}\psi_\nu$$

9 Band matrix

$$U_{nn'} = e^{-i\frac{T}{2}n^2} J_{n-n'}(k)\,(-i)^{n-n'}$$



$$\bar{n} = n + k\sin x$$
$$\bar{x} = x + T\bar{n}$$

$$\langle E_t \rangle = \langle \psi(t) | \frac{\hat{n}^2}{2} | \psi(t) \rangle$$

Classical diffusion

$$\langle E_t^{cl} \rangle \approx \frac{1}{2}\langle n^2 \rangle^{cl} \approx \frac{D}{2} t \approx \frac{k^2}{4} t \quad (K = kT \gg 1)$$

E15

Diffusive time scale

$$t_D \sim q^{\alpha} \sim 1/\hbar^{\alpha} \qquad (\alpha = 2)$$

$$t_D \gg t_E \sim \ln q$$

E16 No exponential correlations decay

$$R(\tau) = \langle 0 | \cos\hat{X}_t \cos\hat{X}_{t+\tau} + \cos\hat{X}_{t+\tau}\cos\hat{X}_t | 0 \rangle_t$$

$$\left.\begin{aligned} \hat{\bar{n}} &= \hat{n} + k\sin\hat{x} \\ \hat{\bar{x}} &= \hat{x} + T\hat{\bar{n}} \end{aligned}\right\} \qquad [\hat{n}, \hat{x}] = -i$$

Heisenberg operators map

$$R(\tau) \sim \langle 0 | \cos\hat{x}_0 \cos\hat{x}_{0+\tau} | 0 \rangle = \langle 0 | \cos x\, U_\tau \cos x\, U_{-\tau} | 0 \rangle \sim$$
$$\sim \frac{1}{(\Delta n_\tau)^{1/2}} \gtrsim \frac{1}{(kt)^{1/2}}$$

E17

Chirikov, Izrailev
Shepelyansky (1981)

$k = 20, \ T = 0.2$
$K = 5$
$E = \langle n^2 \rangle / 2$

t is number of kicks
straight line – classical diffusion rate

F15

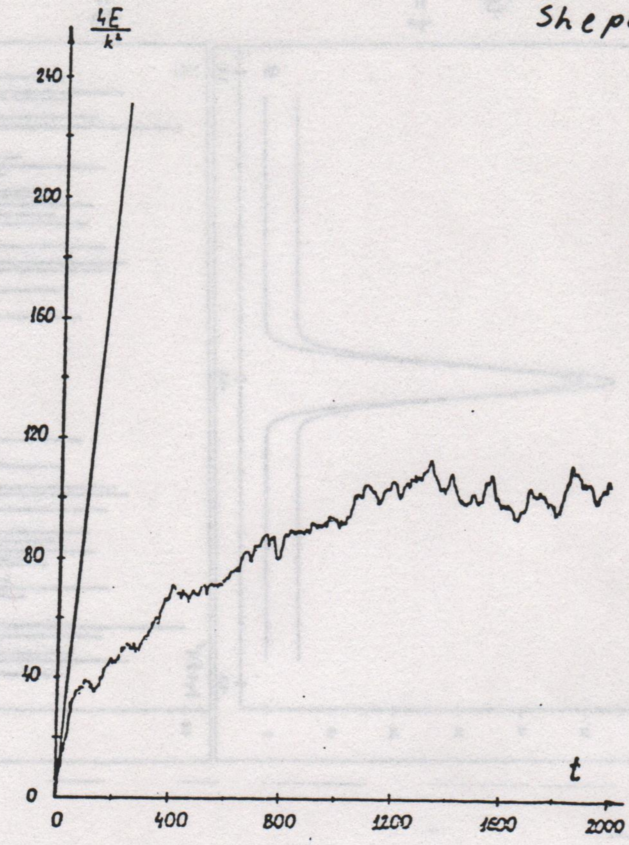
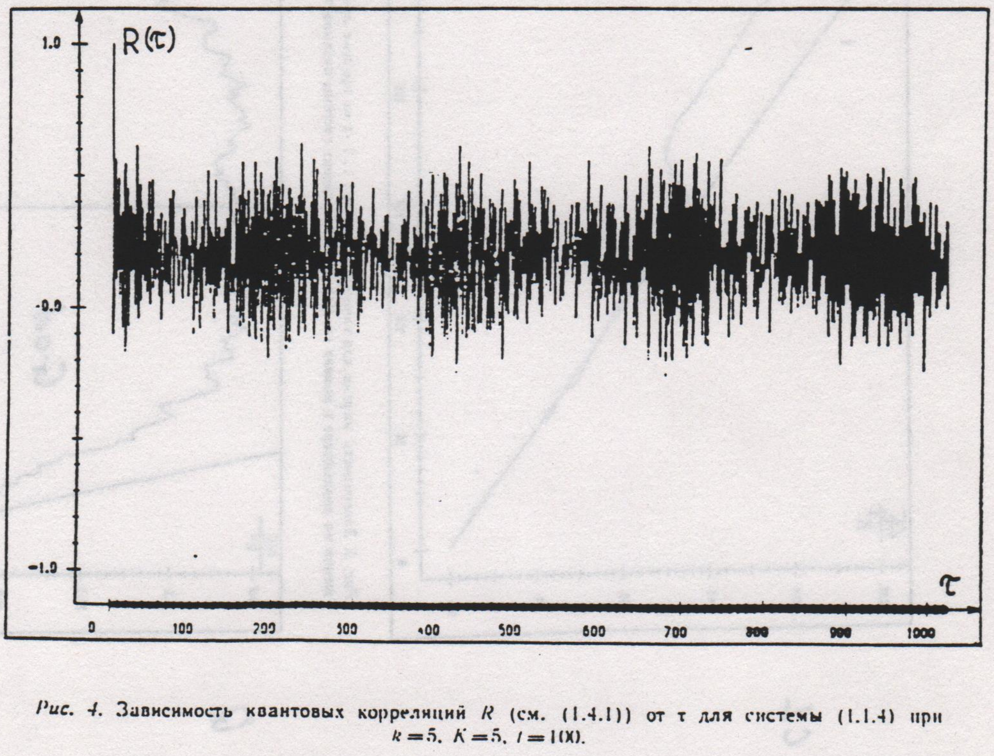


Рис. 1. Зависимость энергии ротатора $E = \langle n^2 \rangle / 2$ от времени для системы (1.1.4) с $k = 20$, $T = 0.25$.
Прямая линия соответствует классической диффузии, ломаная линия — численный результат.

$k = 5, \ K = 5, \ T = 1; \ R = \langle 0 | \cos\hat{X}_t \cos\hat{X}_{t+\tau} + \cos\hat{X}_{t+\tau} \cos\hat{X}_t | 0 \rangle_t$ D. S. (1983)



F16

Рис. 4. Зависимость квантовых корреляций R (см. (1.4.1)) от τ для системы (1.1.4) при $k = 5$, $K = 5$, $T = 100$.

(58)

CL

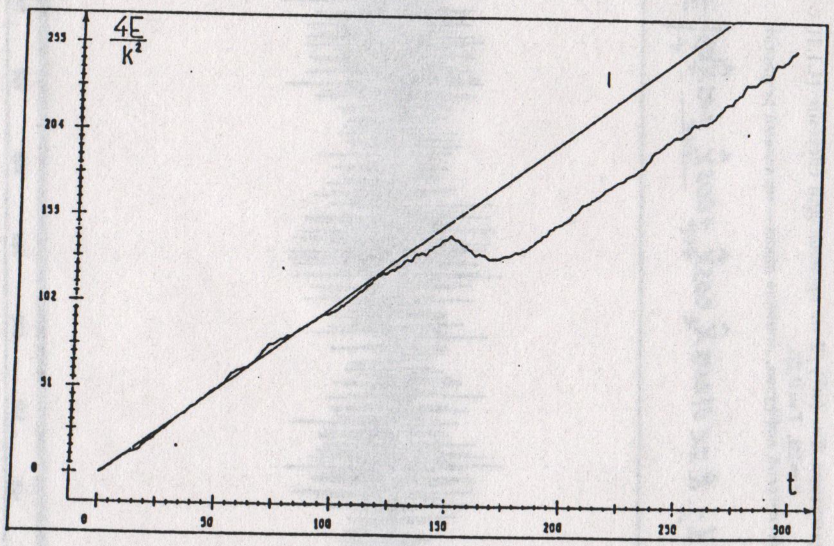


Рис. 5. Зависимость энергии классического ротатора (1.1.4) от времени при обращении движения на компьютере в момент $t=150$; $K=5$. Движение системы оказывается необратимым

Q

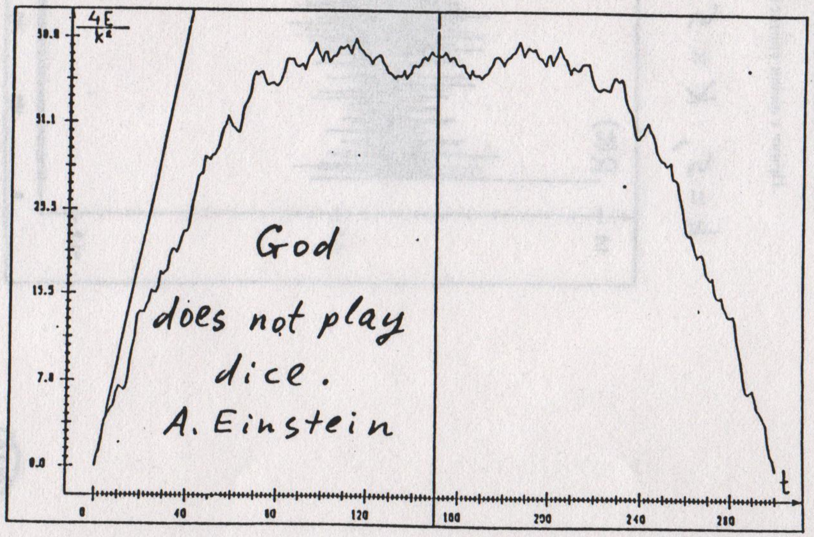


Рис. 6. Зависимость энергии квантового ротатора (1.1.4) от времени при обращении движения и случайной сбивке фаз амплитуд A_n в интервале $\Delta\varphi=0.1$ в момент времени $t=150$; $k=20$, $K=5$.
Прямая линия соответствует классической диффузии, вертикальная прямая отмечает момент обращения времени. Движение квантовой системы обратимо.

F17

D. S. (1983)

$|\psi(\theta)|^2$ as function of θ
initial and final distributions (shifted)

$E = \frac{\langle n^2 \rangle}{2}$; $k = 20$, $K = 5$, $t = 0, 150, 300$

Q

$t = 0; 300$

$t = 150$

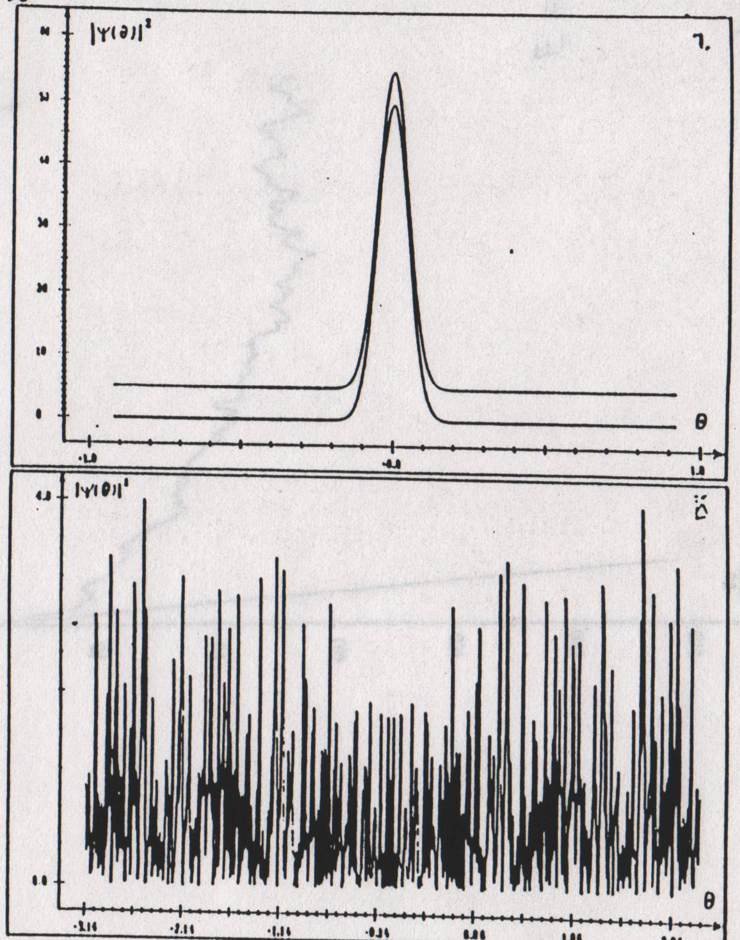


Рис. 8. Распределение вероятности по фазе в модели (1.1.4) для разных моментов времени ($k=20$, $K=5$):
а — начальное гауссово распределение при $t=0$ (нижняя кривая), в момент возврата $t=300$ (верхняя кривая); *б* — «нарезанное» распределение в момент обращения времени $t=150$.

Time reversability of
quantum chaos

($t \to -t$
after 150 kicks)

(30)

Estimate for diffusion suppression time and localization length

Diffusive excitation

Chirikov, Izrailev, Shepelyansky (1981)

$\Delta n \sim (D t_D)^{d/2}$ (d- dimension)

(KR $d=1$)

number of excited levels after time t_D

$(D \simeq k^2/2 > 1 \rightarrow k > 1$ Shuryak border (1976

All frequencies (quasienergies) are **homogeneously** distributed in the interval $[0, 2\pi]$

Distance between lines in the spectrum

$\Delta \nu \sim \frac{1}{\Delta n}$

Due to uncertainty relation $\Delta \nu \Delta t \sim 1$ the distance between lines will be resolved after time

$$t > t_D \sim \frac{1}{\Delta \nu} \sim \Delta n \sim (D t_D)^{d/2}$$

$d=1$ - for $t > t_D \sim D \sim \frac{k^2}{2}$; $\Delta n \sim D \sim \ell \gg 1$

$t_D \sim 1/\hbar^2$ (diffusion rate is measured in number of levels per period of perturbati

$d=2$ - critical dimension ($\ell \sim \exp(D)$)

$d=3$ - delocalization transition Anderson transition ($D \gtrsim 1$)

51

Mapping on a solid-state problem

$$H = H_0(n) + V(x)\,\delta_T(t)$$ Fishman, Grempel Prange (1982)

$$u_\nu = \exp\left(i(\nu - T H_0(\hat{n}))\right) \exp(-iV(x))\, u_\nu$$

$$u = e^{-i\frac{V}{2}}\, u_\nu$$ quasienergy eigenfunction equation

$$e^{i\frac{V}{2}} u = e^{-i(TH_0 - \nu)}\, e^{-i\frac{V}{2}} u \Rightarrow$$

$$= \frac{1 + i\tan\left(\frac{TH_0 - \nu}{2}\right)}{1 - i\tan\left(\frac{TH_0 - \nu}{2}\right)}\, e^{-i\frac{V}{2}} u$$

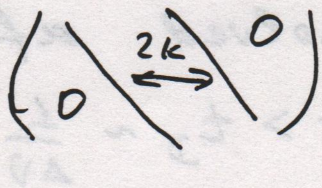
$$\times\cos\frac{V}{2}\Big|\quad 1 - i\tan\left(\frac{TH_0 - \nu}{2}\right)\left(\cos\frac{V}{2} + i\sin\frac{V}{2}\right) u =$$

$$= 1 + i\tan\left(\frac{TH_0 - \nu}{2}\right)\left(\cos\frac{V}{2} - i\sin\frac{V}{2}\right) u$$

$$H_{ss}\, u = \left[\cos\frac{\hat{V}}{2}\,\tan\left(\frac{T\hat{H}_0 - \nu}{2}\right)\cos\frac{\hat{V}}{2} + \frac{1}{2}\sin\hat{V}\right] u = 0$$

band matrix

hermitian



$V = k\cos x$

$$\cos\frac{V}{2} \neq 0 \;\rightarrow\; \text{divide by } \cos\frac{V}{2}$$

$$H_F = \tan\left(\frac{TH_0(n) - \nu}{2}\right) + \tan\frac{V}{2}\;;$$

Lloyd model : $V = 2\arctan(E - 2k\cos x)$

$$E_n u_n + k(u_{n+1} + u_{n-1}) = E u_n$$

$u = \sum_n e^{inx} u_n$; 60 $E_n = \tan\chi_n$; $\chi_n = (\nu - TH_0(n))/2$ – random

2) Lloyd model → exact solution (random χ_n)

$$l = \left[ch^{-1} \left\{ \frac{1}{4k} \left[((2k+E)^2+1)^{1/2} + ((2k-E)^2+1)^{1/2} \right] \right\} \right]^{-1} \approx$$

$$\approx \sqrt{4k^2 - E^2} \qquad (\text{for } l \gg 1)$$

Quasilinear diffusion rate

$$D = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\partial V}{\partial x} \right)^2 dx = 2\sqrt{4k^2 - E^2} > 1$$

$$l = D/2$$ – localization length

$$u_n \sim \exp(-|n-n_0|/l)$$ exponential decay of eigenstates.

Numerical method

→ transfer matrix technique and Lyapunov exponents

$$E_n u_n + \sum_{r=-M}^{M} W_r u_{n-r} = E u_n$$

$$(u_{M-1}, u_{M-2} \ldots u_{-M}) \rightarrow u_M$$

$$\underline{u}_{t+1} = F \, \underline{u}_t \qquad F \rightarrow M \times M$$ symplectic matrix

$\gamma_M > 0$
$\gamma_{-M} < 0$ $\quad \left\langle \frac{\ln \psi_n}{n} \right\rangle = \gamma_1 = 1/l$ minimal postitive Lyapunov exponent

$t = n \rightarrow 10^6 \div 10^7$ levels

33 Kicked rotator (k interacting sites)

$$\ell = \frac{D}{2} = \frac{D_{ql}}{2} \cdot \frac{D}{D_{ql}} \; ; \quad D_{ql} = \frac{k^2}{2} \; ; \; \ell = \frac{D_{cl}}{2T^2}$$

F5 $$\frac{D}{D_{ql}} = f(K \to 2k \sin \frac{T}{2}) \quad \ell \sim 1/\hbar^2 \sim \frac{1}{T^2} g(K)$$

F18 $$\ell \sim \begin{cases} \frac{k^2}{4} & K > K_c \approx 1 \quad (K \gg K_c) \\ k & K \ll K_c \end{cases}$$

?

Tunneling $W \sim \exp(-\frac{2\Delta n}{\ell}) \sim \exp(-\frac{2Tn}{K_{cl}}) \sim$

$\sim \exp(-\frac{2\pi m}{K_{cl}})$ ← no $\hbar$

m - number of periods $\frac{2\pi}{T}$ on a cylinder

Tunneling probability

$W_T \sim \exp(-S/\hbar)$

$S \approx 2\pi m$

Steady-state distribution

$$\overline{f_n} = \overline{|\psi(n,t)|^2} = \sum_m |u_m(0)\, u_m(n)|^2$$

$$\langle |\varphi_m(n)|^2 \rangle \approx \frac{1}{\ell_s} \exp(-\frac{2|n-m|}{\ell_s})$$

$$\overline{f_n} \approx \frac{1}{2\ell_s} \exp(-\frac{2|n|}{\ell_s}) (1 + 2|n|/\ell_s) \; ; \; \overline{n^2} = \ell_s^2$$

$\ell_s \approx D \neq \ell$ fluctuations of Lyapunov exponent

$k = 30,\ K = 10$

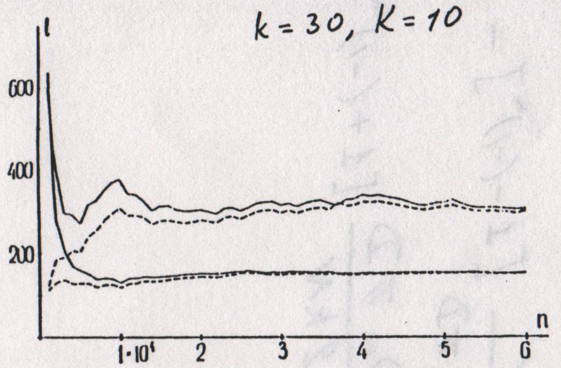


Рис. 9. Пример вычисления длины локализации СФКЭ в модели (1.1.1) при $k = 30$, $K = 10$.

Сплошные линии соответствуют положительным показателям Ляпунова, а штриховые — отрицательным. На рисунке приведены два минимальных показателя ($l = 1/\gamma$)

$k = 2.8,\ T = 4.867$

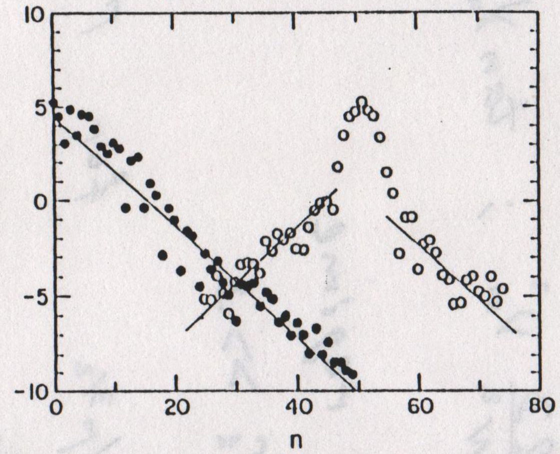


Рис. 10. Локализация СФКЭ в модели (1.1.4) с $k = 2.8$, $T = 4.867$.

Точки и кружки — собственные функции с разными квазиэнергиями (численные данные [31]). Прямые соответствуют значению l, полученному методом ПЛ.

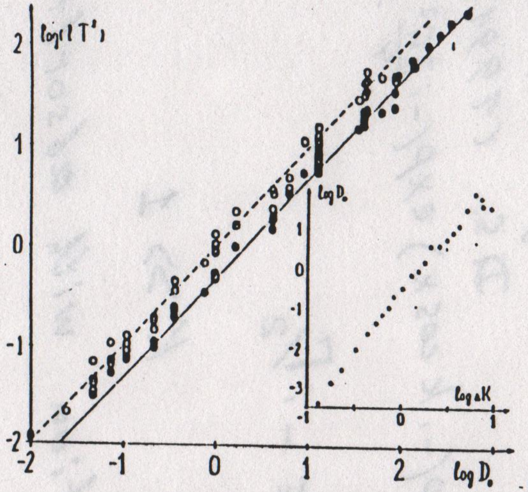


Рис. 12. Зависимость длины локализации в (1.1.4) от скорости диффузии D_0 в стандартном отображении (1.1.5).

Кружки — численные данные для локализации l_s в стационарном распределении (1.7.4). Пунктирная прямая соответствует среднему значению $\langle \alpha_s \rangle = 1.01$. Точки — длина локализации в СФКЭ, полученная методом ПЛ. Прямая — теоретическая формула (1.5.6). На вставке представлены численные данные для зависимости D_0 от $\Delta K = K - K_{cr}$, $K_{cr} = 0.9716...$

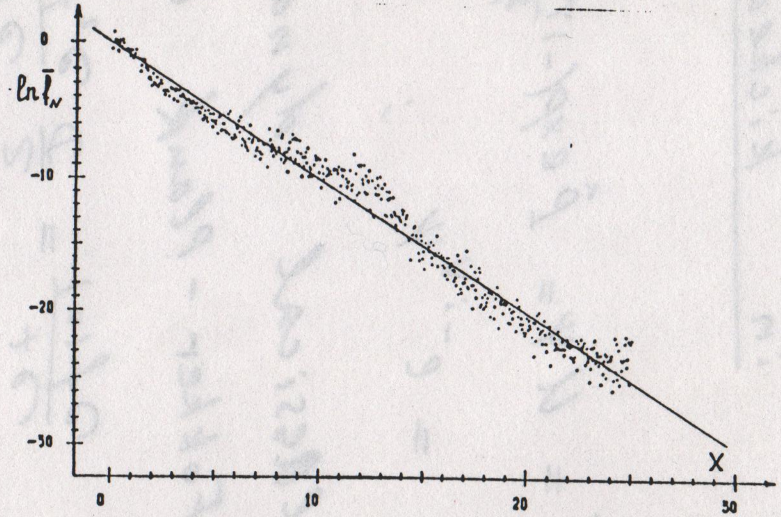


Рис. 16. Пример однородного стационарного распределения для $k = 10$, $T = 0.5$, $K = 5$; $x = 2n/l$, $\bar{f}_N = \bar{f}(n)2l/(1+x)$. Прямая $\bar{f}_N = e^{-x}$.

63

F18

(34)

Statistics of quantum lifetimes in kicked rotator

Borgonovi, Guarneri, D.S. (1991)

$$\overline{\psi} = U\psi = \hat{P} \exp(-i\frac{T\hat{n}^2}{4}) \exp(-ik\cos x) \exp(-i\frac{T\hat{n}^2}{4})\psi =$$

$$= e^{-i\nu}\psi \quad ; \quad \nu = \varepsilon - i\Gamma/2$$

Classical dynamics $K \gg 1$

Fokker - Plank equation with absorbtion:

$$\frac{\partial f(n)}{\partial t} = \frac{D}{2}\frac{\partial^2 f(n)}{\partial n^2}$$

$$-\frac{D}{2}\frac{\partial f}{\partial n}\Big|_{n=\pm N/2} = \pm\frac{k}{\pi} f(\pm N/2)$$

$$\Gamma_m = \frac{2D}{N^2}\nu_m^2 \; ; \; \tan\nu_m = \frac{kN}{2\nu_m \pi D}[1-(-1)^m] - \frac{\nu_m \pi D}{2kN}[1+(-1)^m]$$

diffusive regime

(F19) $t_D \approx 1/\Gamma_1 \gg 1$

$$\frac{dP}{d\Gamma} \sim 1/\Gamma^{3/2} \quad \text{for} \quad \Gamma \gg \Gamma_1$$

estimate

$$n_i^2/D \sim t \sim 1/\Gamma \; ; \; \frac{dn_i}{d\Gamma} \sim \frac{dP}{d\Gamma} \sim \frac{\sqrt{D}}{\Gamma^{3/2}}$$

Level-spacing statistics in complex plane $P(s) \sim s^3$ (64)

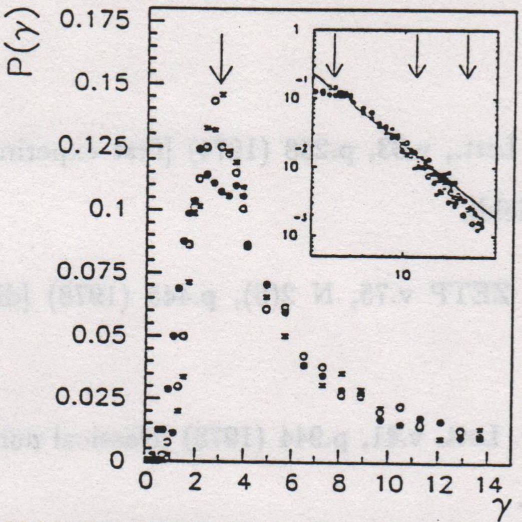


FIG. 1. Probability distribution of level widths for fixed ratio $N/k=10$ and $K=7$, on the x axis; $\bar{\gamma}=N^2\gamma/k^2$. Solid circles, $N=800$; open circles, $N=1600$; stars, $N=2000$. The arrows show the positions of the three lowest classical eigenvalues. Inset: decay of the distribution for large γ in log-log scale, the line is the theoretical $\bar{\gamma}^{(-3/2)}$ law.

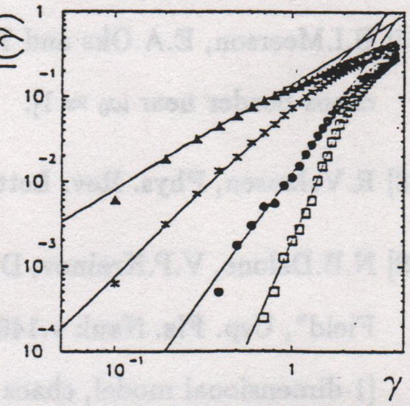


FIG. 2. Log-log plot of the integrated distribution of levels widths for $N/k=10$, $K=7$. Triangles, $N=200$; stars, $N=400$; solid circles, $N=800$; squares, $N=1600$. The straight lines indicate power laws with the exponents 1.25, 2, 3, 5.

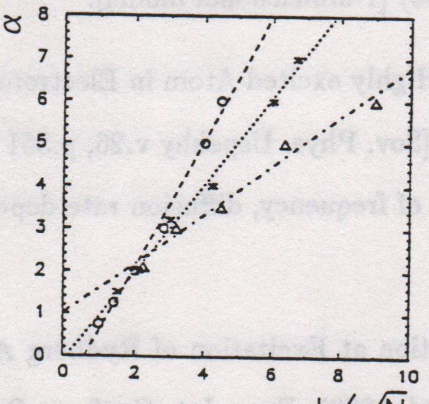


FIG. 3. The exponent of the power laws illustrated in Fig. 2, as a function of $kN^{-1/2}$. Open circles, $N/k=10$; stars, $N/k=6.67$; triangles, $N/k=4.4$. The lines give the least-square linear fitting.

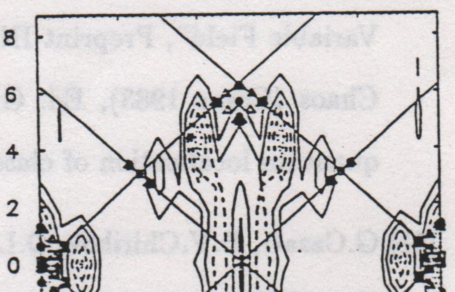


FIG. 4. Contour plot of the Wigner function corresponding to the eigenfunction with the smallest level width for $N=1600$, $k=80$, $K=7$. Points are some classical orbits of period 1 (stars), 2 (open circles), 3 (closed circles), 4 (squares), 5 (triangles). The symmetry lines of the symmetric standard map (Ref. 3) are also shown.

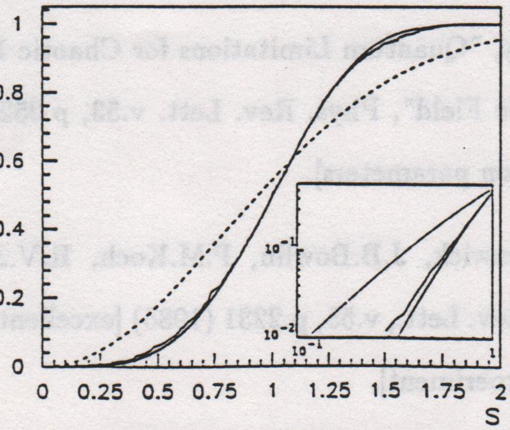


FIG. 5. Integrated distribution of nearest-neighbor spacings in the complex plane for $N=1600$, $k=160$, $K=7$. Dashed line and solid line are the regular distribution and the chaotic one (both from Ref. 8).

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