QUANTUM RESONANCE FOR THE ROTATOR
IN A NON-LINEAR PERIODIC FIELD

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A b s t r a c t

This paper discusses the behavior of a plane quantum rotator under the time-periodic perturbation given as delta-like "kicks" nonlinearly dependent on phase. The case of so-called quantum resonance is considered both analytically and numerically. It is shown that for large times the rotator energy is proportional to t^2 . The structure of quasi-energy spectrum is analysed and its continuity is proved.

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СО АН СССР

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1. INTRODUCTION

In recent years a great number of papers have appeared wherein the attempts have been made to understand quantum properties in the behavior of nonlinear systems. Attention has been focused on the systems being under the periodic perturbation[1÷8]. This is due, first of all, to new possibilities to study experimentally the behavior of atoms and molecules in a field of laser radiation[9÷11].

The present paper is an extension of Ref.12. Our model is a plane quantum rotator with the external nonlinear, periodically time-dependent perturbation (delta-like "kicks"). Up to now the behavior of the corresponding classical system has been studied sufficiently well. In particular, it was shown[13] that under certain condition the motion becomes stochastic even though it is governed by stricktly dynamical equations. On the other hand, if perturbation is small, motion is quasi-periodic. Therefore, there exists a criterion of arising statistical properties in a dynamical system. Numerical investigation carried out in Ref.12 has shown that the behavior of quantum system differs from the classical one even in the strong quasiclassical region. In particular, the diffusion rate of the average rotator energy only for comparatively small times equals the classical one and then decreases sharply. In addition, it has been discovered the specific type of motion (quantum resonance), which have no analogue in the classical system. In this case, the rotator energy grows unlimitedly, independent of the external force value.

The purpose of our work is a careful investigation of the quantum resonance discovered in[12]. It is revealed that in the system there is infinite, dense set of such resonances. The generaé condition of their appearance is also found. The main characteristics of the resonance motion for this system are determined. It is shown analytically that for large times the rotator energy grows as t^2 , what is validated by numerical experiments. The form of asymptotics is independent of perturbation parameter and is universal. The structure of quasi-energy spectrum being continued in the resonance is analysed.

2. Quantum Resonance

The model we choose to study is described by the Hamiltonian:

$$\hat{H} = -\frac{\hbar^2}{2J}\frac{\partial^2}{\partial\theta^2} + \tilde{K}\cos\theta\,\delta_{\tilde{T}}(t) \qquad (2.1)$$

where $\tilde{K}$ is the perturbation parameter, $\delta_{\tilde{T}}(t)=\sum_{n=-\infty}^{\infty}\delta(t-n\tilde{T})$ is the periodic delta function of period $\tilde{T}$ ("kicks"), J is the moment of inertia of the rotator, θ is the angular variable. In the following $J=1$.

One can integrate Schrodinger's equation with Hamiltonian (2.1) to obtain the mapping for a wave function. This mapping involves free rotation during $\tilde{T}$ and a "kick" (see Ref.12):

$$\bar{\Psi}(\theta)=\exp(-iK\cos\theta)\sum_{n=-\infty}^{\infty}A_n\exp(-i\tfrac{Tn^2}{2}+in\theta) \qquad (2.2)$$

where $K=\tilde{K}/\hbar$, $T=\hbar\tilde{T}$ and $\Psi(\theta)=\sum_{n=-\infty}^{\infty}A_n e^{in\theta}$

In what follows, $\hbar=1$. $A_n=\frac{1}{2\pi}\int\Psi(\theta)e^{-in\theta}d\theta$

One can note from eq.(2.2) that the motion does not change

if T is replaced by $T+4\pi m$, where m is integer. It therefore suffices to consider the values of T within the interval $[0, 4\pi]$.

From eq.(2.2) one can find the connection between the Fourier components in one step:

$$\bar{A}_n=\sum_{m=-\infty}^{\infty}F_{nm}A_m \qquad (2.3)$$

where $F_{nm}=(-i)^{n-m}\exp(-i\frac{T}{2}m^2)J_{n-m}(K)$; $J_{n-m}(K)$ is Bessel function.

As was noted in Ref.12, in the case of main quantum resonance ($T=4\pi m$, m is integer)

$$\bar{\Psi}(\theta)=\exp(-iK\cos\theta)\Psi(\theta) \qquad (2.4)$$

and the rotator energy ($\langle E(t)\rangle=-\frac{1}{2}\int_0^{2\pi}\Psi^*(\theta)\frac{\partial^2}{\partial\theta^2}\Psi(\theta)d\theta$) for large times increases as t^2 . So, if the ground state (n = 0) was excited at the initial moment t = 0, then

$$\langle E(t)\rangle=\frac{K^2t^2}{4} \qquad (2.5)$$

Here and below, t is the dimensionless time measured by the number of "kicks".

We investigate now the general case of quantum resonance: $T=\frac{4\pi p}{q}$, p and q are the integer, mutually simple numbers. From eq.(2.2) we have $\bar{\Psi}(\theta)=\exp(-iK\cos\theta)F(\theta)$,

where

$$\begin{aligned} F(\theta)&=\sum_{n=-\infty}^{\infty}A_n\exp(-i\tfrac{2\pi p}{q}n^2+in\theta)=\\ &=\sum_{m=0}^{q-1}\exp(-i\tfrac{2\pi p}{q}m^2)\sum_{\ell=-\infty}^{\infty}A_{m+q\ell}\exp(i(m+q\ell)\theta)=\\ &=\sum_{m=0}^{q-1}\exp(-i\tfrac{2\pi p}{q}m^2)B_m,\\ B_m&=\sum_{\ell=-\infty}^{\infty}A_{m+q\ell}\exp(i(m+q\ell)\theta) \end{aligned} \qquad (2.6)$$

To determine B_m , let us calculate the sum:

$$\sum_{m=0}^{q-1} \exp\left(i\frac{2\pi mn}{q}\right)B_m = \sum_{\ell=-\infty}^{\infty} A_\ell \exp\left(i\left(\theta+\frac{2\pi n}{q}\right)\ell\right) = \psi\left(\theta+\frac{2\pi n}{q}\right) \quad (2.7)$$

Whence, $B_m = \frac{1}{q}\sum_{n=0}^{q-1} \exp\left(-i\frac{2\pi mn}{q}\right)\psi\left(\theta+\frac{2\pi n}{q}\right)$.

As a result, we obtain the main relation for $\overline{\psi}(\theta)$:

$$\overline{\psi}(\theta) = \exp(-ik\cos\theta)\sum_{n=0}^{q-1} \gamma_n \psi\left(\theta+\frac{2\pi n}{q}\right) \quad (2.8)$$

where $\gamma_n = \frac{1}{q}\sum_{m=0}^{q-1} \exp\left(-i\frac{2\pi p}{q}m^2 - i\frac{2\pi mn}{q}\right)$.

Rewrite eq.(2.8) in the form suitable for a further analysis:

$$\overline{\psi}\left(\theta+\frac{2\pi m}{q}\right) = \sum_{n=0}^{q-1} S_{mn}\psi\left(\theta+\frac{2\pi n}{q}\right) \quad (2.9)$$

Here S_{mn} is the matrix of the form:

$$S = \begin{pmatrix} \beta_0 & & 0 \\ & \ddots & \\ 0 & & \beta_{q-1} \end{pmatrix} \cdot \begin{pmatrix} \gamma_0 & \gamma_1 & \dots & \gamma_{q-1} \\ \gamma_{q-1} & \gamma_0 & \dots & \gamma_{q-2} \\ \dots & \dots & \dots & \dots \\ \gamma_1 & \gamma_2 & \dots & \gamma_0 \end{pmatrix} \quad (2.10)$$

$\beta_j = \exp\left(-ik\cos\left(\theta+\frac{2\pi j}{q}\right)\right)$, $S_{mn} = \beta_m \gamma_{n-m}$.

Due to unitarity of the matrix S , its eigenvalues $\lambda_j(\theta) = \exp(i\alpha_j(\theta))$, $|\lambda_j| = 1$. Note, that λ_j depend, in a general case, on θ , as it will be shown below.

It is convenient to represent the matrix S as

$$S_{mn} = \sum_{\ell=0}^{q-1} Q_{m\ell}\, e^{i\alpha_\ell}\, Q^{-1}_{\ell n} \quad (2.11)$$

where Q is some unitary matrix ($Q^{-1} = Q^{+}$) of dimensionality $q \times q$ with the θ -dependent elements Q_{mn} .

Let us introduce the vector-column $\Phi(\theta,t)$ with the elements $\varphi_m(\theta,t) = \psi\left(\theta+\frac{2\pi m}{q},t\right)$. From eq.(2.9) and eq.(2.11) one can find the time dependence of Φ :

$$\varphi_m(\theta,t) = \sum_{n,\ell=0}^{q-1} Q_{mn}\exp(i\alpha_n(\theta)t)\,Q^*_{\ell n}\,\varphi_\ell(\theta,0) \quad (2.12)$$

With a known $\varphi_m(\theta,t)$ one obtains the dependence of the rotator momentum and rotator energy on time:

$$\langle p(t)\rangle = -\frac{i}{q}\sum_{m=0}^{q-1}\int_0^{2\pi} \varphi^*_m(\theta,t)\frac{\partial}{\partial\theta}\varphi_m(\theta,t)\,d\theta \quad (2.13)$$

$$\langle E(t)\rangle = -\frac{1}{2q}\sum_{m=0}^{q-1}\int_0^{2\pi} \varphi^*_m(\theta,t)\frac{\partial^2}{\partial\theta^2}\varphi_m(\theta,t)\,d\theta \quad (2.14)$$

Directly from (2.12) and (2.13) one gets:

$$\langle p(t)\rangle = \langle p(0)\rangle + a_1 t + b_{10} + \sum_{m,m_1=0}^{q-1} P_{mm_1}(t), \quad (2.15)$$

where

$$b_{10} = -\frac{i}{q}\sum_{m,\ell,\ell_1}\int_0^{2\pi} d\theta\left\{\varphi_\ell(\theta,0)\varphi^*_{\ell_1}(\theta,0)Q_{\ell_1 m}Q^{*\prime}_{\ell m}\right\}$$

$$a_1 = \frac{1}{q}\sum_{m,\ell,\ell_1}\int_0^{2\pi} \alpha'_m\left\{\varphi_\ell(\theta,0)\varphi^*_{\ell_1}(\theta,0)Q_{\ell_1 m}Q^*_{\ell m}\right\}d\theta \quad (2.16)$$

$$P_{mm_1}(t) = -\frac{i}{q}\sum_{n,\ell,\ell_1}\int_0^{2\pi} d\theta\left\{\varphi^*_{\ell_1}(\theta,0)\varphi_\ell(\theta,0)Q_{\ell_1 m_1}Q^*_{nm_1}Q'_{nm}Q^*_{\ell m}\exp(i(\alpha_m-\alpha_{m_1})t)\right\}$$

Here and below the dash denotes the derivative over θ .

Since α_m depends in general case on θ , for asymptotically large times $P_{mm_1}(t)$ (for $m \neq m_1$) is expressed by the integral of fastly oscillating function and, hence, for large t $P_{mm_1}(t) = P_{mm_1}(0)\delta_{mm_1} + o(1/t)$. Finally, the asymptotic time dependence of the rotator momentum may be readily found:

$$\langle p(t)\rangle = a_1 t + b_1 + \langle p(0)\rangle \tag{2.17}$$

where $b_1 = b_{10} + \sum_{m=0}^{q-1} P_{mm_1}(0)$

For arbitrary times the dependence $\langle E(t)\rangle$ is determined in a similar fashion:

$$\langle E(t)\rangle = \langle E(0)\rangle + \eta t^2 + a_{20} t + b_{20} + \sum_{m,m_1=0}^{q-1} G_{mm_1}(t) + t\sum_{m,m_1=0}^{q-1} R_{mm_1}(t), \tag{2.18a}$$

and for asymptotically large times:

$$\langle E(t)\rangle = \eta t^2 + a_2 t + b_2 + \langle E(0)\rangle, \tag{2.18b}$$

where

$$\eta = \frac{1}{2q}\int_0^{2\pi} d\theta \Big\{ \sum_{\ell,\ell_1,m=0}^{q-1} (\alpha_m')^2 \varphi_{\ell_1}^*(\theta,0)\varphi_\ell(\theta,0) Q_{\ell_1 m} Q_{\ell m}^* \Big\} \geq 0$$

$$a_{20} = -\frac{i}{q}\sum_{m,\ell,\ell_1=0}^{q-1}\int_0^{2\pi} d\theta \{\alpha_m'' \varphi_{\ell_1}^*\varphi_\ell Q_{\ell_1 m} Q_{\ell m}^* + 2\alpha_m' \varphi_{\ell_1}^*\varphi_\ell Q_{\ell_1 m} Q_{\ell m}^{*\prime} + 2\alpha_m' \varphi_{\ell_1}^*\varphi_\ell' \cdot Q_{\ell_1 m} Q_{\ell m}^*\}; \quad b_{20} = -\frac{1}{q}\sum_{m,\ell,\ell_1=0}^{q-1}\int_0^{2\pi}\{\varphi_{\ell_1}^*\varphi_\ell Q_{\ell_1 m} Q_{\ell m}^{*\prime\prime} + 2\varphi_{\ell_1}^*\varphi_\ell' Q_{\ell_1 m} Q_{\ell m}^{*\prime}\}$$

$$\begin{aligned} G_{mm_1}(t) = -\frac{1}{2q}\sum_{n,\ell,\ell_1=0}^{q-1}\int_0^{2\pi} d\theta \{\exp(i(\alpha_m-\alpha_{m_1})t)\cdot[\varphi_{\ell_1}^*\varphi_\ell Q_{\ell_1 m_1} Q_{nm_1}^* Q_{nm}'' Q_{\ell m}^* + \\ + 2\varphi_{\ell_1}^*\varphi_\ell Q_{\ell_1 m_1} Q_{nm_1}^* Q_{nm}' Q_{\ell m}^{*\prime} + 2\varphi_{\ell_1}^*\varphi_\ell' Q_{\ell_1 m_1} Q_{nm_1}^* Q_{nm}' Q_{\ell m}^*]\} \end{aligned} \tag{2.19}$$

$$R_{mm_1}(t) = -\frac{i}{q}\sum_{n,\ell,\ell_1=0}^{q-1}\int_0^{2\pi} d\theta \{\alpha_m' \varphi_{\ell}^*\varphi_\ell Q_{\ell_1 m_1} Q_{nm_1}^* Q_{nm}' Q_{\ell m}^* \exp(i(\alpha_m-\alpha_{m_1})t)\}$$

$$a_2 = a_{20} + \sum_{m=0}^{q-1} R_{mm}, \quad b_2 = b_{20} + \sum_{m=0}^{q-1} G_{mm}$$

The expressions derived for the rotator energy and rotator momentum are universal and yield the asymtrotic form for large times. Examination of expression (2.19) for the coefficient η, which gives the asymptotic, shows that this coefficient equals zero when all λ_j, and hence α_j also, are independent of θ $(\lambda_j = const)$. Moreover, $\eta = 0$ in the case when at least one value of $\lambda_n = const$ and the initial distribution satisfies the specific condition $\lambda_n \Psi_{\lambda_n}(\theta) = \exp(-ik\cos\theta)\sum_{m=0}^{q-1}\gamma_m \Psi_{\lambda_n}(\theta + \frac{2\pi m}{q})$.

Next, we seek to find the quasi-energy spectrum[14,15]. From eq.(2.8) it follows that the wave functions with a definite quasi-energy at moment $t=0$ are representable in the form:

$$\Psi_{\varepsilon_j(\theta_0)}(\theta,0) = \sum_{n=0}^{q-1} C_n^j(\theta_0)\,\delta(\theta+\theta_0+\tfrac{2\pi n}{q}) \tag{2.20}$$

where $\delta(\theta) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in\theta}$.

The quasi-energy $\varepsilon_j(\theta_0)$ and the coefficients $C_n^j(\theta_0)$ are defined from the following set of linear equations:

$$\exp(-i\varepsilon_j(\theta_0)T)\,C_n^j(\theta_0) = \sum_{m=0}^{q-1}\tilde{S}_{nm} C_m^j(\theta_0), \tag{2.21}$$

where $\tilde{S}_{nm} = \beta_n(\theta_0)\gamma_{n-m}$. The matrix $\tilde{S}$ is unitary and its eigenvalues $\tilde{\lambda}_j(\theta_0) = \exp(i\tilde{\alpha}_j(\theta_0))$ determine the quasi-energy spectrum:

$$\varepsilon_j(\theta_0) = -\frac{\tilde{\alpha}_j(\theta_0)}{T} \tag{2.22}$$

Here θ_0 is a continuous parameter: $0 \leq \theta_0 < 2\pi$

From eq.(2.22) it follows that the quasi-energy spectrum has the discrete levels only if the matrix $\tilde{S}$ has the eigenvalues $\tilde{\lambda}_j = const$. Using the explicit form of $\tilde{S}$, it is easy to show that for any p/q (except the case $p/q = 1/2$ which will be analysed below), $Sp\,\tilde{S}^2 = \sum_{j=0}^{q-1}\tilde{\lambda}_j^2$ depends on the continuous parameter θ_0, i.e. there exist $\tilde{\lambda}_j \neq const$. Thus, the quasi-energy spectrum (2.22) is continuous in resonance. Besides this continuous component the spectrum may have the discrete levels whose full number is the same as that of eigenvalues $\tilde{\lambda}_j = const$. It becomes clear that there are no than $q-1$

discrete levels in the resonance .

Letting $\psi_{\varepsilon_j(\theta_0)}(\theta,0)$ to be known, it is easy to find $\psi_{\varepsilon_j(\theta_0)}(\theta,t)$:

$$\psi_{\varepsilon_j(\theta_0)}(\theta,t)=\exp(-i\varepsilon_j(\theta_0)t)\,\varphi_{\varepsilon_j(\theta_0)}(\theta,t), \qquad (2.23)$$

where $\varphi_{\varepsilon_j(\theta_0)}$ is the quasi-energy eigenfunction, which is time-periodic of period T:

$$\varphi_{\varepsilon_j(\theta_0)}(\theta,t)=\exp(i\varepsilon_j(\theta_0)t-i\kappa\cos\theta\,\vartheta(t-T))\cdot$$

$$\cdot\sum_{n=-\infty}^{\infty}A_n\exp(-i\frac{n^2t}{2}+in\theta);\quad 0\le t\le T \qquad (2.24)$$

A_n are the Fourier components $\psi_{\varepsilon_j(\theta_0)}(\theta,0)$ and $\vartheta(t-T)$ is a single step function.

It is worthwhile to note that in resonance $\langle|n|\rangle$ is proportional to the time $(\langle|n|\rangle\sim t)$. Therefore, if the unperturbed system possessed the spectrum $E_n\sim n^m$ ($m>1$ is the integer), then its energy would increse with time by the law $\langle E(t)\rangle\sim t^m$.

The explicit form of $\lambda_j(\theta)$ has been found for three cases.

1) $P/q=1$ is the main resonance. The time-energy dependence (when at $t=0$ the ground state $n=0$ is excited) is determined by formula (2.5). The quasi-energy spectrum has the form:

$$\varepsilon(\theta_0)=\frac{K}{4\pi}\cos\theta_0 \qquad (2.25)$$

As $\psi_{\varepsilon(\theta_0)}(\theta,0)=\delta(\theta+\theta_0)$, the quasi-energy eigenfunction is determined, according to (2.24), by the expression:

$$\varphi_{\varepsilon(\theta_0)}(\theta,t)=\exp(i\varepsilon(\theta_0)t-i\kappa\cos\theta\,\vartheta(t-T))\cdot$$

$$\cdot\sum_{n=-\infty}^{\infty}\exp(-i\frac{n^2t}{2}+in(\theta+\theta_0)), \qquad (2.26)$$

where $0\le t\le T;\ T=4\pi$

2) $P/q=1/4$. From eq.(2.8) it follows

$$\overline{\psi}(\theta)=\exp(-i\kappa\cos\theta)\frac{1}{\sqrt{2}}(e^{-i\frac{\pi}{4}}\psi(\theta)+e^{i\frac{\pi}{4}}\psi(\theta+\pi)) \qquad (2.27)$$

(If $q=4\ell$, ℓ is the integer, the coefficients $\gamma_{2m+1}=0$ and the dimensionality of S is $q/2\times q/2$). The eigenvalues $\lambda_\pm=\tilde{\lambda}_\pm=\exp(\pm i\alpha(\theta)-i\frac{\pi}{4})$, where

$$\cos(\alpha(\theta))=\frac{1}{\sqrt{2}}\cos(K\cos\theta) \qquad (2.28)$$

At $K\ll 1$ we have $\frac{\partial\alpha}{\partial\theta}\approx-\mathrm{sign}(\alpha)\frac{K^2}{2}\sin 2\theta$, and if at the initial moment $\psi(\theta)=\frac{1}{\sqrt{2\pi}}$ (ground state), then $\eta\approx\frac{K^4}{16}$. Under the same initial conditions, at $K\gg 1$ we have $\eta\approx\frac{K^2}{12}$. From eq.(2.22) we find the quasi-energy spectrum:

$$\varepsilon_\pm(\theta_0)=\frac{1}{4}\mp\frac{\alpha(\theta)}{\pi} \qquad (2.29)$$

For $K\ll 1$ the spectrum $\varepsilon_\pm(\theta_0)\approx\frac{1}{4}\mp(\frac{1}{4}+\frac{K^2}{4\pi}\cos^2\theta_0)$ is two narrow zones of $\sim K^2$ in size. For $K\geq\pi$ we have two wide bands: $\frac{1}{2}\le\varepsilon(\theta_0)\le 1,\ \frac{3}{2}\le\varepsilon(\theta_0)\le 2$. The time dependence $\varphi_{\varepsilon_\pm(\theta_0)}$ is given by formula (2.24), where A_n are Fourier components of the function $\psi_{\varepsilon_\pm(\theta_0)}(\theta,0)=C_1(\theta_0)\delta(\theta+\theta_0)+C_2(\theta_0)\delta(\theta+\theta_0+\pi)$; here (C_1,C_2) is the eigenvector of the matrix $\tilde{S}$.

3) If $P/q=1/2$, from eq.(2.8) one obtains:

$$\overline{\psi}(\theta)=\exp(-i\kappa\cos\theta)\psi(\theta+\pi) \qquad (2.30)$$

It is seen that the system returns to the initial state in two "kicks". The eigenvalues are equal to $\lambda_{1,2}=\lambda_{3,2}=\pm 1$. The quasi-energy spectrum consists of two discrete levels with the quasi-energies $\varepsilon_1=0;\ \varepsilon_2=1/2$. The eigenfunctions of the level ε_1 , to be more precise, their values at moment $t = mT$, are the functions $\psi_{\varepsilon_1}(\theta) = g_\pm(\theta)(1 \pm \exp(-ik\cos\theta))$ where $g_\pm(\theta)$ is an arbitrary function satisfying $g_\pm(\theta+\pi) = \pm g(\theta)$. The eigenfunctions of the level ε_2 are $\psi_{\varepsilon_2}(\theta) = g_\pm(\theta)(1 \mp \exp(-ik\cos\theta))$. Each level is degenerate infinitely-fold and the functions ψ_{ε_1} and ψ_{ε_2} form the total set.

Apparently, the degeneration of eigenvalues (when $\lambda_j = const$) is accidental and there is nocase for other resonances. Indeed, at the arbitrary initial distribution $\psi(\theta)$ if some $\lambda_j = const$ were available, then a fraction of the energy would possess to the discrete component of the quasi-energy discrete spectrum. In accordance with this, the time-energy dependence would include periodic time-undamping oscillations (just as in the case of $P/q = 1/2$). In our numerical experiments such an effect has not been observed. At $K \gg 1$ the dependence $\langle E(t)\rangle$ was a smooth function of time, at $K < 1$ the slope size grew as the time increased (see Figs 1 and 2 respectively).

Thus, in quantum resonance ($P/q \neq 1/2$) the energy of the system grows infinitely by the asymptotic law $\langle E(t)\rangle \sim t^2$, the quasi-energy spectrum being continuous.

For the quantities $(\alpha')^2_{max} \equiv \max\limits_{0\le j\le q-1} \overline{\left|\frac{\partial\alpha_j(\theta)}{\partial\theta}\right|}$ (the line indicates the averaging over θ) one has succeeded in obtaining the following estimates (they are valid for $\tilde\alpha_j$ of the matrix $\tilde S$ as well) from the explicit form of S :

a) If $K \ll q$, then

$$\eta \sim \frac{(\alpha')^2_{max}}{2} \geq q\,|J_q(K)|^2 \sim \left(\frac{K}{q}\right)^{2q} \qquad (2.31)$$

ark, that this estimate is that of the lowest possible value

of $(\alpha')^2_{max}$. One might expect that the exact value in the order of magnitude coincides with this lowest estimate (see section 3).

b) If $K \gg q$ and since $\eta \sim \frac{(\alpha')^2_{max}}{2}$, we have

$$\eta \approx \frac{K^2}{\xi q} \qquad (2.32)$$

where ξ is some quantity dependent on the initial conditions and, in practice, independent of K and q. Under the smooth initial conditions $\psi(\theta, 0)$ the estimate for ξ yields $\xi \approx 5$.

From the estimates obtained for $(\alpha')^2_{max}$ it follows that at $K \ll q$ (p, q are any mutually simple numbers) the quasi-energy spectrum consists of q exponentially narrow zones of $\Delta\varepsilon \sim \left(\frac{K}{q}\right)^q$ in size. In case $K \gg q$, to find the zone structure, it is required a comprehensive knowledge of the eigenvalues of the matrix $\tilde S$. Unfortunately, the explicit form of $\tilde\lambda_j(\theta)$ has failed.

From the said above (see eqs.(2.31) and (2.32)) it follows that $\eta \to 0$ at $p \to \infty$, $q \to \infty$, $K = const$. This means that for irrational values of $T/4\pi$ the quantity η is equal to zero. In this case the motion of the system has quite another character (see Ref.12).

Let $T = \frac{4\pi p}{q} + \delta$ where $|\delta| \ll 1$. Then during $t_r \sim 1/d\delta$

$$d \sim \max_{0\le t\le t_r} \langle E(t)\rangle$$

the system characteristics vary in time just as the case of exact resonance $T = \frac{4\pi p}{q}$, what was observed clearly in the numerical experiment (Fig.3). From $\eta t_r \gtrsim K^2 t_r$ let us find such a detuning δ_z at which the resonance p/q influences greatly (it is suggested that $d \sim K^2 t_z$). At $K > q$ (otherwise, the detuning is exponentially small) we have

$$\delta_z \sim \frac{1}{K^2 q^2} \tag{2.33}$$

For the main resonance ($q = 1, p = 0$) $T = \delta$ and from the condition (2.33) it follows that $T \lesssim 1/K^2$. As is seen, within the quasiclassic region ($K \to \infty$, $T \to 0$, $KT = const$) inequality (2.33) is not satisfied, i.e. an influence of the main resonance is unessential. One may find the summary of all the detunings:

$$\delta_\Sigma \simeq \sum_{q=1}^{K} \sum_{p=1}^{q-1} \delta_z(p,q) \sim \frac{\ln K}{K^2} \tag{2.34}$$

Since $\delta_z << 1$, then in the case of irrational $T/4\pi$ the resonances influence weakly the system motion.

3. Numerical Experiments

In addition to theoretical analysis of our model, the numerical studies have been carries out also. In computation the Fourier components of the wave function have been found by formula (2.3). Although the summation in (2.3) has been made from $-\infty$ to $+\infty$, the sum contains $\sim 2K$ terms since $|J_n(K)|$ falls exponentially down with increase of n at $n > K$ (the "kick" covers $\simeq 2K$ levels). In view of this, the finite number ($\approx 2K$) of Bessel functions has been used in our calculations. The control for computation accuracy is to test the cormalization condition of the wave function: $W = \int_0^{2\pi} |\psi(\theta)|^2 d\theta = 1$. In all cases the errors do not exceed $\delta W \lesssim 3.10^{-3}$. The major limitation on a run is imposed by the finiteness of the chosen number of levels. Under a quite large perturbation the fast excitation of the system, high levels occur and computation errors become essential. The program has been improved (as compared to that in Ref.12), what made it possible to increase the computation rate approximately by a factor of 2 and the number of levels of model system - up to 2001. In the main experiments the real computation time was $\approx$ 10 min at BESM-6 (Tables 1,2). The consideration of symmetric initial distributions ($\psi(\theta) = \psi(-\theta)$)enabled us to increase additionally the number of levels up to 4001 (-2000, +2000). Nevertheless, the computation was carried out with the 2001st level, due to symmetry of initial conditions and Hamiltonian (2.1).

The initial conditions were varied from excitation of one level (ground state) to excitation of about 20 levels (Gaussian packet). In all cases the asymptotic form of motion depended slightly on a choice of the initial state. During each run the rotator energy $<E> = -\frac{1}{2}\int_0^{2\pi} \psi^* \frac{\partial^2}{\partial\theta^2} \psi \, d\theta$ was calculated. At the same time, the time-energy dependence was plotted and the least--square fit $<E(t)>$ for a squared polinomial was performed.

For $K < q$ the squared time-energy dependence (see, e.g., Fig.2,4)was observed well. The fit was there carried out by the formula $<E(t)> = \eta t^2 + <E(0)>$. Table 1 contains the data for η at different K and q . At $K \ll q$ the values of η are too small and the squared energy growth for finite times $t \lesssim 200$ is not always noticeable (see, Fig5). So, it is difficult to talk about quantitative agreement with estimate (2.31) but one can assert that η decreases much more quickly than K/q .

For $K \gtrsim q$ the dependence $\eta(K,q)$ is approximated by analytical estimate (2.32). Experimetal data (Table 2) show a quite good agreement with this formula. The value of ξ varies slightly and is independent explicitly of K and q. The average value of ξ is $\langle \xi \rangle = 2.4$.

It was varified separately in what extent η depends on the values of p at the same value of q. As was expected, the dependence on p, according to (2.31) and (2.32), is negligibly small.

In the quasiclassical region ($K \to \infty, T = \frac{4\pi}{q} \to 0$ at $KT = const \gg 1$) experimental data show for $K \gtrsim q$ that for small times the dependence $\langle E(t) \rangle$ in dimensionless variables is described well by the semiemphirical formula:

$$\langle E(t) \rangle = \frac{K^2 t^2}{\xi q} + \frac{K^2}{4} t + \langle E(0) \rangle \tag{3.1}$$

At t^2 the coefficient is in agreement with the theoretical estimate (2.32) for η. The second term in eq.(3.1), linear in time, corresponds exactly to classical diffusion[12]. Nevertheless, the coefficient $K^2/4$ differs from the asymptotic value of a_1 in (2.18). Therefore, the term $K^2 t/4$ is not, strictly speaking, diffusional. It follows from the experimental dependence (3.1) that for the times $t < t^*$, where $t^* \approx \frac{q \xi}{4} \sim 1/T$ (in dimensional variables $\tau < \tau^* = t^* \tilde{T} \lesssim 1/k$), the energy grows mainly due to the "diffusion" term $K^2 t/4$. For $t > t^*$ the squared term being a purely quantum becomes dominating.

4. Conclusive remarks

Our study shows that in the case of quantum resonances whose system is everywhere dense, the asymptotic dependence of the rotator energy on time is universal and is described by the squared law (2.18). This implies that there is no quantum stability border ($K \approx 1$), which was predicted[3] and observed in the non--resonance case[12]. It is important to note that the classical criterion of stability ($KT \approx 1$) is absent too, although the system can be in the strong quasiclassical region. At the same time, for the non-linear system,which is governed by the classical Hamiltonian corresponding to (2.1), the KAM theory (Kolmogorov-Arnold-Moser[16÷18]) is applicable. This theory points out (just as numerical experiments[13]) the motion stability under a small perturbation. In our case this means that there is the distinction in behavior of the quantum system in comparison to the classical one, at least, for large times.

For relatively small t, when the asymptotic properties do not yet arise, the character of the system behavior can be complicated enough and strongly depends on the parameters K and T. For example, the squared growth of the system energy is not clearly observed if $K \gtrsim q \approx 1$ and $T \gtrsim 1$. In this case, the energy is proportional to t^2. If $K < q$, then, in practice, the energy oscillates and the squared growth of ηt^2 is small because of $\eta \ll 1$.

It is interesting to observe the motion at $p \to \infty, q \to \infty$, $p/q = const$. what corresponds to the non-resonant value of T. As numerical experiments have shown[12], the motion of the system is quite different as compared to that in resonance. The analytical study faces excessive difficulties due to necessity to know the exact solution in resonance for any time rather than asymptotically only. On the other hand, one can also investigate the quasi-energy spectrum structure under transition to high resonances $q \gg 1$. It has been showed that at $q > K$ the width of each quasi-energy zone is exponentially small ($\Delta\varepsilon \sim \sqrt{\eta} \approx (\frac{K}{q})^q$). Moreover, the total width of all q zones is small as well. This apparently indicates that the quasi-energy spectrum becomes discrete in the non-resonant case. If $K \gg q$, the question on the overlapping and the quasi-energy spectrum zones remains open.

In closing, we would like to make a remark concerning the feasibility of quantum resonances in the systems under delta-like in time perturbation (for the one-dimensional case, the latter is representable as follows: $f(x)\delta_T(t)$, where $f(x)$ is arbitrary function of x-coordinate). There is no difficulty to show that for the existence of a resonance it is necessary that the spectrum of unperturbed Hamiltonian H_0 be discrete and have the form of a polynomial of the quantum number with rational numbers. In addition, it is also required that the condition of the form $\psi_m \psi_n = \psi_{m+n}$ be satisfied for eigenfunctions of the Hamiltonian H_0. We then have: $\exp(i\frac{2\pi\ell(m+n)}{q}) \times \psi_{n+m} = (\exp(i\frac{2\pi\ell m}{q})\psi_m)(\exp(i\frac{2\pi\ell n}{q})\psi_n)$. Following from this equality, one succeeds in reducing the mapping in one period to multiplication by the matrix. Probably, the last condition may be relaxed.

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Table 1

K/q	η	K	P/q
0.025	6.10^{-6}	0.1	1/4
0.058	10^{-4}	1	1/17
0.058	10^{-4}	1	4/17
0.099	10^{-3}	10	1/101
0.100	2.10^{-5}	0.5	2/5
0.150	10^{-2}	1.2	1/8
0.176	10^{-3}	3	1/17
0.200	6.10^{-4}	1	1/5
0.235	0.08	4	1/17
0.235	0.144	4	4/17
0.353	0.36	6	1/17

Table II

K/q	η	ξ	K	P/q
0.706	3.0	2.8	12	1/17
1.76	23.4	2.3	30	1/17
3.33	52.4	3.2	50	1/15
5.00	181	2.4	85	1/17
5.44	284	1.7	87	1/16
6.21	239	1.3	87	1/14
7.69	404	1.9	100	1/13
7.73	246	2.7	85	1/11
8.57	206	2.5	60	1/7
10.7	245	3.3	75	1/7
12.4	452	2.3	87	1/7

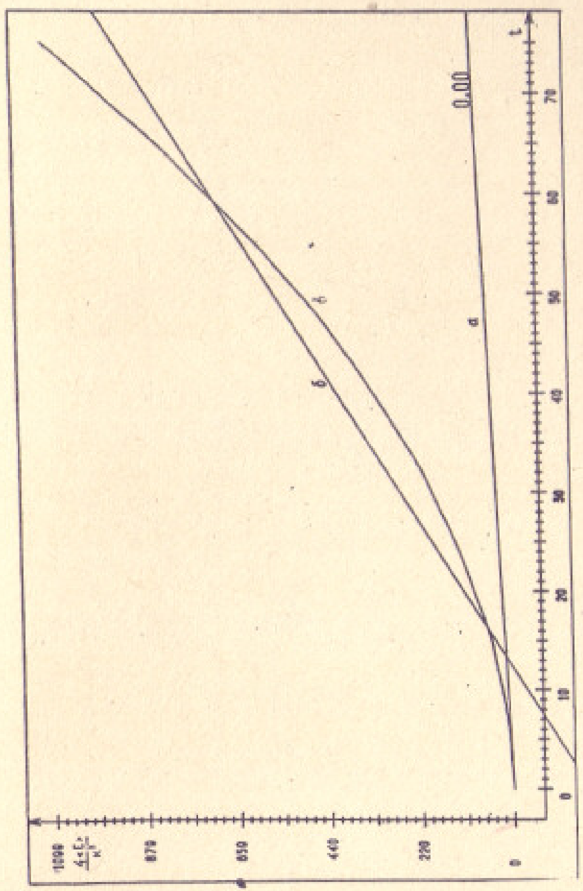

Fig.1 The time dependence of rotator energy $\langle E \rangle$ in the case of quantum resonance, $T = 4\pi/17$, $K = 19$, $\tau = 75$. The straight line "a" corresponds to classical diffusion $\langle E \rangle = K^2 t/4$, the straight line "б" - to linear interpolation at the moment τ (run end), "б" - to experimental result.

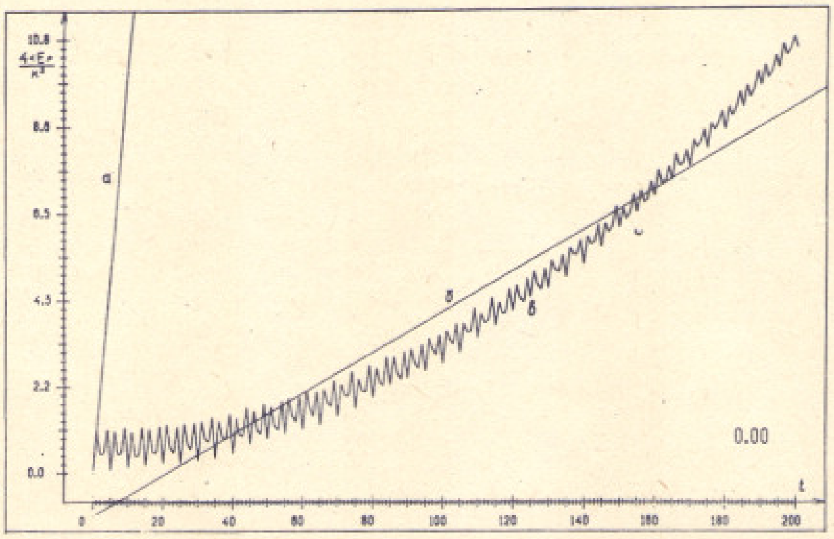

Fig.2 The same as in Fig.1 at $T = 4\pi \cdot \frac{2}{5}$, K = 0.5, t = 200.

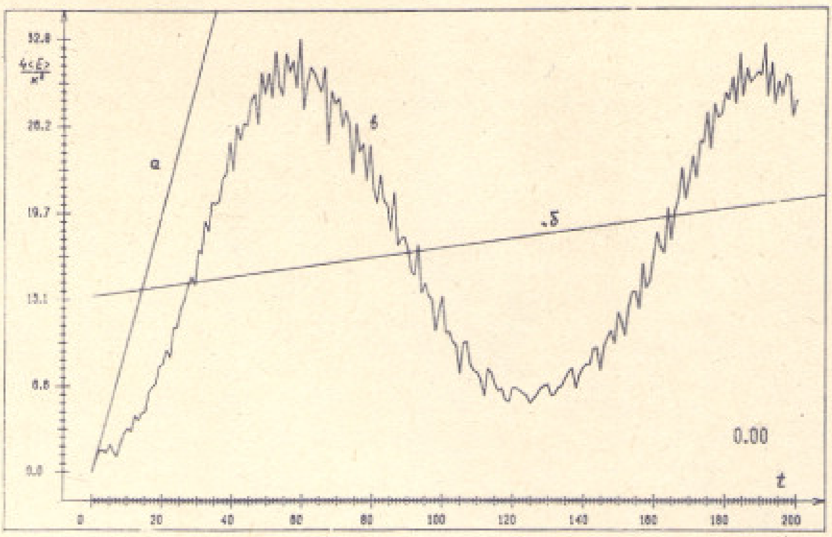

Fig.3 The same as in Fig.1 at $T = \frac{4\pi}{3}(1 + 0.001)$, K = 1, t = 200.

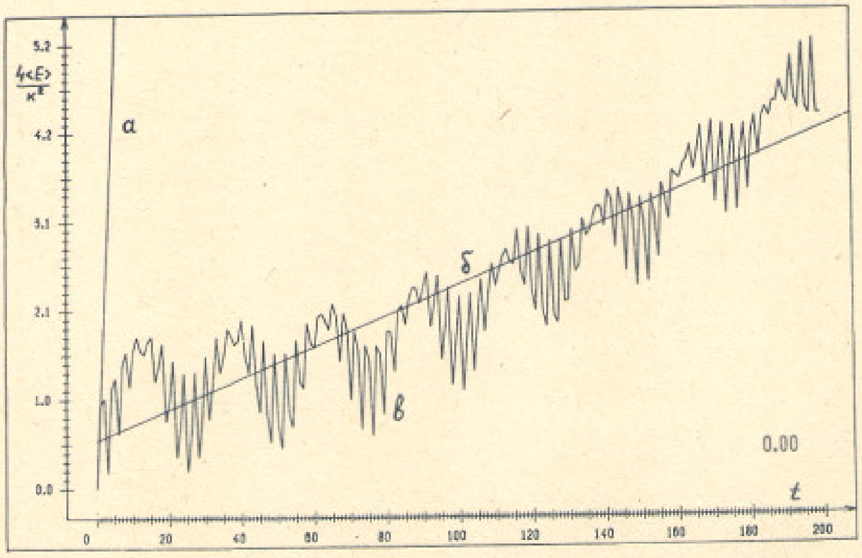

Fig.4 The same as in Fig.1 at $T = \frac{4\pi}{3}$, K = 0.25, t = 200.

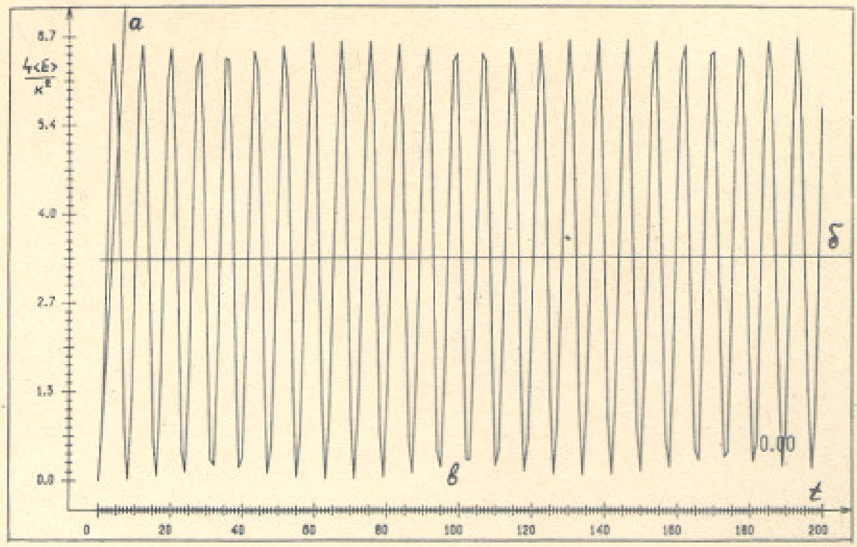

Fig.5 The same as in Fig.1 at $T = \frac{4\pi}{8}$, K = 0.1, t = 200.

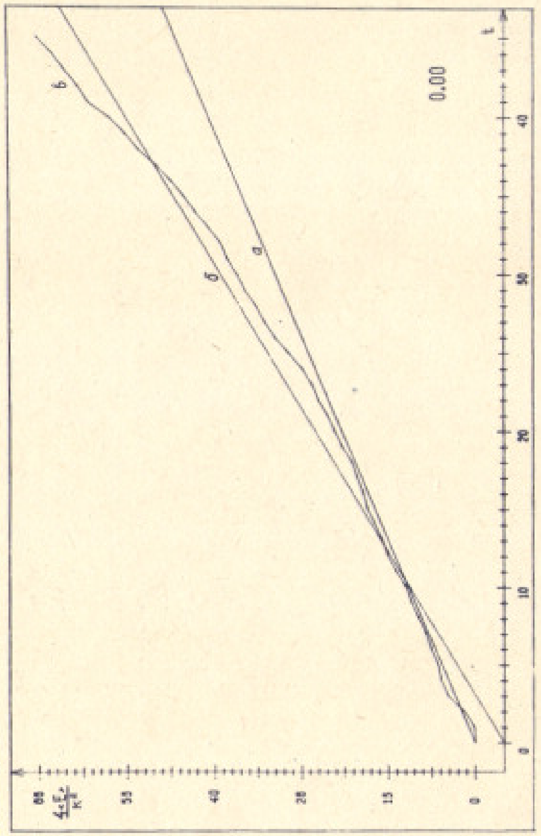

Fig.6 The same as in Fig.1 at $\eta = \frac{4\pi}{101}$, $K = 40$, $\tau = 45$.

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