QUANTUM RESONANCE FOR A ROTATOR IN A NONLINEAR PERIODIC FIELD

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An investigation is made into the behavior of a planar quantum rotator under the influence of an external perturbation which is periodic in the time and takes the form of delta-function "kicks" which depend nonlinearly on the phase. The case of so-called quantum resonance is studied analytically and numerically. It is shown that the rotator energy at large times increases in proportion to t^2. The structure of the spectrum of quasi-energies is analyzed and shown to be continuous.

1. Introduction

Many papers have been published recently containing attempts to understand the quantum features in the behavior of nonlinear systems. The main attention has been devoted to systems subject to an external periodic perturbation (see, for example, [1-8]). This is due in the first place to the new possibilities of experimental investigation into the behavior of atoms and molecules in a field of laser radiation [9-11].

In the present paper, we continue the investigation started in [12]. Our model is a planar quantum rotator with an external linear perturbation which depends periodically (in the form of delta-function kicks) on the time. The behavior of the corresponding classical system has been well studied. In particular, it has been shown [13] that if a certain condition is satisfied the motion becomes stochastic, although it is also described by purely dynamical equations. On the other hand, when the perturbation is small, the motion preserves a quasiperiodic nature. Thus, there exists a criterion which indicates when statistical properties can arise in the dynamical system. The numerical investigation made in [12] showed however that the behavior of a quantum system differs significantly from a classical one even in the deep quasiclassical region. In particular, the diffusion rate of the mean energy of the rotator agrees with the classical rate only for relatively short times, after which it decreases rapidly. In addition, a new type of motion, called quantum resonance, was found, and it has no analog in a classical system. In this regime, the energy of the rotator increases unboundedly irrespective of the magnitude of the external force.

The aim of the present paper is to investigate in detail the quantum resonance discovered in [12]. We have found that in the system there is an infinite, everywhere dense set of such resonances, and we have found the general condition for their occurrence. In the paper, we determine the main characteristics of the motion of the system in resonance. We show analytically that at large times the rotator energy increases quadratically with the time, which is also confirmed by numerical experiments. The asymptotic behavior does not depend on the magnitude of the perturbation and is universal. We analyze the structure of the quasi-energy spectrum, which is continuous in resonance.

2. Quantum Resonance

The chosen model is described by the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2J}\frac{\partial^2}{\partial\theta^2} + \tilde{k}\cos\theta\cdot\delta_{\tilde{T}}(t), \tag{2.1}$$

where $\tilde{k}$ is the parameter which characterizes the magnitude of the perturbation, $\delta_{\tilde{T}}(t)=\sum_{n=-\infty}^{\infty}\delta(t-n\tilde{T})$ is a string of delta functions of the time (kicks), J is the moment of inertia of the rotator, and θ is the angular variable. In what follows, we assume J = 1.

Solving the Schrödinger equation with the Hamiltonian (2.1), we obtain a mapping for the wave function after one step, this including the free rotation during the time $\tilde{T}$ and the kicks (see [12]):

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$$\bar{\psi}(\theta)=\exp(-ik\cos\theta)\sum_{n=-\infty}^{\infty}A_n\exp\left(-i\frac{Tn^2}{2}+in\theta\right), \tag{2.2}$$

where $k=\tilde{k}/\hbar$, $T=\hbar\tilde{T}$, $A_n=\frac{1}{2\pi}\int\limits_0^{2\pi}\psi(\theta)e^{-in\theta}\,d\theta$. In what follows, $\hbar = 1$.

Note that in accordance with (2.2) the motion is not changed under the substitution $T \to T + 4\pi m$, where m is an integer. It is therefore sufficient to consider the values of T on the interval $[0, 4\pi]$.

From (2.2), we find the connection between the Fourier components separated by one step:

$$\bar{A}_n=\sum_{m=-\infty}^{\infty}F_{nm}A_m, \tag{2.3}$$

where $F_{nm}=(-i)^{n-m}\exp\left(-i\frac{T}{2}m^2\right)J_{n-m}(k)$; $J_{n-m}(k)$ is a Bessel function.

As was noted in [12], in the case of the fundamental quantum resonance ($T = 4\pi m$, m integral)

$$\bar{\psi}(\theta)=\exp(-ik\cos\theta)\psi(\theta) \tag{2.4}$$

and the rotator energy $E(t)=-\frac{1}{2}\int\limits_0^{2\pi}\psi^*(\theta)\frac{\partial^2}{\partial\theta^2}\psi(\theta)\,d\theta$ at long times increases quadratically with the time. Thus, if only the ground state ($n = 0$) is excited at the initial time $t = 0$, then

$$E(t)=k^2t^2/4. \tag{2.5}$$

Here and in what follows, t is the dimensionless time measured in the number of kicks.

We investigate now the general case of quantum resonance: $T = 4\pi p/q$. From (2.2) we have $\bar{\psi}(\theta)=\exp(-ik\cos\theta)\cdot F(\theta)$, where

$$F(\theta)=\sum_{n=-\infty}^{\infty}A_n\exp\left(-i\frac{2\pi p}{q}n^2+in\theta\right)=\sum_{m=0}^{q-1}\exp\left(-i\frac{2\pi p}{q}m^2\right)\sum_{l=-\infty}^{\infty}A_{m+ql}\exp(i(m+ql)\theta)=$$

$$\sum_{m=0}^{q-1}\exp\left(-i\frac{2\pi p}{q}m^2\right)B_m;\quad B_m=\sum_{l=-\infty}^{\infty}A_{m+ql}\exp(i(m+ql)\theta). \tag{2.6}$$

To find B_m, we calculate the sum

$$\sum_{m=0}^{q-1}\exp\left(i\frac{2\pi mn}{q}\right)B_m=\sum_{l=-\infty}^{\infty}A_l\exp\left(i\left(\theta+\frac{2\pi n}{q}\right)l\right)=\psi\left(\theta+\frac{2\pi n}{q}\right). \tag{2.7}$$

Hence $B_m=\frac{1}{q}\sum_{n=0}^{q-1}\exp\left(-i\frac{2\pi mn}{q}\right)\psi\left(\theta+\frac{2\pi n}{q}\right)$.

Finally, we obtain the basic relationship for $\bar{\psi}(\theta)$:

$$\bar{\psi}(\theta)=\exp(-ik\cos\theta)\sum_{n=0}^{q-1}\gamma_n\psi\left(\theta+\frac{2\pi n}{q}\right), \tag{2.8}$$

where

$$\gamma_n=\frac{1}{q}\sum_{m=0}^{q-1}\exp\left(-i\frac{2\pi p}{q}m^2-i\frac{2\pi mn}{q}\right).$$

We rewrite (2.8) in a form convenient for further analysis:

$$\bar{\psi}\left(\theta+\frac{2\pi m}{q}\right)=\sum_{n=0}^{q-1}S_{mn}\psi\left(\theta+\frac{2\pi n}{q}\right), \tag{2.9}$$

where S_{mn} is a matrix of the form

$$S=\begin{pmatrix}\beta_0 & & \\ & \cdot & 0 \\ 0 & & \cdot \\ & & & \beta_{q-1}\end{pmatrix}\begin{pmatrix}\gamma_0 & \gamma_1 & \ldots & \gamma_{q-1} \\ \gamma_{q-1} & \gamma_0 & \ldots & \gamma_{q-2} \\ \ldots & \ldots & \ldots & \ldots \\ \gamma_1 & \gamma_2 & \ldots & \gamma_0\end{pmatrix}, \quad \beta_j=\exp\left(-ik\cos\left(\theta+\frac{2\pi j}{q}\right)\right), \quad S_{mn}=\beta_m\gamma_{n-m}. \tag{2.10}$$

Because the matrix S is unitary, its eigenvalues satisfy $\lambda_j(\theta)=\exp(i\alpha_j(\theta))$, $|\lambda_j|=1$. We emphasize that λ_j in the general case depends on θ, as will be shown below.

The matrix S can be conveniently represented in the form

$$S_{mn}=\sum_{l=0}^{q-1} Q_{ml}e^{i\alpha_l}Q_{ln}^{-1}, \tag{2.11}$$

where Q is a q × q unitary matrix ($Q^{-1}=Q^{+}$) with elements Q_{mn} that depend on θ.

We introduce the vector column $\Phi(\theta, t)$ with elements $\Phi_m(\theta, t)=\psi(\theta+2\pi m/q, t)$. From (2.9) and (2.11) we find the time dependence of Φ_m:

$$\Phi_m(\theta, t)=\sum_{n,l=0}^{q-1} Q_{mn}\exp(i\alpha_n(\theta)t)Q_{ln}{}^{*}\Phi_l(\theta, 0). \tag{2.12}$$

Knowing $\Phi_m(\theta, t)$, we can find the time dependence of the momentum and energy of the rotator:

$$P(t)=-\frac{i}{q}\sum_{m=0}^{q-1}\int_0^{2\pi}\Phi_m{}^{*}(\theta, t)\frac{\partial}{\partial\theta}\Phi_m(\theta, t)\,d\theta, \tag{2.13}$$

$$E(t)=-\frac{1}{2q}\sum_{m=0}^{q-1}\int_0^{2\pi}\Phi_m{}^{*}(\theta, t)\frac{\partial^2}{\partial\theta^2}\Phi_m(\theta, t)\,d\theta. \tag{2.14}$$

We obtain directly from (2.12) and (2.13)

$$P(t)=P(0)+a_1t+b_{10}+\sum_{m,m_1=0}^{q-1} R_{mm_1}(t), \tag{2.15}$$

where

$$b_{10}=-\frac{i}{q}\sum_{m,l,l_1}\int_0^{2\pi} d\theta\{\Phi_l(\theta,0)\Phi_{l_1}{}^{*}(\theta,0)Q_{l_1m}Q_{lm}^{*\prime}\}, \qquad a_1=\frac{1}{q}\sum_{m,l,l_1}\int_0^{2\pi}\alpha_m{}'\{\Phi_l(\theta,0)\Phi_{l_1}{}^{*}(\theta,0)Q_{l_1m}Q_{lm}{}^{*}\}\,d\theta,$$
$$R_{mm_1}(t)=-\frac{i}{q}\sum_{n,l,l_1}\int_0^{2\pi} d\theta\{\Phi_{l_1}{}^{*}(\theta,0)\Phi_l(\theta,0)Q_{l_1m_1}Q^{*}_{nm_1}Q_{nm}{}'Q_{lm}{}^{*}\exp(i(\alpha_m-\alpha_{m_1})t)\}. \tag{2.16}$$

Here and in what follows, the prime denotes the derivative with respect to θ.

Since α_m in the general case depends on θ, at asymptotically large times $R_{mm_1}(t)$ (for $m \neq m_1$) can be expressed in terms of an integral of a rapidly oscillating function and, therefore, at large $t: R_{mm_1}(t)=R_{mm_1}(0)\cdot\delta_{mm_1}+o(1/t)$. On the basis of what we have said, we can readily find the asymptotic time dependence of the momentum of the rotator:

$$P(t)=a_1t+b_1+P(0), \tag{2.17}$$

where $b_1=b_{10}+\sum_{m=0}^{q-1} R_{mm}(0)$.

Similarly, the dependence E(t) at large times can be determined:

$$E(t)=\eta t^2+a_2t+b_2+E(0). \tag{2.18}$$

We give expressions only for a_2 and η, the structure of b_2 being similar to that of b_1:

$$a_2=-\frac{i}{q}\int_0^{2\pi} d\theta\left\{\sum_{m,l,l_1=0}^{q-1}\left[\frac{1}{2}\alpha_m{}''\Phi_{l_1}{}^{*}(\theta,0)\Phi_l(\theta,0)Q_{l_1m}Q_{lm}{}^{*}+\alpha_m{}'\Phi_{l_1}{}^{*}(\theta,0)Q_{l_1m}(\Phi_l(\theta,0)Q_{lm}{}^{*})'\right]+\right.$$
$$\left.\sum_{m,l,l_1,n=0}^{q-1}[\alpha_m{}'\Phi_{l_1}{}^{*}(\theta,0)\Phi_l(\theta,0)Q_{l_1m}Q_{nm}{}^{*}Q_{nm}{}'Q_{lm}{}^{*}]\right\}, \quad \eta=\frac{1}{2q}\int_0^{2\pi} d\theta\left\{\sum_{l,l_1,m=0}^{q-1}(\alpha_m{}')^2\Phi_{l_1}{}^{*}(\theta,0)\Phi_l(\theta,0)Q_{l_1m}Q_{lm}{}^{*}\right\}\geqslant 0. \tag{2.19}$$

The obtained expressions for the energy and momentum of the rotator are universal and give the asymptotic behavior at large times. Analysis of the expression (2.19) for the coefficient η, which determines the asymptotic behavior, shows that it vanishes when all the λ_j, and hence also α_m, are independent of θ (λ_j = const). In addition, $\eta = 0$ when at least one λ_n = const and the initial distribution satisfies the special condition

$$\lambda_n \psi_{\lambda_n}(\theta) = \exp(-ik\cos\theta) \sum_{m=0}^{q-1} \gamma_m \psi_{\lambda_n}\left(\theta + \frac{2\pi m}{q}\right).$$

We now try to find the quasienergy spectrum [14-15]. It follows from (2.8) that wave functions with definite quasienergy at the time t = 0 can be represented in the form

$$\psi_{\varepsilon_j(\theta_0)}(\theta, 0) = \sum_{n=0}^{q-1} C_n^j(\theta_0)\,\delta\left(\theta + \theta_0 + \frac{2\pi n}{q}\right), \tag{2.20}$$

where

$$\delta(\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in\theta}.$$

The quasienergy $\varepsilon_j(\theta_0)$ and the coefficients $C_n^j(\theta_0)$ are determined by the system of linear equations

$$\exp(-\varepsilon_j(\theta_0)T)\,C_n^j(\theta_0) = \sum_{m=0}^{q-1} \tilde{S}_{nm} C_m^j(\theta_0), \tag{2.21}$$

where $\tilde{S}_{mn} = \beta_n(\theta_0)\gamma_{n-m}$. The matrix $\tilde{S}$ is unitary, and its eigenvalues $\tilde{\lambda}_j(\theta_0) = \exp(i\tilde{\alpha}_j(\theta_0))$ determine the quasienergy spectrum

$$\varepsilon_j(\theta_0) = -\tilde{\alpha}_j(\theta_0)/T. \tag{2.22}$$

Here, θ_0 is a continuous parameter: $0 \leqslant \theta_0 < 2\pi$.

It follows from (2.22) that the spectrum of quasienergies has discrete levels only when $\tilde{S}$ has eigenvalues $\tilde{\lambda}_j$ = const. Using the explicit form of $\tilde{S}$, we can readily show that for all p/q (except the case $p/q = 1/2$, which will be considered separately) $\operatorname{Sp}\tilde{S}^2 = \sum_{j=0}^{q-1} \tilde{\lambda}_j^2$ depends on the continuous parameter θ_0, i.e., there exist $\tilde{\lambda}_j \neq$ const. Therefore, the quasienergy spectrum (2.22) in resonance is continuous. Besides this continuous component, the spectrum may have discrete levels, whose number is equal to the number of eigenvalues $\tilde{\lambda}_j$ = const, from which it is clear that in resonance p/q ($p/q \neq 1/2$) has not more than q − 1 discrete levels.

Knowing $\psi_{\varepsilon_j(\theta_0)}(\theta, 0)$, we can readily find $\psi_{\varepsilon_j(\theta_0)}(\theta, t)$:

$$\psi_{\varepsilon_j(\theta_0)}(\theta, t) = \exp(-i\varepsilon_j(\theta_0)t)\,\varphi_{\varepsilon_j(\theta_0)}(\theta, t), \tag{2.23}$$

where $\varphi_{\varepsilon_j(\theta_0)}$ is an eigenfunction of the quasienergy $\varepsilon_j(\theta_0)$ that is periodic in the time with period T:

$$\varphi_{\varepsilon_j(\theta_0)}(\theta, t) = \exp(i\varepsilon_j(\theta_0)t - ik\cos\theta\cdot\vartheta(t-T)) \sum_{n=-\infty}^{\infty} A_n \exp\left(-i\frac{n^2 t}{2} + in\theta\right); \quad 0 \leqslant t \leqslant T, \tag{2.24}$$

A_n are the Fourier components of $\psi_{\varepsilon_j(\theta_0)}(\theta, 0)$ and $\vartheta(t - T)$ is the unit step function.

It is interesting to note that $\langle |n| \rangle$ in resonance increases in proportion to the time ($\langle |n| \rangle \sim t$), and therefore if the unperturbed system has spectrum $E_n \sim n^m$ ($m > 1$, integral), its energy will increase with the time in accordance with the law $E(t) \sim t^m$.

We find the exact form of $\lambda_j(\theta)$ in three cases.

1. p/q = 1, which is the fundamental resonance. The dependence of the energy on the time is given by (2.5). The quasienergy spectrum has the form

$$\varepsilon(\theta_0) = (k/4\pi)\cos\theta_0. \tag{2.25}$$

Since $\psi_{\varepsilon(\theta_0)}(\theta, 0) = \delta(\theta + \theta_0)$, the eigenfunction of the quasienergy is determined in accordance with (2.24) by the expression

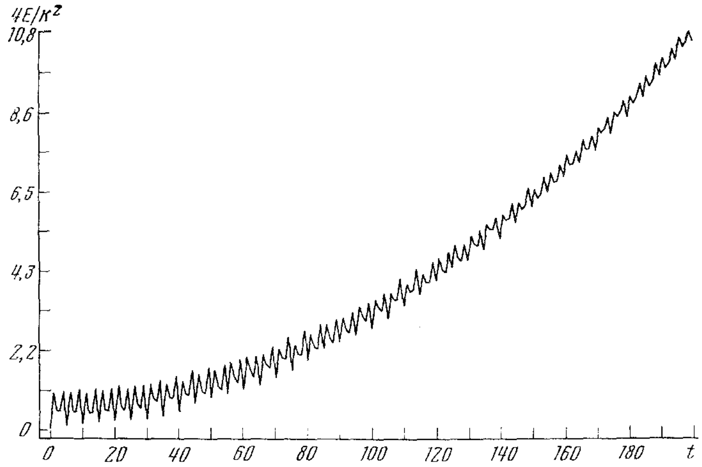

Fig.1

$$\varphi_{\varepsilon(\theta_0)}(\theta,t)=\exp(i\varepsilon(\theta_0)t-ik\cos\theta\cdot\vartheta(t-T))\sum_{n=-\infty}^{\infty}\exp\left(-i\frac{n^2}{2}t+in(\theta+\theta_0)\right), \tag{2.26}$$

where $0\leqslant t\leqslant T$, $T=4\pi$.

2. $p/q = \frac{1}{4}$. It follows from (2.8) that

$$\bar{\psi}(\theta)=\exp(-ik\cos\theta)\frac{1}{\sqrt{2}}(e^{-i\pi/4}\psi(\theta)+e^{i\pi/4}\psi(\theta+\pi)) \tag{2.27}$$

(for $q=4l+2$, l integral, the coefficients $\gamma_{2m+1}=0$, and S has dimension $\frac{q}{2}\times\frac{q}{2}$). The eigenvalues $\lambda_\pm=\tilde{\lambda}_\pm=\exp\left(\pm i\alpha(\theta)-i\frac{\pi}{4}\right)$, where

$$\cos(\alpha(\theta))=(1/\sqrt{2})\cos(k\cos\theta). \tag{2.28}$$

For $k \ll 1$, we have $\partial\alpha/\partial\theta\approx-\mathrm{sign}(\alpha)(k^2/2)\sin 2\theta$, and if at the initial time $\psi(\theta)=1/\sqrt{2\pi}$ (ground state), then $\eta \approx k^4/16$. For the same initial conditions when $k \gg 1$ we have $\eta \approx k^2/12$. From (2.22), we find the quasi-energy spectrum

$$\varepsilon_\pm(\theta_0)={}^1/_4\mp\alpha(\theta_0)/\pi. \tag{2.29}$$

For $k \ll 1$, the spectrum $\varepsilon_\pm(\theta_0)\approx\frac{1}{4}\mp\left(\frac{1}{4}+\frac{k^2}{4\pi}\cos^2\theta_0\right)$ consists of two narrow bands of widths $\sim k^2$. For $k \gtrsim \pi$, we have two broad bands: ${}^1/_2\leqslant\varepsilon(\theta_0)\leqslant1$; ${}^3/_2\leqslant\varepsilon(\theta_0)\leqslant2$. The time dependence of $\varphi_{\varepsilon\pm(\theta_0)}$ is given by (2.24), where A_n are the Fourier components of the function $\psi_{\varepsilon\pm(\theta_0)}(\theta,\ 0)=C_1(\theta_0)\delta(\theta+\theta_0)+C_2(\theta_0)\delta(\theta+\theta_0+\pi)$; here, (C_1, C_2) is an eigenvector of the matrix $\tilde{S}$.

3. In the case $p/q = \frac{1}{2}$, we obtain from (2.8)

$$\bar{\psi}(\theta)=\exp(-ik\cos\theta)\psi(\theta+\pi). \tag{2.30}$$

It can be seen that the system returns to the original state between two kicks. The eigenvalues satisfy $\lambda_{1,2}=\tilde{\lambda}_{1,2}=\pm1$. The quasienergy spectrum consists of two discrete levels with quasienergies $\varepsilon_1=0$, $\varepsilon_2={}^1/_2$. The eigenfunctions of the level ε_1, or rather their values at the time $t = mT$, are the functions $\psi_{\varepsilon_1}(\theta)=g_\pm(\theta)(1\pm\exp(-ik\cos\theta))$, where $g_\pm(\theta)$ is an arbitrary function satisfying the relation $g_\pm(\theta+\pi)=\pm g_\pm(\theta)$. The eigenfunctions of the level ε_2 are $\psi_{\varepsilon_2}(\theta)=g_\pm(\theta)(1\mp\exp(-ik\cos\theta))$. Every level is infinitely degenerate, and the functions ψ_{ε_1} and ψ_{ε_2} form a complete set.

It is evident that the degeneracy of the eigenvalues (when certain λ_j = const) is fortuitous and does not occur for other resonances. Indeed, in the presence of certain λ_j = const and for an arbitrary initial distribution of $\psi(\theta)$, some of the energy would belong to the discrete component of the quasienergy spectrum. Accordingly, the time dependence of the energy would contain periodic undamped (in time) oscillations (as in the case $p/q = \frac{1}{2}$). Such an effect was not found in the numerical experiments. For $k \gg 1$, the dependence $E(t)$ was a smooth function of the time; for $k \ll 1$, the size of the inflections decreased with the time (see Fig.1, in which we have plotted the rotator energy E as a function of the time for quantum resonance when $T=4\pi\cdot{}^2/_5$, $k=0{,}5$, $t=200$).

Thus, in quantum resonance ($p/q = \frac{1}{2}$) the energy of the system increases to infinity in accordance with the asymptotic law $E(t) \sim t^2$, and the quasienergy spectrum is continuous.

For the quantities $(\alpha')^2_{\max} \equiv \max\limits_{0\leqslant j\leqslant q-1} \overline{\left|\frac{\partial\alpha_j(\theta)}{\partial\theta}\right|^2}$ (the bar denotes averaging over θ) we can obtain from the explicit form of S the following estimates (they also hold for $\tilde{\alpha}_j$ of the matrix $\tilde{S}$):

a) $k \ll q$, and then

$$\eta \sim \frac{(\alpha')^2_{\max}}{2} \geqslant q|J_q(k)|^2 \sim \left(\frac{k}{q}\right)^{2q}. \tag{2.31}$$

Note that this estimate is an estimate of the smallest possible value of $(\alpha')^2_{\max}$. It is to be expected that the true value is in order of magnitude equal to this lower bound (see Sec. 3);

b) $k \gg q$, for which we have

$$\eta \sim \frac{(\alpha')^2_{\max}}{2} \approx \frac{k^2}{\xi q}, \tag{2.32}$$

where ξ depends on the initial conditions and is almost independent of k and q. For smooth initial conditions $\psi(\theta, 0)$ the estimate for ξ gives $\xi \approx 5$.

It follows from the estimates obtained for $(\alpha')^2_{\max}$ that for $k \ll q$ (p and q are any mutually prime numbers) the quasienergy spectrum consists of q exponentially narrow bands of width $\Delta\varepsilon \sim (k/q)^q$. In the case $k \gg 1$, to find the band structure one needs knowledge of the detailed properties of the eigenvalues of $\tilde{S}$. Unfortunately, it was not possible to find the explicit form of $\lambda_j(\theta)$.

It follows from what we have said above (see (2.31) and (2.32)) that $\eta \to 0$ as $p \to \infty$, $q \to \infty$, $k = \text{const}$. This means that for irrational values of $T/4\pi$ the value of η is zero, and in this case the motion of the system is entirely different in nature (see [12]).

Now suppose $T=4\pi p/q+\delta$, where $|\delta|\ll 1$; then because the advance of the phases is small, it follows that during a time $t_r \sim 1/\delta d$ ($d \approx \max E(t_r)$), $0 \leqslant t \leqslant t_r$, the characteristics of the system change in time in the same way as in the case of the exact resonance $T = 4\pi p/q$, and this was also clearly observed in the numerical experiment. From the condition $\eta t_r^2 \geq k^2 t_r$ we find the amount of the departure from resonance δ_r at which the influence of the resonance of p/q is significant (we assume that $d \sim k^2 t_r$). For $k > q$ (otherwise, the departure from resonance is negligibly small, $\delta \sim (k/q)^{4q}$) we have

$$\delta_r \leqslant 1/k^2 q^2. \tag{2.33}$$

For the fundamental resonance ($q = 1$, $p = 0$) $T = \delta$, and it follows from the condition (2.33) that $T \leq 1/k^2$. It can be seen that in the quasiclassical region ($k \to \infty$, $T \to 0$, $kT = \text{const}$) the inequality (2.33) is not satisfied, i.e., the influence of the fundamental resonance is small. One can find the total value of all the δ_r:

$$\delta_\Sigma \approx \sum_{q=1}^{k} \sum_{p=1}^{q-1} \delta_r(p, q) \sim \frac{\ln k}{k^2}. \tag{2.34}$$

Since $\delta_\Sigma \ll 1$, it follows that in the case of irrational $T/4\pi$ the resonances have little influence on the motion of the system.

3. Numerical Experiments

Besides the theoretical analysis, we also made a numerical investigation of the model. In the process of solution of the problem, we found the Fourier components of the wave function in accordance with formula (2.3). Although the summation in (2.3) is from $-\infty$ to $+\infty$, the sum actually contains ~2k terms, since $|J_n(k)|$ decreases exponentially with increasing n for $n > k$ (the kicks cover $\approx$2k levels). Because of this, a finite number ($\approx$2k) of Bessel functions was used in the calculations. A control on the accuracy of the calculation consisted of verifying the wave function normalization condition $W = \int\limits_0^{2\pi} |\psi(\theta)|^2 d\theta = 1$. In all cases, the errors did not exceed $\delta W \leqslant 3\cdot 10^{-2}$. The finiteness of the chosen number of levels imposes the main

TABLE 1

k/q	η	k	p/q
0,025	$6\cdot10^{-6}$	0,1	1/4
0,058	10^{-4}	1	1/17
0,058	10^{-4}	1	4/17
0,099	10^{-3}	10	1/101
0,100	$2\cdot10^{-5}$	0,5	2/5
0,150	10^{-2}	1,2	1/8
0,176	10^{-3}	3	1/17
0,200	$6\cdot10^{-4}$	1	1/5
0,235	0,08	4	1/17
0,235	0,144	4	4/17
0,353	0,36	6	1/17

TABLE 2

k/q	η	ξ	k	p/q
0,706	3,0	2,8	12	1/17
1,76	23,4	2,3	30	1/17
3,33	52,4	3,2	50	1/15
5,00	181	2,4	85	1/17
5,44	284	1,7	87	1/16
6,21	239	1,3	87	1/14
7,69	404	1,9	100	1/13
7,73	246	2,7	85	1/11
8,57	206	2,5	60	1/7
10,7	245	3,3	75	1/7
12,4	452	2,3	87	1/7

restriction on the computing time. In the case of a sufficiently large perturbation, there is a rapid excitation of high levels of the system and the computing errors become appreciable. The program was improved compared with [12], which made it possible to increase the computing rate by about a factor two, and also raise the number of levels of the model system to 2 001. The actual computing time on a BÉSM-6 in the typical experiments (Tables 1 and 2) was about 10 min. The additional increase in the number of levels was achieved by considering symmetric initial distributions ($\psi(\theta)=\psi(-\theta)$). The number of levels was taken equal to N = 4 001 (−2 000, +2 000), but because of the symmetry of the initial conditions and the Hamiltonian (2.1) the calculation was actually made with 2 001 levels.

The initial conditions were varied from excitation of one level (the ground state) to the excitation of about 20 levels (Gaussian packet). In all cases, the asymptotic form of the motion depended weakly on the choice of the initial state. In the evaluation of the numerical results, we calculated the rotator energy $E=-\frac{1}{2}\int_0^{2\pi}\psi^*\frac{\partial^2}{\partial\theta^2}\psi\,d\theta$. We simultaneously plotted the graph of the energy as a function of the time and used the least-squares method to fit $E(t)$ to a quadratic polynomial.

For $k < q$, one can follow experimentally a good quadratic dependence of the energy on the time (see Fig.1). The fitting was done in this case in accordance with the formula $E(t) = \eta t^2 + E(0)$. The data for η for different k and q are given in Table 1. For $k \ll q$, the values of η are too small, and the quadratic growth of the energy at finite times $t \leq 200$ is not always noted. Therefore, it is difficult to speak of a quantitative agreement with the estimate (2.31), but it can be asserted that η decreases much faster than k/q.

For $k \gg q$, the dependence $\eta(k, q)$ can be approximated by the analytic estimate (2.32). The experimental data (Table 2) show reasonable agreement with this formula, and the value of ξ changes weakly and does not depend explicitly on k and q. The mean value of ξ is $\langle\xi\rangle = 2.4$.

We tested separately the extent to which η depends on the values of p for the same value of q. As expected, there is hardly any dependence on p, in agreement with (2.31)-(2.32).

In the quasiclassical region ($k\to\infty$, $T=4\pi/q\to0$ for kT = const $\gg 1$) the experimental data for $k \gg q$ show that at short times the dependence $E(t)$ in dimensionless variables can be well described by the semi-empirical formula

$$E(t)=k^2t^2/\xi q+k^2t/4+E(0). \tag{3.1}$$

The coefficient of t^2 agrees with the theoretical estimate (2.32) for η. The second term in (3.1), which is linear in the time, corresponds exactly to classical diffusion [12], though the coefficient $k^2/4$ differs in general from the asymptotic value of a_2 in (2.18), so that the term $k^2t/4$ is in fact not a diffusion term. It follows from the experimental dependence (3.1) that at times $t < t^*$, where $t^*\approx q\xi/4\sim 1/T$ (in dimensionless variables $\tau<\tau^*=t^*\bar{T}\leqslant J/\hbar$) the energy increases basically through the "diffusion" term $k^2t/4$. For $t > t^*$, the quadratic term, which is purely quantum, becomes dominant.

4. Concluding Remarks

Our investigations show that for quantum resonances, the system of which is everywhere dense, the asymptotic dependence of the rotator energy on the time is universal and described by the quadratic law (2.18). This means that in resonance there is no quantum stability boundary ($k \approx 1$) as predicted in [3] and observed in the nonresonance case [12]. It is also important to note that there is no classical stability criterion ($kT \approx 1$), although the system can then be in the deep quasiclassical region. At the same time, for a nonlinear system with classical Hamiltonian corresponding to (2.1) it follows from the Kolmogorov–Arnol'd–Moser theory [16-18] and the numerical experiments [13] that the motion in the case of a small perturbation is stable and the energy of the system bounded. All this indicates that there is an important difference between the behavior of a quantum system and a classical system, at least at large times.

For relatively small t, when asymptotia has not yet been reached, the system may have a fairly complicated behavior which depends strongly on the parameters k and T. For example, the quadratic growth of the energy of the system can be most clearly followed if $T \gg 1$ and $k \gg q \approx 1$. In this case, the energy immediately increases in proportion to t^2. But if $k < q$, the energy of the system in practice oscillates, and the quadratic growth ηt^2 is small because $\eta \ll 1$.

It is interesting to see what happens to the motion in the limit $p \to \infty$, $q \to \infty$, $p/q = \text{const}$, which corresponds to a transition to a nonresonance value of T. In this case, as is shown by the numerical experiments in [12], the motion of the system is quite different from what it is in resonance. However, the analytic investigation leads to great difficulties because it is necessary to know the exact solution in resonance at all times and not only asymptotically. On the other hand, one can consider how the structure of the quasienergy spectrum changes on the transition to high resonances $q \gg 1$. Here we have been able to show that for $q > k$ the width of each quasienergy band is exponentially small ($\Delta\varepsilon \sim \sqrt{\eta} \approx (k/q)^q$), and so is the total width of all the q bands. This may indicate that in the nonresonance case the quasienergy spectrum becomes discrete. In the case $k \gg q$, the question of the overlapping and width of the quasienergy bands remains open.

In conclusion, we make a comment about systems which can manifest quantum resonances under the influence of a delta-function perturbation that is periodic in the time (for the one-dimensional case, this can be represented in the form $f(x)\delta_T(x)$, where $f(x)$ is an arbitrary function of the coordinate x). It is easy to show that for the existence of a resonance it is necessary that the spectrum of the unperturbed Hamiltonian be discrete and have the form of a polynomial in the quantum number with rational coefficients. It is also required that a condition of the form $\psi_m \cdot \psi_n = \psi_{m+n}$ hold for the eigenfunctions of the Hamiltonian H_0. Then

$$\exp\left(i\frac{2\pi l(m+n)}{q}\right)\psi_{m+n} = \left(\exp\left(i\frac{2\pi lm}{q}\right)\psi_m\right)\left(\exp\left(i\frac{2\pi lm}{q}\right)\psi_n\right),$$

and on the basis of this equation it is possible, as in the considered case, to reduce the representation after a period to multiplication by a matrix. It is quite possible that the last condition could be weakened.

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