Quantum resonance for a rotor in a nonlinear periodic field

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1. The behavior of nonlinear quantum systems under the action of an external periodic perturbation has recently received much attention.[1-7] This is primarily related to new possibilities of experimental studies of the behavior of atoms and molecules in a laser-radiation field.[8,9] Basic difficulties in the theoretical analysis are encountered when the external action cannot be assumed to be small, in which case perturbation theory no longer applies. In this case numerical studies become increasingly important. Such a study for a simple and well-studied model in classical mechanics[10] (a planar rotor under the action of δ-function pulses) was performed in Ref. 11. The basic result was the discovery of significant deviations in the behavior of a quantum system (in comparison with the classical one) when the motion becomes stochastic. These differences persist even if the quantum system is in the strongly quasiclassical region. In particular, the diffusion rate of the mean energy of the rotor coincides with the classical value only over limited times, and then drops off quickly. In addition, a distinctive type of motion, named quantum resonance, was discovered, and has no analog in the classical system. The present work is devoted to a detailed study of the quantum resonance discovered in Ref. 11.

2. The model selected is described by the Hamiltonian

$$\hat{H} = \frac{\hbar^2}{2J}\frac{\partial^2}{\partial\theta^2} + \tilde{k}\cos\theta\,\delta_{\tilde{T}}(t), \tag{1}$$

where $\tilde{k}$ is a parameter characterizing the size of the perturbation, $\delta_T(t) = \sum_{n=-\infty}^{\infty}\delta(t-n\tilde{T})$ is a periodic δ-function, J is moment of inertia of the rotor, and θ is the angular variable; in what follows we put J = 1. Solving the Schrödinger equation with the Hamiltonian (1), we obtain for the wave function after one step a representation that includes the free rotation during the time $\tilde{T}$ and the pulse (see Ref. 11):

$$\bar{\psi}(\theta) = \exp(-ik\cos\theta)\sum_{n=-\infty}^{\infty} A_n \exp\left(-i\frac{Tn^2}{2} + in\theta\right), \tag{2}$$

where $k = \tilde{k}\hbar$, $T = \hbar\tilde{T}$, $A_n = \frac{1}{2\pi}\int_0^{2\pi}\psi(\theta)e^{-in\theta}d\theta$ Below we take $\hbar = 1$. We note that according to (2) the motion does not change when we replace $T \to T + 4\pi m$, where m is an integer. Therefore, it is sufficient to consider T on the interval $[0, 4\pi]$.

As was noted in Ref. 11, in the case of the principal quantum resonance ($T = 4\pi m$, m is an integer) the rotor energy $E(t) = -\frac{1}{2}\int_0^{2\pi}\psi^*(\theta)\frac{\partial^2}{\partial\theta^2}\psi(\theta)d\theta$ increases quadratically with time at large values of t. We study the general case of quantum resonance: $T = 4\pi p/q$, where p and q are mutually prime integers. Following certain transformations, (2) can be represented in the form

$$\bar{\psi}(\theta) = \exp(-ik\cos\theta)\sum_{n=0}^{q-1}\gamma_n\psi\left(\theta + \frac{2\pi n}{q}\right),$$
$$\gamma_n = \frac{1}{q}\sum_{m=0}^{q-1}\exp\left(-i\frac{2\pi p}{q}m^2 - i\frac{2\pi mn}{q}\right). \tag{3}$$

We write (3) in a form convenient for the following analysis:

$$\bar{\psi}\left(\theta + \frac{2\pi m}{q}\right) = \sum_{n=0}^{q-1} S_{mn}\psi\left(\theta + \frac{2\pi n}{q}\right); \tag{4}$$

where S_{mn} is a matrix represented in the form of a product of a diagonal and a cyclic matrix: $S_{mn} = \beta_m \cdot \gamma_{n-m}$; $\beta_m = \exp[-ik\cos(\theta + 2\pi m/q)]$. Owing to the unitarity of the matrix S_{mn} its eigenvalues are of the form $\lambda_j(\theta) = \exp[i\alpha_j(\theta)]$, $|\lambda_j| = 1$. We emphasize that in the general case the λ_j depend on θ. From (4) one can find the time dependence of the rotor energy (t is the dimensionless time, measured by the number of pulses):

$$E(t) = E(0) + \eta t^2 + a_{10}t + b_{10} + \sum_{m,m_1=0}^{q-1} G_{mm_1}(t) + t\sum_{m,m_1=0}^{q-1} R_{mm_1}(t), \tag{5}$$

where η, a_{10}, b_{10} are time-independent constants. As an example we give the expressions for η and $R_{mm_1}(t)$:

$$\eta = \frac{1}{2q}\int_0^{2\pi} d\theta\left\{\sum_{l,l_1,m=0}^{q-1}(\alpha'_m)^2\Phi^*_l(\theta,0)\Phi_l(\theta,0)Q_{l_1m}Q^*_{lm}\right\} \geqslant 0,$$

$$R_{mm_1}(t) = -\frac{i}{q}\left\{\sum_{n,l,l_1=0}^{q-1}\int_0^{2\pi} d\theta\left\{\alpha'_m\Phi^*_{l_1}(\theta,0)\Phi_l(\theta,0)Q_{l_1m_1}Q^*_{nm_1}Q'_{nm}Q^*_{lm}\right.\right.$$
$$\times \exp(i(\alpha_m - \alpha_{m_1})t)\Big\}, \tag{6}$$

where $\Phi_m(\theta, t) = \psi(\theta + 2\pi m/q, t)$, Q is the unitary matrix that reduces S_{mn} to diagonal form, and the prime denotes a derivative with respect to θ. Since the α_m depend on θ, at asymptotically long times $R_{mm_1}(t)$ and $G_{mm_1}(t)$ ($m \neq m_1$) are expressed in terms of an integral of a rapidly oscillating function and tend to $R_{mm}(0)$, $G_{mm}(0)$ with increasing time. Thus, at asymptotically long times we have

$$E(t) = \eta t^2 + a_1 t + b_1 + E(0). \tag{7}$$

The asymptotic time dependence of the rotor momentum is determined analogously: $P(t) = a_2 t + b_2 + P(0)$. The expressions obtained for the energy and momentum are universal, and give the form of the asymptotic behavior at long times.

3. We study the structure of the quasienergy spectrum. At time t = 0 the wavefunctions with definite quasienergy can be represented in the form

$$\psi_{\epsilon_j(\theta_0)}(\theta,0)=\sum_{n=0}^{q-1} C_n^j(\theta_0)\delta_{2\pi}\left(\theta+\theta_0+\frac{2\pi m}{q}\right). \qquad (8)$$

The quasienergy $\varepsilon_j(\theta_0)$ is determined by the eigenvalues $\tilde{\lambda}_j(\theta_0)=\exp[i\tilde{\alpha}_j(\theta_0)]$ of the unitary matrix $\tilde{S}_{nm}=\beta_n(\theta_0)\cdot\gamma_{n-m}$:

$$\epsilon_j(\theta_0)=-\tilde{\alpha}_j(\theta_0)/T. \qquad (9)$$

The coefficients $C_n^j(\theta_0)$ are elements of eigenvectors of the matrix $\tilde{S}_{nm}$, and θ_0 is a continuous parameter: $0\le\theta_0<2\pi$. It is seen from an analysis of (9) that the quasienergy spectrum has discrete levels when the matrix $\tilde{S}_{nm}$ has eigenvalues $\tilde{\lambda}_j$ that do not depend on θ_0. It follows from the explicit form of $\tilde{S}_{nm}$ that for any p/q ($p/q\neq{}^1/_2$) the spectrum of energies (9) is continuous at resonance. One easily find from (8)

$$\psi_{\epsilon_j(\theta_0)}(\theta,t)=\exp(-ik\cos\theta\cdot\vartheta(t-T))\sum_{n=-\infty}^{\infty}A_n\exp\left(-i\frac{n^2t}{2}+in\theta\right),$$

$$0\leqslant t\leqslant T;$$

where A_n are the Fourier components of $\psi_{\epsilon_j(\theta_0)}(\theta,0)$, $\vartheta(t-T)$ is the unit step function. It is interesting to note that $\langle|n|\rangle$ increases at resonance linearly with time; therefore, if the unperturbed system possessed a spectrum $E_n\sim n^m$ (m > 1 is an integer), its energy would increase with time according to the law $E(t)\sim t^m$.

The explicit form of $\tilde{\lambda}_j(\theta)$ is found in three cases:

a) p/q = 1 (the principal resonance). The time dependence of the energy (when the ground state n = 0 is initially excited) is given by the formula $E(t)=k^2t^2/4$. The quasienergy spectrum is $\varepsilon(\theta_0)=k\cos\theta_0/4\pi$.

b) $p/q={}^1/_2$. From (3) we obtain $\tilde{\psi}(\theta)=\exp(-ik\cos\theta)\cdot\psi(\theta+\pi)$. After two pulses the system returns to the initial state. The eigenvalues $\tilde{\lambda}_{1,2}=\pm1$. The quasienergy spectrum consists of two levels with quasienergies $\varepsilon_1=0$, $\varepsilon_2={}^1/_2$. This degeneracy of eigenvalues (when the $\tilde{\lambda}_j$ are independent of θ) is accidental, and obviously does not occur for other resonances.

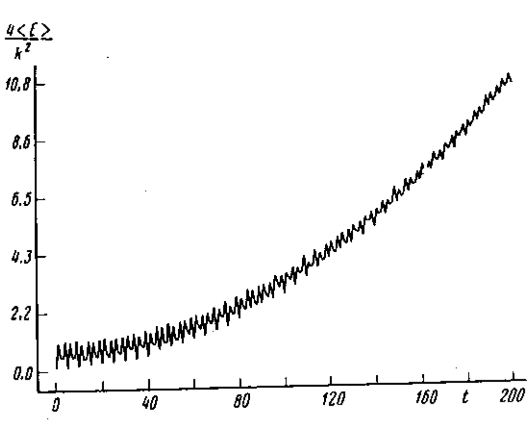

FIG. 1. Time dependence of rotor energy E in case of quantum resonance: $T=4\pi\cdot{}^2/_5$; k = 0.5; t = 200.

c) $p/q={}^1/_4$. It follows from (3) that $\tilde{\psi}(\theta)=\exp(-ik\cdot\cos\theta)\cdot2^{-1/2}\,[e^{-i\pi/4}\psi(\theta)+e^{i\pi/4}\psi(\theta+\pi)]$. The eigenvalues $\lambda_\pm=\tilde{\lambda}_\pm=\exp[\pm i\alpha(\theta)-i\pi/4)$, where $\cos[\alpha(\theta)]\cdot2^{-1/2}\cos(k\cos\theta)$. For $k\ll1$, when the ground state is initially excited, we obtain $\eta\approx k^4/16$. For $k\gg1$ we have $\eta\approx k^2/12$. The quasienergy spectrum is $\epsilon_\pm(\theta_0)={}^1\!/\!_4\mp\alpha(\theta_0)/\pi$.

From the estimates obtained for η it follows that for $k\ll q$, $\eta\sim(k/q)^{2q}$ and the quasienergy spectrum consists of q exponentially narrow bands of width $\Delta\varepsilon\sim(k/q)^q$. In the case $k\gg q$ ($\eta\sim k^2/q$), to find the band structure it is required to know the detailed properties of the eigenvalues of the matrix S_{nm}.

4. Along with the theoretical analysis we also investigated the model numerically. In solving the problem the Fourier components of the wave function were found from the formula $\bar{A}_n=\sum_{m=-\infty}^{\infty}F_{nm}A_m$, where $F_{nm}=(-i)^{n-m}\cdot\exp(-iTm^2/2)J_{n-m}(k)$; $J_{n-m}(k)$ is a Bessel function. In effect the sum contains $\sim2k$ terms, since $|J_n(k)|$ falls off exponentially with increasing n' for n' > k (the pulse encompasses $\sim2k$ levels). The accuracy of the calculation was controlled by checking the condition of normalization of the wave function: $W=\sum_{n=-\infty}^{\infty}|A_n|^2=1$. In all cases the errors did not exceed $\delta W\leqslant3\cdot10^{-3}$. The errors of the calculation increases with time because of the finite number of levels N = 4001. The initial conditions were varied from the excitation of one level (the ground state) to the excitation of ~20 levels (a Gaussian packet). In all cases the asymptotic form of motion depended weakly on the choice of the initial state. The rotor energy was calculated from the formula $E=\sum_{n=-\infty}^{+\infty}\frac{n^2}{2}|A_n|^2$. The numerical results for a very wide range of k, q, p show that the dependence E(t) is quite well approximated by a quadratic polynomial (see Fig. 1). The experimental values of η agree in order of magnitude with the analytic estimates.

5. The studies performed show that for quantum resonances whose system is everywhere dense the asymptotic time dependence of the rotor energy is universal and is described by the quadratic law (7). This implies that the quantum stability limit ($k\approx1$) predicted in Ref. 3 and observed in the nonresonance case[11] is absent at the resonance. It is also important to note that there exists no classical stability criterion ($kT\approx1$) either, even though the system may be in the strongly quasiclassical region. At the same time, for a nonlinear system with a classical Hamiltonian corresponding to (1), according to the Kolmogorov–Arnold–Moser theory[12–14] and the numerical experiments of Ref. 10, when the perturbation is small the motion is stable and the energy of the system is bounded. All this indicates an essential difference between the behavior of the quantum system and that of the classical one.

It is easily shown that for a perturbation of type $f(x)\cdot\delta_T(t)$ it is necessary for the existence of resonance that the spectrum of the unperturbed Hamiltonian H_0 be discrete and have the form of a polynomial in the quantum number with rational coefficients. In this case it is also required that the condition $\psi_m\psi_n=\psi_{m+n}$ be satisfied for the eigenfunctions of the Hamiltonian H_0. It is quite likely that the latter condition can be weakened.

The quantum-resonance effect considered here can be useful in the study of rapid excitation of quantum systems by means of sufficiently short periodic laser pulses.

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The effect of plane gravitational waves of arbitrary frequency on an oscillator

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The problem of detecting gravitational waves (GW) introduces the very important task of developing methods of effective investigation of the behavior of various systems (detectors) in the GW field. We consider one of such possibilities, based on the well known averaging method of Krylov and Bogolyubov.[1,2]

We will proceed from the Golikov–Sherman equations,[3] which describe the displacements of particles in a gravitational field relative to a reference line,

$$(\delta^\alpha_\beta - \overset{0}{u}{}^\alpha \overset{0}{u}{}_\beta)\left\{\frac{\delta^2\xi^\beta}{ds^2} - \frac{\delta\Phi^\beta}{d\xi} + \xi^\delta(2\overset{0}{\Phi}{}^\beta\overset{0}{\Phi}{}_\delta + R^\beta{}_{\gamma\delta\rho}\,\overset{0}{u}{}^\gamma\overset{0}{u}{}^\rho)\right\} = 0, \quad (1)$$

where $R^\alpha_{\beta\gamma\delta}$ is the Riemann–Christoffel tensor, u^α is the particle velocity ($\alpha, \beta, \gamma = 0, 1, 2, 3$), $\Phi^\alpha = (1/mc^2)\,F^\alpha$, $F^\alpha(x, u)$ is the four-dimensional force field, ξ^α is the relative displacement vector, and the quantity

$$\delta\Phi^\alpha/d\xi = \tilde{\Phi}^\alpha - \overset{0}{\Phi}{}^\alpha + \overset{0}{\Gamma}{}^\alpha_{\beta\gamma}\,\overset{0}{\Phi}{}^\beta\xi^\gamma$$

is the absolute derivative of the four-force vector Φ^α in the direction of the vector ξ. All the quantities appearing in (1) are defined on the reference line.

Let a gravitational wave

$$ds^2 = (dx^0)^2 - (dx^3)^2 - g_{ab}(x^0 + x^3)\,dx^a dx^b, \quad a, b = 1, 2, \quad (2)$$

which is propagating along the coordinate x^3, impinge on a detector located in the coordinate plane (x^1x^2). (The actual form of the detector as well as the type of GW, which can be weak or strong, is not of concern at the moment.)

Let us take as the reference line the geodesic of the center of mass ($\overset{0}{\Phi}{}^\alpha = 0$) and use the frame of reference co-moving with the center of mass ($\overset{0}{u}{}^\alpha = \delta^\alpha_0$). Then the original equations assume the form

$$\frac{d^2\xi^a}{dx^2} + 2\overset{0}{\Gamma}{}^a_{ob}\frac{d\xi^b}{dx} = \tilde{\Phi}^a, \quad x = x^0 + x^3. \quad (3)$$

This does not contradict the conclusion of Ref. 4 that the displacements of particles in a field of plane waves are longitudinal, although we consider the co-movement condition only in the local neighborhood of the reference line and not over the whole region of space-time.

We expand the force $\tilde{\Phi}^\alpha$ on the world line $\tilde{x}^\alpha(\tau)$ in a Taylor series:

$$\tilde{\Phi}^\alpha = \overset{0}{\Phi}{}^\alpha + \frac{\partial\Phi^\alpha}{\partial x^\gamma}(\overset{0}{x})(\tilde{x}^\gamma - \overset{0}{x}{}^\gamma) + \frac{\partial\Phi^\alpha}{\partial u^\gamma}(\overset{0}{x})(\tilde{u}^\gamma - \overset{0}{u}{}^\gamma) + \ldots = -K^\alpha_\gamma\xi^\gamma - D^\alpha_\gamma\frac{d\xi^\gamma}{ds} + \ldots, \quad (4)$$

where the quantities

$$K^\alpha_\gamma = \frac{1}{mc^2}k^\alpha_\gamma \equiv -\frac{\partial\Phi^\alpha}{\partial x^\gamma}(\overset{0}{x}), \quad D^\alpha_\gamma = \frac{1}{mc}d^\alpha_\gamma \equiv -\frac{\partial\Phi^\alpha}{\partial u^\gamma}(\overset{0}{x})$$

are interpreted as the stiffness constant and internal resistance of the detector. We next introduce the small positive parameter $\varepsilon = 1/c$ and reduce Eq. (3) to the following form:

$$d\xi^a/dx = \epsilon G^a(x, \eta), \quad d\eta^a/dx = H^a_0 + \epsilon H^a_1, \quad (5)$$

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