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QUASICLASSICAL APPROXIMATION FOR STOCHASTICAL QUANTUM SYSTEMS

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QUASICLASSICAL APPROXIMATION FOR STOCHASTICAL QUANTUM SYSTEMS

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A b s t r a c t

The conditions of the validity of the quasiclassical approximation for quantum systems stochastical in the classical limits are studied. It is shown that the quasiclassical approximation is applicable on the times $t_0 \sim n_{ch} \sim 1/\hbar$ greatly exceeding the time of spreading the quasiclassical packet $\tau_0 \sim \ln n_{ch} \sim \ln(1/\hbar)$. The general estimate for the times t_0 is obtained. Numerical experiments for simple quantum systems confirm the theoretical predictions.

1. Introduction

In recent years the so-called stochastical motion has been discovered and studied, that allows one to understand in what way a random motion can arise in the classical mechanics systems described by purely dynamical equations (see, e.g., /1-6/). It has been shown that the stochasticity appears when some restrictions imposed on the interaction and the system nonlinearity are met. It is connected with the local instability of motion, i.e. with the exponentially rapid divergence of close trajectories in a phase space. If the stochasticity criterion is satisfied, the system motion is mixing, that means the tripping of the time correlations of dynamical variables.

The stochasticity in classical systems has been analysed adequately, meanwhile the study of quantum dynamical systems stochastical in the classical limit ($\hbar = 0$) has begun quite recently (7-11). Investigation of such systems is of considerable interest both for the construction of a statistical mechanics with no use of additional hypotheses and for the explanation of various phenomena in quantum systems. So, the new possibilities of an experimental study of the behaviour of atoms and molecules in the field of a strong electromagnetic wave /12-14,24-25/ have stimulated the publication of numerous papers in which the quantum specific features of the behaviour of nonlinear systems under the action of an external time-periodical perturbation are examined (see, for example, /15-21/). One of the methods of considering such problems is the quasiclassical approximation /7-9, 15,19,26/. At the same time, it is known that in nonlinear systems the corrections to quasiclassical expressions increase with time (see, e.g.,19) and after some time period t_0 the quasiclassical expansion becomes inapplicable. For integrable systems these times are proportional, as a rule, to a characteristic quantum number or to a certain other quantum parameter of the problem ($t_0 \sim n_{ch} \sim 1/\hbar$). This follows directly from the Ehrenfest theorem and is a result of the fact that until the packet is not spreaded, it moves over (along) the classical trajectories. For stochastical systems the question concerning the times on which the WKB approximation (Wenzel-Kramers-Brillouin) method is valid, is a more complicated one because of the local instability of the classical trajectories, which leads to the exponentially rapid spreading of the quasiclassical packet. Thus, the quasiclassical approximation is entirely conserved on the times $\tau_0 \sim \ln n_{ch} \sim \ln(1/\hbar)$. The behaviour of stochastical quantum systems (SQSs), by such systems one means the quantum systems which are stochastical in the classical limit, on such times was examined analytically in Ref./8/ on the example on a nonlinear oscillator with external periodical perturbation. In /9/ the SQSs were analyzed in the quasiclassical approximation but the validity of the method on large times was not substantiated.

The analytical and numerical analysis of the SQSs in Refs./10,20/ was exemplified by the quantum rotator under the action of periodical perturbation. The main result /10/ is that the motion of the SQS under consideration is similar, under certain conditions, to the stochastical motion of the classical system. In particular, the rotator energy grew diffusively with time. However, significant differences in the diffusion rate on large times were observed. In Ref./20/ for an infinite series of parameters the asymptotic energy-time dependence has been found, which differs from the classical one.

The aim of the present paper is to study the conditions under which the WKB method is valid for the SQS. On the example of simple models, the times are found on which the deviations from the classical values are small. The conditions are found under which the quantum corrections keeps small in comparison with the classical values on all times. The general condition of the validity of the quasiclassical approximation for the SQS is obtained, basing upon Maslov's results /22,23/.

2. The model

Let us consider a model of the rotator in an external field with Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2J}\frac{\partial^2}{\partial\theta^2} + \tilde{K}\cos\theta\,\delta_{\tilde{T}}(t) \qquad (2.1)$$

where $\tilde{K}$ is a parameter characterizing the perturbation, $\delta_{\tilde{T}}(t) = \sum_{n=-\infty}^{\infty}\delta(t - n\tilde{T})$ is the time paling of delta-functions (kicks), J is the momemt of inertia of the rotator, θ is the angular variable. In what follows we assume that $J = 1$.

The associated calssical problem is described by the Hamiltonian :

$$H = \frac{p^2}{2} + \tilde{K}\cos\theta\,\delta_{\tilde{T}}(t) \qquad (2.2)$$

and in view of the periodicity of perturbation, the rotator motion may be described by the mapping :

$$\begin{aligned} \bar{p} &= p + \tilde{K}\sin\theta \\ \bar{\theta} &= \theta + \tilde{T}\bar{p} \end{aligned} \qquad (2.3)$$

where $\bar{p}$, $\bar{\theta}$ are the values of the variables after a kick.

The mapping (2.3) has been studied in detail in Ref./6/, where it has been shown that the value $\tilde{K}\tilde{T} \approx 1$ is the border of stability. At $\tilde{K}\tilde{T} < 1$ the motion is stable, and the change of the quantity p is limited ($|\Delta p| \lesssim \sqrt{\tilde{K}/\tilde{T}}$). At $\tilde{K}\tilde{T} \gg 1$, it is the case already at $\tilde{K}\tilde{T} = 5$, the motion becomes stochastic. In this case θ is a random variable, Δp changes according to the diffusion law, and the pulse-distribution function has the Gaussian form :

$$\begin{aligned} \langle (p(t) - p_0)^2 \rangle &= D\,\frac{\tilde{K}^2}{2}\,t \\ f(p) &= \frac{1}{\sqrt{\pi t}\,\tilde{K}}\exp\left[-\frac{p^2}{\tilde{K}^2 t}\right] \end{aligned} \qquad (2.3A)$$

$\langle p \rangle = p_0$, $D \approx 1$. Here and below t is dimensionless time measured by the number of kicks. The brackets ⟨ ⟩ imply the averaging over a large number of the trajectories corresponding to different initial data. Note also that such simple mapping (2.3) describes approximately the total number of interesting mechanical systems, for instance, the motion of a charged particle in a magnetic trap and, what is much more essential, the motion in the vicinity of the separatrix of a nonlinear resonance of the quite general form /6/.

Let us proceed now to the quantum-mechanical consideration. Solving the Schrodinger equation with the Hamiltonian (2.1), we obtain the mapping for a wave function in one step, which includes free rotation during the time $\tilde{T}$ and the kick (the action of the kick is reduced to the product of the wave function by $\exp(-i\,\tilde{k}/\hbar \cos x)$ /10/)

$$\bar{\Psi}(x) = (2\pi i T)^{-1/2} \int_{-\infty}^{\infty} \Psi(x_1) \exp\left[-ik \cos x_1 + i\,\frac{(x-x_1)^2}{2T}\right] dx_1 \qquad (2.4)$$

where $k = \tilde{k}/\hbar$, $T = \hbar\tilde{T}$, and $\hbar = 1$.

Because of periodicity $\Psi(x)$ with period 2π, $\bar{\Psi}(x)$ is also periodical and normalized over the period $\int_0^{2\pi} |\bar{\Psi}(x)|^2 dx = 1$. From (2.4) we obtain the expression for Ψ at the time moment t :

$$\Psi(x,t) = = (2\pi i T)^{-t/2} \int \cdots \int_{-\infty}^{\infty} \varphi_0(x_0) \exp(iS(x,x_{t-1},\ldots,x_0))\, dx_0 \ldots dx_{t-1} \qquad (2.5)$$

where

$$S(x,x_{t-1},\ldots,x_0) = \frac{(x-x_{t-1})^2}{2T} - k \cos x_{t-1} + \ldots + \frac{(x_1-x_0)^2}{2T} - k \cos x_0 + S_0(x_0), \qquad \Psi(x,0) = \varphi_0(x_0) \exp(iS_0(x_0)).$$

In what follows we are interested in the quasiclasscal approximation in the stochasticity region, that corresponds to $k \gg 1$ (the number of quanta in a kick is large), $T \ll 1$, $kT \gg 1$. In this case t-fold integral is calculated by the

stationary phase method /27/ :

$$\Psi(x,t) = \sum_{k=1}^{N} |J_k|^{-1/2} \exp\left[iS_k(x) + i\,\frac{\pi}{4}\mu_k - i\,\frac{\pi}{4}t\right] \cdot \left\{\sum_{m=0}^{\infty} \frac{1}{m!} L_k^m[\varphi_0(x_0)\exp(i\tilde{S})]\Big|_{\vec{x}=\vec{x}^k}\right\} + O(k^{-\infty}) \qquad (2.6)$$

where $\vec{x} = \vec{x}^k$ is the stationary point in the t-dimensional space, which is a solution of the set of equations

$$\frac{\partial S(x,x_{t-1},\ldots,x_0)}{\partial x_i} = 0\ ; \quad i = 0,\ldots,t-1$$

N is the total number of stationary points ;

$$H_{ij}^k(x) = \frac{\partial^2 S}{\partial x_i \partial x_j}\Big|_{\vec{x}=\vec{x}^k} =$$

$$= \frac{1}{T}\left\{\left[-1 + T\,\frac{\partial^2 S_0(x_0)}{\partial x_0^2}\right]\delta_{i,0} + (2 + kT\cos x_i)\delta_{i,j} - \delta_{i,j+1} - \delta_{i+1,j}\right\}\Big|_{x_i = x_i^k}, \quad J_k(x) = \det(TH_{ij}^k(x)),$$

$$\mu_k = \operatorname{sgn} H^k, \qquad L_k = \frac{1}{2}\sum_{i,j=0}^{t-1} (H^k)_{ij}^{-1} \frac{\partial^2}{\partial x_i \partial x_j},$$

$$\tilde{S} = S - S_k - \frac{1}{2}\sum_{i,j=0}^{t-1} H_{ij}^k (x_i - x_i^k)(x_j - x_j^k)$$

It follows from the set of equations for $S(x,x_{t-1},\ldots,x_0)$ and also from (2.3) that at the statinary point $\vec{x}^k$ is the coordinates of the classical trajectory of the system (2.3) at the time moment $i\,(0 \leqslant i \leqslant t-1)$, which satisfies the conditions

$$x = x(x_0,t), \qquad p(x_0,0) = \frac{\partial S_0(x_0)}{\partial x_0} \qquad (2.7)$$

The number of solutions of (2.7) is equal to the number of stationary points N corresponding to different trajectories $x^k(x_0,t)$ from the initial distribution $p(x_0,0)$, arriving to the point x at a time t with various different momenta. For stochastic systems because of the local instability of trajectories, the number of the terms in the sum (2.6) N and the Jacobian $J_k(x) = (\partial x^k(x_0,t))/\partial x_0$ (see /1-3/) grow exponentially with time $N \sim \exp(ht)$ (see Fig.1), $J_k \sim \exp(ht)$, where h = KS is the entropy /28,29/. For (2.3) $h \approx \ln(kT/2)$ at $kT > 4$ /6/.

It is seen from (2.6) that the quasiclassical expansion is unvalid near the degenerated stationary points at which $J_k(x) = 0$. In this case, the wave function at the point x has the caustic (really, the caustic is on the k-th trajectory and $\partial p_k(x,t)/\partial x = \infty$). But even in the presence of q degenerated stationary points in (2.6) there are N - q quasiclassical terms in x. Both q and N grow exponentially with time and since the size of the caustic is finite ($\sim k^{-2/3}$) and they are distributed over x more or less uniformly, $q \sim k^{-2/3}N \ll N$. It is worth noting that in a vicinity of the degenerated point the amplitude peak decreases exponentially with time ($\sim k^{1/6}\exp(-ht)$). It is already clear from the fact that a generated stationary point appears in one step from some quasiclassical term of the sum (2.6), which has an exponentially small pre-exponent. Hence, $q \sim k^{-2/3}\exp(ht)$ classical trajectories with caustics will arrive at an arbitrary point x in the period $t > \tau_0 \sim \ln(k^{2/3})/h$ (see Fig.1), whereas the number of classical trajectories arriving at this point will be much larger $N - q \sim \exp(ht)$. It is seen from (2.5) and (2.6) that the main contribution to the t-dimensional integral comes from the region $\Delta V \sim k^{-t/2}$, i.e. the stationary points are well isolated. From the said above, we make the conclusion that the influence of the caustics at all times is insignificant and can be neglected (cf./11/). This result is borne out by numerical experiments (see § 3).

Assuming the quantum corrections to the main quasiclas-

sical term (m = 0) in (2.6) to be small, let us find the time dependence of the rotator energy

$$E(t) = \frac{1}{2}\int_0^{2\pi} \left|\frac{\partial\Psi(x,t)}{\partial x}\right|^2 dx .$$

The main energy contribution is given by differentiation of the action S_k in the exponent. Differentiation of the pre-exponent and the phase shift μ_k yields non-increasing-with-time correcting terms of the order of k^{-1} from the main contribution :

$$E(t) = \frac{1}{2}\int_0^{2\pi} dx \left\{\sum_{k,k_1=1}^{N} p_k(x)p_{k_1}(x)|J_kJ_{k_1}|^{-1/2}\cdot \right. \tag{2.8}$$

$$\left.\cdot\exp[i(S_k(x)-S_{k_1}(x)+\frac{\pi}{4}(\mu_k-\mu_{k_1}))]\cdot\varphi_0(x_0^k(x,t))\varphi_0(x_0^{k_1}(x,t))\right\}$$

where $p_k(x) = (\partial S_k(x))/\partial x$ is the classical momentum along the k-th trajectory arriving at the point x ; $x_0^k(x,t)$ is the initial point of the k-th trajectory as a function of the final point. Let us try to evaluate the contribution from the interference terms with $k \neq k_1$. Their number is $N_{int} \sim N^2$ and the value of each term is $A \sim J^{-1}\int_0^{2\pi} dx \exp(i(S_k(x)-S_{k_1}(x))) \sim \exp(-2ht) \sim N^{-2}$. The integral entering A is a typical correlation function exponentially time-damping because of randomness of the classical action. Hence, due to the fact that the classical problem is stochastic, we have the sum N_{int} of random quantities with amplitude $A \sim N^{-2}$, which give the value quantity $\Sigma_{int} \sim \sqrt{N_{int}}\, A \sim N^{-1}$. Moreover, at $k \neq k_1$ the integrant in (2.8) has no stationary point. Indeed, the phase stationary nature gives $\partial S_k/\partial x = p_k(x) = \partial S_{k_1}/\partial x = p_{k_1}(x)$ but since the momenta now coincide, this means that iether $k = k_1$ or x is the caustic, whose contribution, as already shown, can be neglected. Because of the absence of stationary points, the terms with $k \neq k_1$ are of the order of $O(k^{-\infty})$. The interference terms with $k \neq k_1$ in (2.8) can therefore be neglected and the sum over $k \neq k_1$ remaines which gives the classical energy value (see /22,23/).

Thus, to find the times on which the characteristics of the quantum problem coincides with the classical ones with an accuracy up to $O(k^{-1})$ in their relative magnitude, it is necessary to know on what times the terms with $m \neq 0$ in (2.6) keeps small in comparison with the main quasiclassical terms ($m = 0$). Note that at $kT \gg 1$ the matrix elements $(H^k)^{-1}$ are readily evaluated from the matrix explicit form : $(H^k)^{-1}_{i,i+m} \sim (kT)^{-m-1}$. Let us find first of all the correction $\delta_1 \sim k^{-1}$ to the term with $m = 0$ in (2.6) with an accuracy up to the terms with due redard for the action L_k only at $\exp(i\tilde{S})$, because this gives the corrections rapidly increasing with time. We represent the sum over m in the form :

$$\sum_{m=0}^{\infty} \frac{1}{m!} L_k^m[\varphi_0(x_0)\exp(i\tilde{S})]\Big|_{\vec{x}=\vec{x}^k} = \sum_{j=0}^{\infty} \delta_j^{(k)}\varphi_0 \qquad (2.9)$$

where $\delta_0=1$, $\delta_{j+1} \sim \delta_j/k$.
Calculations lead to the following expression for δ_1 (the contribution to it is given by the terms with $m = 2,3$) :

$$\delta_1^{(k)} = \frac{1}{8} \sum_{j=0}^{t-1} \frac{3\cos^2 x_j^k + 5\sin^2 x_j^k}{k \cos^3 x_j^k} + O(1/kT) \qquad (2.10)$$

Note that the singularity in the denominator arises with the presence of the caustic at the point x ($J_k(x) = 0$) but since the influence of the caustics is insignicant, upon summing over all N trajectories the contribution from these divergences is small too. Due to the randomness of the classical trajectory, the sum over j grows as $t^{1/2}$ and, therefore, on average

$$\delta_1 \sim i\chi t^{1/2}/k \qquad (2.11)$$

where the numerical coefficient is $\chi \sim 1$. It is worth mentioning that the subsequent successive terms over $(kT)^{-1}$ in the expression for δ_1 grow not more radidly $t^{1/2}$ and, hence, they

can be ignored. Analysis of the subsequent terms of the expansion over k^{-1} shows that $\delta_j \sim (t/k^2)^{j/2}$. Hence, at the times

$$t < t_0 \sim k^2 \qquad (2.12)$$

the quantum corrections are small and the characteristics of the quantum system coincide with the classical characteristics with an accuracy up to $O(k^{-1})$. It should however be noted that the quantities exponentially decreasing in the classics, for example, time-different correlator, during the time $\tau_0 \sim \ln k$ becomes of the order of $O(k^{-1})$ and the quantum corrections must be taken into account for their further calculation.

On the times $t \sim t_0$ all the corrections $\delta_j \sim 1$ and the quasiclassical approximation becomes inapplicable. Thus, beginning with the times $t > t_0$ the characteristic of the quantum problem, for instance the rotator energy, could be expected to highly differ from their classical values. Following from this, it is possible the estimation for the times t^* at which the energy diffusion observed in /10/ slows down :

$$t^* \sim t_0 \sim k^2 \qquad (2.13)$$

Unfortunately, it is difficalt to make the accurate experimental check of the functional dependence of t^* on k because of a sharp increase of the count for large k($k > 100$). The numerical results obtained are discussed in §3.

Note that in the system (2.1) the quasiclassical parameters are k, T and , for this reason, the diffusion does not improve the quasiclassic approximation. However, in many systems the quasiclassical approximation is improved with increasing the level number. It could therefore expect that at a quite rapid diffusion the quantum corrections in such systems will increase much slower than in (2.1). Let us consider as a model the system with Hamiltonian (2.1) in which k depends on time according to the law $k(t) = k_0(1+t)^{\alpha}$. As a rule, k is an increasing function of action k= k(I) and, hence of time, since with the presence of stochasticity I the number grows with time. This circumstance is reflected in the chosen model which is also convenient for numerical study. Nevertheless, it should be noted that in realistic systems with $k = k(I)$ the situation may be more complicated. In view of this, the model proposed should be regarded only as the first approximation.

At $k_0 T > 1$ the variable x becomes, already after a few kicks, random and

$$|\Delta p(t)|^2 = k_0^2 t^{2\alpha} < \sin^2 x(t) > = \frac{k_0^2 t^{2\alpha}}{2} \qquad (2.14)$$

From (2.14) we find the law of diffusion energy growth :

$$E(t) = \frac{k_0^2 t^{2\alpha+1}}{4(2\alpha+1)} + E(0) \qquad (2.15)$$

The expression for δ_1 is the same as (2.10) but with $k = k(j)$. On average, δ_1^2 grows as

$$\langle \delta_1^2 \rangle \sim \sum_{j=0}^{t-1} \frac{1}{k^2(j)} \approx \int_1^t \frac{dt}{k_0^2 t^{2\alpha}} \approx \frac{t^{1-2\alpha}}{k_0^2(1-2\alpha)} \qquad (2.16)$$

It follows from (2.16) that at $\alpha > 1/2$ the corrections always are small and the time-energy dependence is described by (2.15) on all times. The limiting value is $\alpha = 1/2$. In this case

$$\langle \delta_1^2 \rangle \sim \frac{\ln t}{k_0^2} \qquad (2.17)$$

and Eq.(2.15) holds on exponentially large times $t_0 \sim \exp(k_0^2)$. At $0 \le \alpha < 1/2$ the quasiclassical approximation is true for

$$t < t_0 \sim [k_0^2(1-2\alpha)]^{1/(1-2\alpha)} \qquad (2.18)$$

and during this period the energy grows according to the classical law (2.15).

3. Numerical experiments

In order to verify the results obtained the numerical study of the model (2.1) has been performed. Solving the Schrodinger equation, we have used the fact that the action of a kick is reduced to multiplication of the wave function by $\exp(-iV(x))$. Then, its Fourier components A_n have been found with the aid of the fast Fourier transform (FFT), for which the free rotation has resulted in a phase shift $-i(Tn^2/2)$. The wave function has been found from the obtained values A_n with the use of the FFT and so on. The number of the levels was $N_L = 4097$ (-2048,2048), but because of symmetry of the initial conditions ($\Psi(x) = \Psi(-x)$) and the Hamiltonian (2.1) 2049 levels of virtually have been taken in our computations. A high accuracy of the computation (up to 1%) was provided by coincidence of the results when changing N_L by a factor of 2 and when the population $|A_n|^2$ of the upper levels was small. The major limitation on a run was imposed by the finiteness of the selected number of levels. In addition, a test run was performed for agreement of the results (up to 10^{-7}) with those given in /10/ under the same conditions. Upon a quite large perturbation, especially at $k = k(t)$ a fast excitation of the upper levels occurs and the computation errors become significant. In typical experiments ($N_L = 1025$, $t = 300$) the computation time at BESM-6 was 10 minutes.

The initial conditions were ranging from excitation of a level (the ground state) to excitation of about 200 levels (the Gaussian packet). The noticeable dependence on the initial conditions was not observed. When processing of the computation results the rotator energy was calculated $E = 1/2 \sum_{n=-\infty}^{\infty} n^2 |A_n|^2$ and the time-energy dependences and the distribution functions f(n) were plotted in the normalized coordinates :

$$x = \frac{n^2(2\alpha+1)}{k_0^2 t^{2\alpha+1}}$$

$$f_N(n) = f(n)\sqrt{\frac{\pi k_0^2}{(2\alpha+1)}\, t^{2\alpha+1}} \qquad (3.1)$$

In case of the classical diffusion Eg.(3.1) is reduced to

$$f_N(n) = \exp(-x) \qquad (3.2)$$

In Figures this distribution is depicted by the line "a".

The results of numerical experiments for $f_N(n)$ were interpolated according to the formula (see /10/) :

$$f_N(n) = A\exp(-Bx) \qquad (3.3)$$

where A and B are the parameters termined by the least square fit. The fraction of the diffusible component was determined as

$$W_d = A\int_0^\infty e^{-Bx}\frac{dx}{\sqrt{\pi x}} = \frac{A}{\sqrt{B}} \qquad (3.4)$$

The results for matching experimental distribution over (3.3) are given in Figures by lines "b" (at the end of computation).

To verify whether the influence of the caustics and islands of stability /6/ is essential in the quantum consideration, the system was studied, which had no caustics and in which a measure of the stable components (islands) was strictly equal to zero /1/. This is the model of the same rotator but with another external perturbation over x :

$$V(x) = \begin{cases} -x^2/2 + \pi^2/8, & 0 \le x \le \pi/2 \\ \dfrac{(x-\pi)^2}{2} - \pi^2/8, & \pi/2 \le x \le \pi \end{cases} \qquad (3.5)$$

$$V(x) = V(x+2\pi), \quad V(x) = V(-x)$$

The classical dynamics of the system can be described by the mapping similar to (2.3) :

$$\overline{p} = p - k\frac{\partial V}{\partial x}, \quad \overline{x} = x + T\overline{p} \qquad (3.6)$$

At kT > 4 the motion becomes completely stochastic and the energy grows according to the diffusion law :

$$E(t) = \frac{\pi^3 k^2}{12}\, t + E(0) \qquad (3.7)$$

The study of the quantum problem (3.5) shown that the qualitative behavour of the system remained the same as the behavour of the system (2.1)(see Fig.2) - at some time moment t^* the diffusion rate degun to decrease sharply. This indicates the fact the slowing down of diffusion is not associated with the presence of the caustics and islands of stability. Note, that in contrast to (2.1), the Fourier spectrum of a kick $\exp(-ikV(x))$ decreases only in a power manner ($\sim 1/n^3$)(the FFT turns out to be indispensable) and already one kick connects all levels, whereas in (2.1) a kick covers $\sim$ 2k levels with the exponential accuracy.

For the system (2.1) the comparison was made of the more fine characteristics of the classic and quantum problems and namely : of the dependence of the diffusion coefficient D on the kT parameter on the times smaller than t^*. The results of the experiments (see Fig.3A) indicates a good agreement of the dependences D(kT) for the classic (these results were taken from /6/) and quantum cases. For the quantum problem the same oscillations D with the same period are observed as those in the classic case. An insignificant difference in values is accounted for by the fact that the initial conditions in /6/ and in the present work are, generally speaking, different. In case of the same initial conditions (in the classic - the line $p(x) = 0$, $0 \le x \le 2\pi$, and in the quantum problem - the ground state with $n = 0$, $k \approx 40$) experimental data for D(kT) are given in Fig.3B. The insignificant difference is only observed at the level of quantum corrections.

The time period t^* during which the energy grows according to the law similar to the classic one, was calculated on the basis of the experimental data. In this case, as t_a^* and t_b^*, it was taken the time period t, begining from which the energy of the quantum operator differed by 25% and 50% from the energy in the classic case (see Tables 1 and 2). In order to verify the functional dependence (2.12),(2.18) the quantity $\delta_{a,b}=[(t^*_{a,b})^{1-2\alpha}/(k_0^2(1-2\alpha))]^{1/2} \approx \text{const}$ was also calculated. The dependence $\Delta_{a,b}=(t^*_{a,b})^{1-\alpha}/k_0(1-\alpha) \approx \text{const}$ was also verified, which corresponds to the linear growth of quantum corrections ($\delta_1 \sim \int_0^t dt/k(t)$). Some cases of the motion of a quantum rotator at $k=k(t)$ are presented in Figs.4-6. The experimental results show that, according to the predictions (section 2), the time t^* increases sharply with increasing k and growing α. At $\alpha \geq 0{,}35$ the computer reserves do not already allow the slowing down of the diffusion rate to be observed, that makes difficult the experimental verification of the relation (2.18). It is worth emphasizing that at $\alpha \geq 0{,}35$ not only the diffusion law coincides with the classic law, but also the distribution function are similar to the classic Gaussian distribution.

4. The general criterion

Let's make use of V.P.Maslov's /22,23/ quasiclassic expression for the wave function, which satisfies the Schrödinger with the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\Delta + V_0(x) + \varepsilon V(x,\tau) \qquad (4.1)$$

and the initial condition $\Psi|_{\tau=0} = \varphi_0(x)\exp(iS(x)/\hbar)$ (note that φ_0 is independent of $\hbar$ and S is real) :

$$\Psi(x,\tau) = \sum_{k=1}^{N} |J_k|^{-1/2} \exp[\frac{i}{\hbar} S_k(x,\tau) - i\frac{\pi}{2}\mu_k] \cdot \cdot \left\{\sum_{m=0}^{\infty}[\hat{L}_k^m\varphi_0(x_0)]\,\Big|_{x_0=x_0^k(x,\tau)}\right\} + O(\hbar^\infty) \qquad (4.2)$$

Here, just as in (2.6) the summation is performed over all classical trajectories reaching tHe point x at the time moment τ and satisfying the initial conditions :

$$x_0(x,\tau) = x_0^k, \qquad p_0(x_0^k) = \frac{\partial S(x_0)}{\partial x_0}\Big|_{x_0=x_0^k}$$

$$J_k = \det\frac{\partial x(x_0,\tau)}{\partial x_0}\Big|_{x_0=x_0^k} \qquad (4.3)$$

(the problem can be n-dimensional), $S_k(x,\tau)$ is the action along the classic trajectory connecting the points x_0^k, x and μ_k is the Morse index of this trajectory; the operator $\hat{L}$ acts on an arbitrary function as follows

$$\hat{L}_k\varphi(x_0,\tau) = \frac{i\hbar}{2m}\int_0^\tau |J_k(x_0,\tau_1)|^{1/2}\,\Delta_x(\varphi(x_0,\tau_1)|J_k(x_0,\tau_1)|^{-1/2})d\tau_1 \qquad (4.4)$$

where $\Delta_x = \sum_{j=1}^{n}[\sum_{i=1}^{n}(\partial x_{0i}/\partial x_j)(\partial/\partial x_{0i})]^2$ is the Laplace operator in curvilinear coordinates.

Formally, the expansion (4.2) is true at the points x not being caustics /22,23/. However, as shown in section 2, although the classical trajectories with caustics ($\tau_0 \sim \sim \ln n_{ch} \sim \ln(1/\hbar)$) always arrive at x in a short time, their relative number is small and their contribution can be neglected.

The sum over m in (4.2) is an expansion in powers of $\hbar$. The term with $m=0$ gives the classical value for the average /22,23/ and the following terms with $m \neq 0$ are quantum corrections to them. Therefore, if on all times

$$\delta_m^{(k)} = [\hat{L}^m\varphi_0(x_0)]\Big|_{x_0=x_0^k(x,\tau)} \ll \varphi_0 \qquad (4.5)$$

the corrections to the averages will be always small. Thus, the question on what times the quasiclassic is applicable, reduces to the study of the time dependence of δ_m . From (4.4) and (4.5) we obtain an expression for δ_1 in the one-dimensional case :

$$\delta_1^{(k)} = \frac{i\hbar}{2m}\int_0^\tau \left\{\left[J_k^{-2}\frac{\partial^2\varphi_0}{\partial x_0^2} - 2J_k^{-3}\frac{\partial J_k}{\partial x_0}\frac{\partial\varphi_0}{\partial x_0}\right] + \left[\frac{5}{4}J_k^{-4}\left(\frac{\partial J_k}{\partial x_0}\right)^2 - \frac{1}{2}J_k^{-3}\frac{\partial^2 J_k}{\partial x_0^2}\right]\varphi_0\right\} d\tau_1 \tag{4.6}$$

Because $J_k \sim \exp(h\tau)$ and $\partial J_k/\partial x_0 \sim \exp(2h\tau)$, the first term yields the not-time-increasing correction, and its contribution can be neglected. Hence, the condition (4.5) thus reduces to

$$|\tilde{\delta}_1^{(k)}| = \left|\frac{i\hbar}{2m}\int_0^\tau \left[\frac{5}{4}J_k^{-4}\left(\frac{\partial J_k}{\partial x_0}\right)^2 - \frac{1}{2}J_k^{-3}\frac{\partial^2 J_k}{\partial x_0^2}\right] d\tau_1\right| \ll 1 \tag{4.7}$$

The verification shows (see Application A) that for the system (2.1) the correction (4.7) coincides with (2.10). It should be taken into account that the time integral in (4.2-7) should be understand as a difference of the protoplastic functions at the time moments τ and 0. Since at the intermediate time moments J_k vanishes the integral neither from the first nor from the second term in (4.7) is sign-varying (cf.(2.10)). Estimation of the higher corrections shiws that $\delta_j \sim \delta_1^j$ and the condition of their smallness is equivalent to (4.7).

Due to the stochasticity of the classical system the following estimates take place :

$$J_k \sim e^{h\tau}, \quad \frac{\partial J_k}{\partial x_0} \sim e^{2h\tau}/L, \quad \frac{\partial^2 J_k}{\partial x_0^2} \sim e^{3h\tau}/L^2 \tag{4.8}$$

where L is a certain typical size. It follows from (4.7) and (4.8) that the exponential time dependence in (4.7) decreases and, hence, the quantum corrections in SQS grow with time

not more rapidly than according to the degree law. To find the time dependence δ_1 it is convenient to proceed to the variables of the angle-action unperturbed problem in the classical Hamiltonian. In the case when the perturbation is small and the condition of "moderate" nonlinearity /2/ is fulfilled, it sufficies to confine oneselves to the expansion of the Hamiltonian near the initial I_0 up to terms $(\Delta I)^2$. At $I_0/\hbar \gg 1$ the standard quantization (see, for example, 15, 19) leads to a Hamiltonian :

$$\hat{H} = \omega\hat{I} + \gamma\hat{I}^2 + \varepsilon[V_0(I_0,\theta,\tau) + \frac{1}{2}(\hat{I}V_1(I_0,\theta,\tau) + V_1(I_0,\theta,\tau)\hat{I}) + \frac{1}{2}\hat{I}V_2(I_0,\theta,\tau)\hat{I}] \tag{4.9}$$

where $\omega = \left.\frac{dE(I)}{dI}\right|_{I=I_0}$, $2\gamma = \left.\frac{d\omega}{dI}\right|_{I=I_0}$, $V_1 = \frac{dV}{dI}$, $V_2 = \frac{d^2V}{dI^2}$,

$$\hat{I} = -i\hbar\frac{\partial}{\partial\theta}$$

Following V.P.Maslov /22,23/ we find the solution of the Schrodinger equation with the Hamiltonian (4.9) in the form $\Psi = \varphi J^{-1/2}\exp(iS/\hbar)$, where J,S is the Jacobian and the classical problem action. Substituting Ψ into the equation and taking into account the fact that J and S are the solutions of the Liouville and Hamiltonian-Jacobi equations, respectively, we get an equation for $\varphi(\theta_0,\tau)$:

$$\frac{\partial\varphi}{\partial\tau} = \hat{G}(\theta_0,\tau)\varphi$$

$$\hat{G}(\theta_0,\tau) = i\hbar\left(\gamma + \frac{\varepsilon}{2}V_2(\theta_0,\tau)\right)J^{1/2}\left(J^{-1/2}\frac{\partial}{\partial\theta_0}\right)^2 J^{-1/2} + \frac{i\varepsilon}{2}\hbar J^{-1/2}\left(\frac{\partial V_2(\theta_0,\tau)}{\partial\theta_0}\right)\frac{\partial}{\partial\theta_0}J^{-1/2} \tag{4.10}$$

with the initial condition $\varphi(\theta_0,0) = \varphi_0(\theta_0)$. Introducing the operator $\hat{F} = \int_0^{\tau} \hat{G}(\theta_0, \tau_1)d\tau_1$ similar to (4.4), it is possible represent $\varphi(\theta_0,\tau)$ as a series over F or, which is the same, over $\hbar$:

$$\varphi(\theta_0,\tau) = \sum_{m=0}^{\infty} \hat{F}^m \varphi_0(\theta_0) \tag{4.11}$$

Hence, $\Psi(\theta,\tau)$ is of the form (4.2) with substitution of x by θ , $\hat{L}$ by $\hat{F}$. The expression for δ_1 is analogous to (4.7) :

$$|\delta_1^{(k)}| = \left| i\hbar \left\{ \int_0^{\tau} (\gamma + \frac{\varepsilon}{2} V_2(\theta_0,\tau_1)) \left[\frac{5}{4} J_k^{-4} \left(\frac{\partial J_k}{\partial \theta_0}\right)^2 - \frac{1}{2} J_k^{-3} \frac{\partial^2 J_k}{\partial \theta_0^2} \right] d\tau_1 - \right.\right.$$

$$\left.\left. - \frac{\varepsilon}{4} \int_0^{\tau} J_k^{-3} \frac{\partial V_2(\theta_0,\tau_1)}{\partial \theta_0} \frac{\partial J_k}{\partial \theta_0} d\tau_1 \right\} \right| \ll 1 \tag{4.12}$$

Taking into account of the following terms of the expansion of the Hamiltonian over ΔI leads only to an insignificant change (by the quantity $\sim \varepsilon$) of γ and V_2 in (4.12). This change can be neglected. Thus, the quantum correction for the system described by the Hamiltonian $H = H_0(I) + \varepsilon V(I,\theta,\tau)$ in the classic is given by Eq.(4.12), where

$$\gamma = \frac{1}{2} \frac{d^2 H_0}{dI^2} \bigg|_{I=I_k}, \quad V_2 = d^2V/dI^2 \Big|_{I=I_k}, \quad J_k = \partial\theta^{(k)}(\theta_0,\tau)/\partial\theta_0$$

Let's consider, as an example, a system with the Hamiltonian

$$H = H_0(I) + \varepsilon V(I,\theta) g(\tau) \tag{4.13}$$

where $g(\tau)$ have the form of the kicks acting during the time T_0 and each kick is followed by the other with the interval $T (T \gg T_0)$. As we'll see below, at small ε the contribution V_2 to (4.12) can be neglected. For this reason only the term with γ is taken into account. Let's denote the change of the action during a kick by ΔI (at ωT_0, $\gamma T_0 \ll 1$, $\Delta I \sim \varepsilon V T_0$)

and the stochasticity criterion is assumed to be fulfilled $K = \gamma T \Delta I \gg 1$. To calculate δ_1 we divide the integral in (4.12) into a sum of integrals over T_0 and T :

$$|\delta_1^{(k)}| = \hbar \left| \sum_{m=0}^{t} (\xi_m^{(k)} + \eta_m^{(k)}) \right| ,$$

$$\xi_m^{(k)} = \int_{mT}^{mT+T_0} \gamma^{(k)} q_k(\tau,\theta_0) d\tau \sim \gamma_m^{(k)} T_0$$

$$\eta_m^{(k)} = \int_{mT+T_0}^{mT+T} \gamma^{(k)} q_k(\tau,\theta_0) d\tau \tag{4.14}$$

$$q_k(\tau,\theta_0) = \left[\frac{5}{4} J_k^{-4} \left(\frac{\partial J_k}{\partial \theta_0}\right)^2 - \frac{1}{2} J_k^{-3} \frac{\partial^2 J_k}{\partial \theta_0^2} \right]$$

Since

$$J(mT + T_0 + \tau) \sim J(mT + T_0)(1 + 2\gamma\Delta I\tau),$$

$$J^4(mT + T_0 + \tau) \sim J^2(mT + T_0)(1 + 2\gamma\Delta I\tau)$$

and so on (the similar relations hold for the system (2.1) with $\Delta I \sim k$), $\eta_m^{(k)} \sim \hbar/\Delta I_k(mT)$. Since the terms of the sum over m in (4.14) are stochastically independent because of the stochasticity of the classical system, δ_1 grows, on average, according to the law

$$\langle |\delta_1^{(k)}|^2 \rangle \sim \hbar^2 \sum_{m=0}^{t} \left[\frac{1}{\overline{\Delta I}_k(mT)} + \gamma_m^{(k)} T_0 \right]^2 \ll 1 \tag{4.15}$$

where $\overline{\Delta I}_k(mT) = \langle (\Delta I_k(mT))^2 \rangle^{1/2}$ is the change of the action during a kick averaged over a random phase θ . Just as in the system (2.1) the condition of quasiclassical approximation is fulfilled, when the number of the levels captured by a kick is large.

If the classical system is a flow /2/, it is impossible to reduce its dynamics to the action of kicks and then $J' \sim J^2$, $J'' \sim J^3$ and so on. In this case the integral in (4.12) can be divided into a sum of integrals over timing intervals $\Delta\tau \sim \tau_e = 1/h$ (τ_e is the inverse KS - entropy) which will be time statistically independent and, hence, we obtain :

$$|\delta_1^{(k)}|^2 \sim \hbar^2 \int (\gamma^{(k)}/h)^2 h d\tau \ll 1 \qquad (4.16)$$

At $\gamma = const$ Eq.(4.16) gives $|\delta_1| \sim \frac{\hbar\gamma}{h} (h\tau)^{1/2}$.

5. Conclusion

The main aim of the present work was to determine on what times the quasiclassic approximation for SQS is applicable. The performed study has shown that despite an exponentially fast spreading of the quasiclassic packet the quasiclassic is valid on the times $t_0 \sim n_{ch} \sim 1/\hbar$. Moreover, as a result of the classic diffusion up the levels, the growth of quantum corrections can slow down with time. Basing on V.P.Maslov's results /22,23/ the general criteriod (4.12) is obtained to determine the times on which the quasiclassic is applicable is valid for SQS .

The analysis of the system (2.1) which has bee studied analytically and numerically in /10,20/, has shown that in this system the quantum corrections grow ~~with~~ during the time $t_0 \sim k^2$. The estimate (2.12) has been obtained for the time t^*, beginning from which the diffusion rate sharply slows down /10/. Unfortunately, the avaiable experimental data do not aloows the predicted relation to be verify with enough accuracy. At the same time the numerical study has shown that on the times $t < t^*$ the quasiclassic approximation well describes even fine characteristics, such as the dependence of the diffusion coefficient on the stochasticity parameter (see Fig.3).

Numerical experiments for the ~~mode~~ rotator model with perturbation (3.5) confirm the above made (section 2) conclusion on the insignificant influence of caustics and islands of stability. In case of the system (2.1) with $k = k(t)$ the results of numerical counting substantiate the conclusion

on a sharp increase of t^* with growth of α . In this case already at $\alpha \geq .35$ t^* is not observed during the run counting. Unfortunately the qualitative testing verification of the dependence (2.18) has failed because of a sharp increase of t^* at $\alpha \to .5$. On the whole the conclusion may be made that numerical experiments confirm the predictions of the developed theory.

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Appendix A.

Let us calculate the correction $\tilde{\delta}_1$ for the system (2.1) using (4.7). According to (2.1) and (2.3) we derive the expressions for p, θ between t and t+1 kicks ($\hbar = 1$, $m = 1$) :

$$p(\tau) = p_t + k \sin \theta_t \qquad\qquad (A.1)$$
$$\theta(\tau) = \theta_t + p_t\tau + k\tau \sin \theta_t$$

The following relation is obtained from (2.3) :

$$\frac{\partial\theta_t}{\partial\theta_0} = (kT)^{t-1}\cos\theta_{t-1}\cos\theta_{t-2}\cdots\cos\theta_0 + O(1/kT) \qquad (A.2)$$

According to (A.2) we have :

$$\frac{\partial\theta(\tau)}{\partial\theta_0} = \frac{\partial\theta_t}{\partial\theta_0}\,(1 + k\tau \cos\theta_t) + O(1/kT)$$

$$\frac{\partial^2\theta(\tau)}{\partial\theta_0^2} = -\left[\frac{\partial\theta_t}{\partial\theta_0}\right]^2 k\tau \sin\theta_t + O(1/kT) \qquad (A.3)$$

$$\frac{\partial^3\theta(\tau)}{\partial\theta_0^3} = -\left[\frac{\partial\theta_t}{\partial\theta_0}\right]^3 k\tau \cos\theta_t + O(1/kT)$$

After partition of the integral (4.7) into a sum of integrals from t to t + 1 we get

$$\tilde{\delta}_1^{(k)} = \frac{1}{2}\sum_{j=0}^{t}\int_0^T \left\{\frac{5}{4}\,\frac{(k\tau)^2\sin^2 x_j^k}{(1+k\tau\cos x_j^k)^2} + \frac{1}{2}\,\frac{k\tau\cos x_j^k}{(1+k\tau\cos x_j^k)^3}\right\}d\tau + O(1/kT)$$

After a simple integration we derive the expression (2.10).

Table 1.

$$\alpha = 0, \quad k_0 = k$$

$$\langle\Delta_a\rangle = 2.4, \quad \langle(\Delta_a - \langle\Delta_a\rangle)^2\rangle^{1/2}/\langle\Delta_a\rangle = .58$$
$$\langle\Delta_b\rangle = 5.7, \quad \langle(\Delta_b - \langle\Delta_b\rangle)^2\rangle^{1/2}/\langle\Delta_b\rangle = .59$$
$$\langle\delta_a\rangle = .27, \quad \langle(\delta_a - \langle\delta_a\rangle)^2\rangle^{1/2}/\langle\delta_a\rangle = .32$$
$$\langle\delta_b\rangle = .49, \quad \langle(\delta_b - \langle\delta_b\rangle)^2\rangle^{1/2}/\langle\delta_b\rangle = .28$$

kT	k	t_a^*	t_b^*	Δ_a	Δ_b	δ_a	δ_b
5	5	5	10	1	2	.45	.63
5	10	25	35	2.5	3.5	.50	.59
5	20	30	70	1.5	3.5	.27	.42
5	22.5	35	300	1.6	13	.26	.77
5	25	50	> 300	2	12	.28	.69
5	27.5	80	200	2.9	7.3	.33	.51
5	30	80	245	2.7	8.2	.30	.52
5	35	75	> 300	2.1	8.6	.25	.49
5	37.5	225	-	6	-	.41	-

Cont-ed

5	40	120		3		.27	
5	42.5	75		1.8		.20	
5	45	165		3.7		.29	
5	50	305		6.1		.35	
5	55	320		5.8		.33	
5	60	225		3.75		.25	
5	65	170		2.6		.20	
5	70	240		3.4		.22	
5	75	260		3.5		.21	
5	80	> 215	400	2.7		.18	

Cont-ed

kT	k	t_a^*	t_b^*	Δ_a	Δ_b	δ_a	δ_b
6	40	130	> 300	3.25		.29	
7	10	5	10	.5	1	.22	.32
7	20	10	55	.5	2.75	.16	.37
7	30	40	95	1.3	3.17	.21	.32
10	10	20	30	2.0	3	.45	.55
10	20	20	50	1.0	2.5	.22	.35
10	30	35	135	1.2	4.5	.20	.39
10	80	85	> 400	1.1		.12	
11.28	11.28	10	50	.89	4.4	.28	.63
11.28	23.85	80		3.4		.38	
12.5	40	115	260	2.9	6.5	.27	.40
20	20	50	205	2.5	10.3	.35	.72
20	30	55	225	1.8	7.5	.25	.50
20	40	115	320	2.9	8.0	.27	.45
20	60	240	> 400	4.		.26	

Cont-ed

23.85	23.85	15 / 50	.63 / 2.1	.16 / .30
36.42	36.42	35	.96	.16
36.42	40	30	.75	.14

Table 2.

$$k = k(t), \quad \alpha \neq 0$$

$$\langle\Delta_a\rangle = 16, \quad \langle(\Delta_a - \langle\Delta_a\rangle)^2\rangle^{1/2} = .53\cdot\langle\Delta_a\rangle$$
$$\langle\Delta_b\rangle = 24.7, \quad \langle(\Delta_b - \langle\Delta_b\rangle)^2\rangle^{1/2} = .59\cdot\langle\Delta_b\rangle$$
$$\langle\delta_a\rangle = 1.1, \quad \langle(\delta_a - \langle\delta_a\rangle)^2\rangle^{1/2} = .31\cdot\langle\delta_a\rangle$$
$$\langle\delta_b\rangle = 1.5, \quad \langle(\delta_b - \langle\delta_b\rangle)^2\rangle^{1/2} = .33\cdot\langle\delta_b\rangle$$

k_o	T	α	t^*_a / t^*_b	Δ_a / Δ_b	δ_a / δ_b
5	1	.10	170 / 265	22,6 / 33.7	1,7 / 2.1
5	1	.15	40 / 60	5.4 / 7.6	.87 / 1.0
5	1	.20	330 / ≈ 700	25.9 / 47	1.5 / 1.8
5	1	.25	500	28	1.3
5	1	.35	>500		
7	1	.10	20 / 35	2.4 / 3.9	.53 / .66
7	1	.15	190 / 430	14.5 / 29	1.1 / 1.4
7	1	.20	400	21.6	1.1
10	1	.10	160 / 450	10.7 / 27	.85 / 1.8
10	1	.15	260	13.3	.84
10	1	.20	>600		

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Figure Captions

Fig. 1. Image of line $p(\theta, t=0)=0$, $0 \le \theta \le 2\pi$ after action of transformation (2.3) A) t = 3, B) t = 4, $\tilde{k}$ = 5, $\tilde{T}$ = 1. Phase space is reduced in square with dimension 2π .

Fig. 2A. The time dependence of rotator energy E in the case of perturbation type "saw" (3.5). k = 5, T = 1, t = 200. The straight line "a" corresponds to classical diffusion (3.7), the straight line "b" - to linear interpolation at the moment t (run end), the broken line - to experimental result.

Fig. 2B. Distribution function of system (3.5) in normalized co-ordinates $f_N(n)$ and x (see (3.1) with $k^2 \to \pi^2 k^2/3$) for values of Fig. 2A. The straight line "a" corresponds to the theoretical formula (3.2), "b" - to linear interpolation according to formula (3.3), the broken line - to experimental result.

Fig. 3A. Dependence of attitude of experimental coefficient of diffusion D_E to theoretical D_T = 1, from fractional part $\{ kT/2\pi \}$: • - for classical system (2.3); + - for quantum system (2.1) with $k \approx 40$; figures near some pointes-values of kT; initial conditions are different.

Fig. 3B. The same as in Fig. 3A, but for equal initial conditions in classical and quantum cases.

Fig. 4A. The time dependence of rotator energy E for the system (2.1) with $k = k_0(1+t)^{\alpha}$, k_0 = 5, T = 1, α = 0.2, t = 600. The smooth line corresponds to classical diffusion (2.15). The broken line - to experimental result.

Fig. 4B. Distribution function of system (2.1) with $k = k_0(1+t)^{\alpha}$ in normalized co-ordinates $f_N(n)$ and x (see (3.1)) for values of Fig. 4A. The straight line "a" corresponds to the theoretical formula (3.2), "b" - to linear interpolation according to formula (3.3) the broken line - to experimental result.

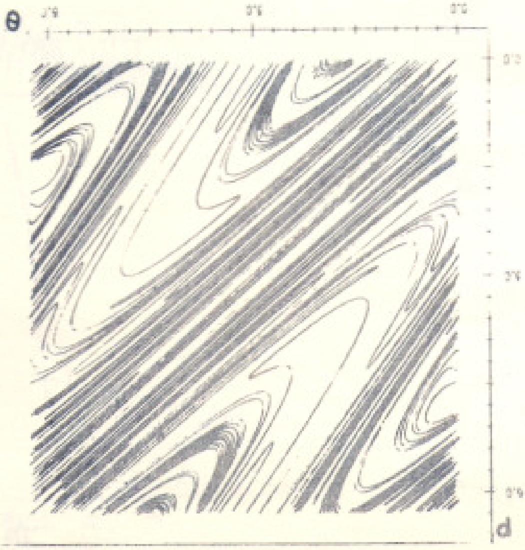
Fig. 5A. The same as in Fig. 4A for k_0 = 5, T = 1, α = 0.35, t = 500.

Fig. 5B. The same as in Fig. 4B for values from Fig. 5A.

Fig. 6A. The same as in Fig. 4A for k_0 = 5, T = 1, α = 0.5, t = 200.

Fig. 6B. The same as in Fig. 4B for values from Fig. 6A.

Fig. 1B



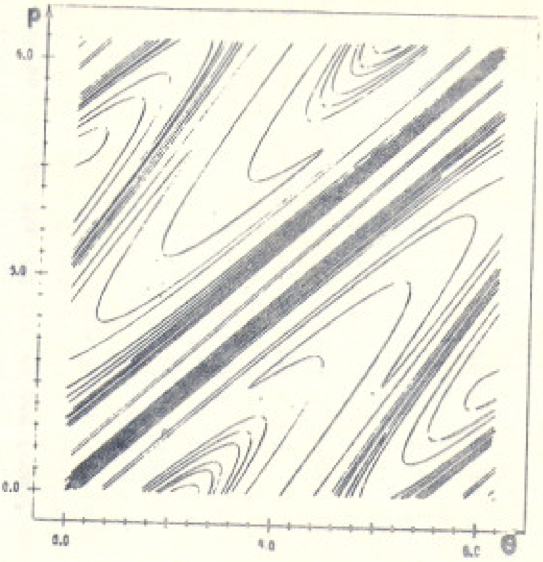


Fig. 1A

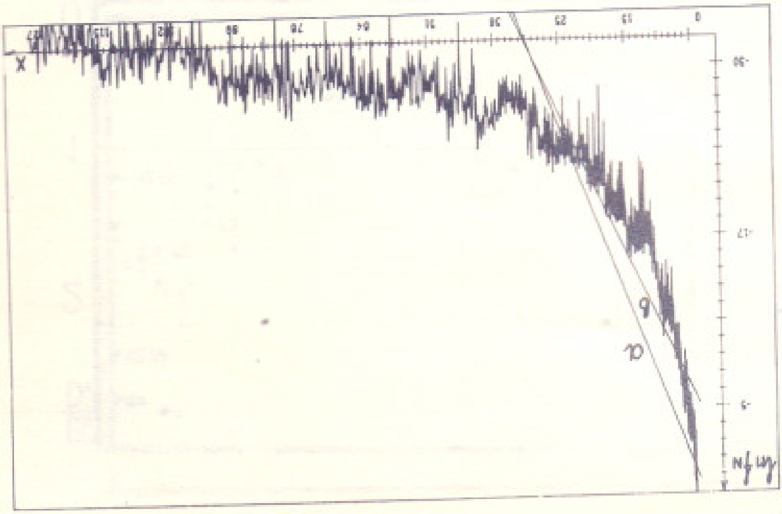


Fig. 2B

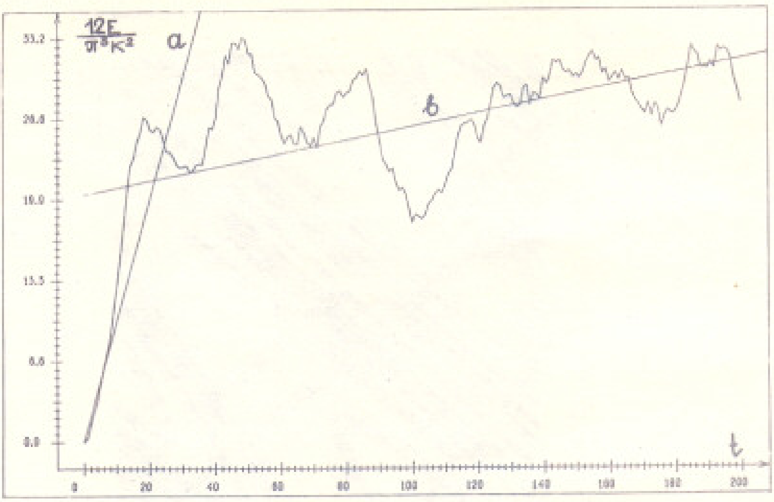
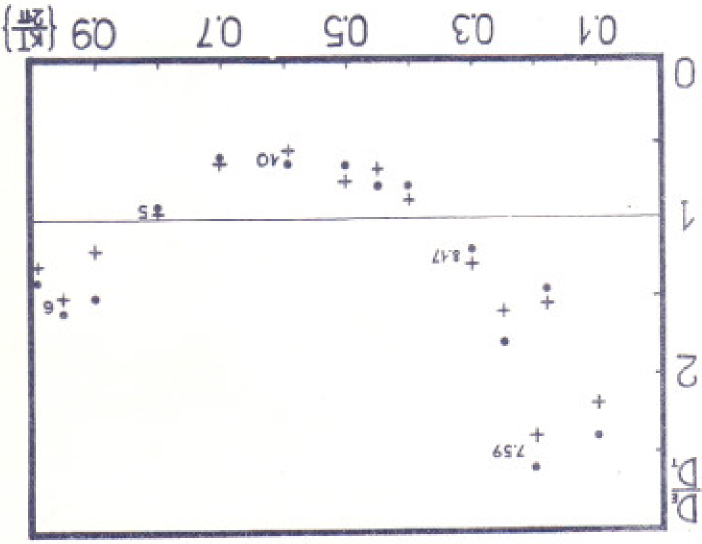
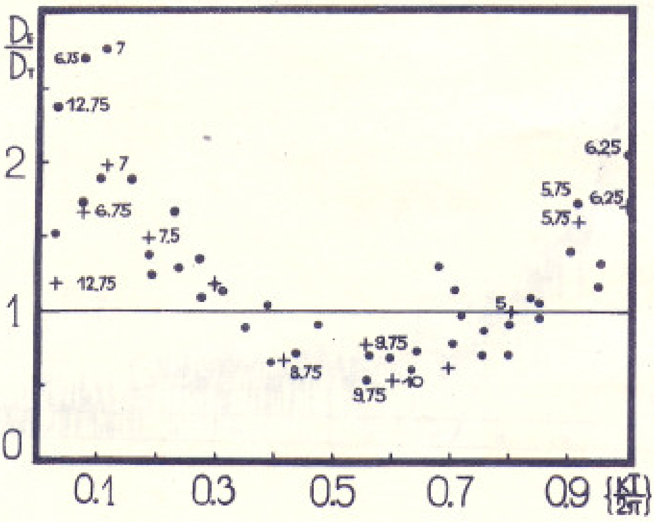


Fig. 2A



0.1
0.3
0.5
0.7
0.9
0
1
2
7.59
8.17
5
10
6

Fig. 3B



D₁/D₇
0
1
2
0.1
0.3
0.5
0.7
0.9
12.75
7
6.75
7.5
8.75
9.75
10
5
5.75
6.25

Fig. 3A

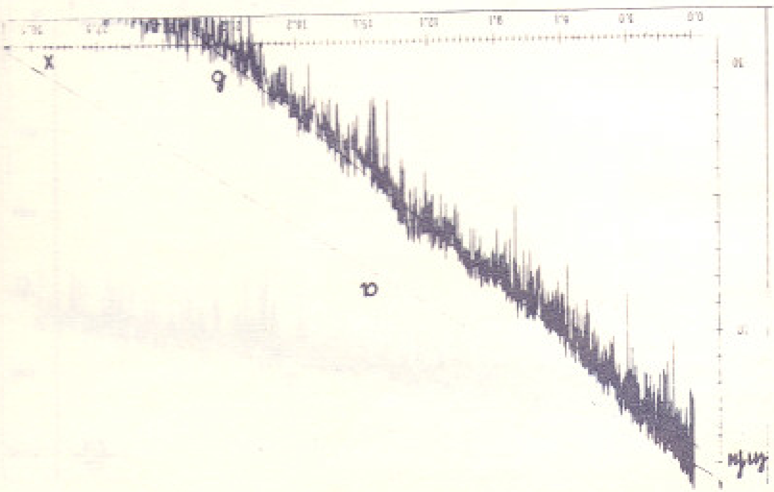


Fig. 4B

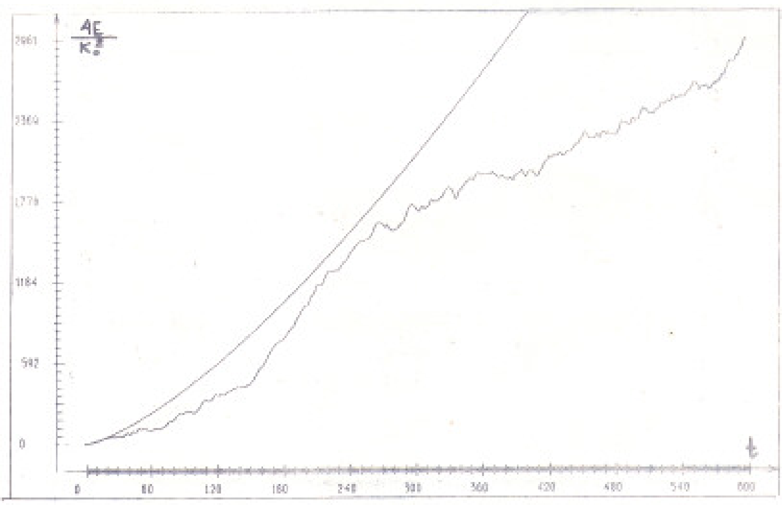
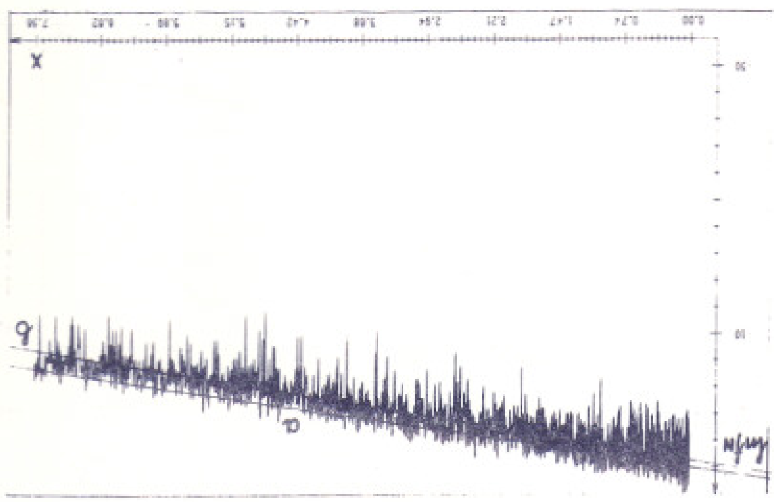

2961
2369
1776
1184
592
0
t
0
60
120
180
240
300
360
420
480
540
600

Fig. 4A

Fig. 5B



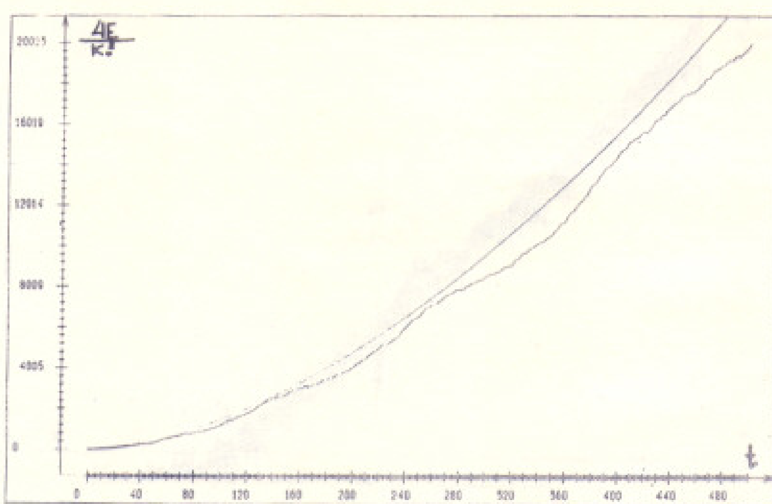
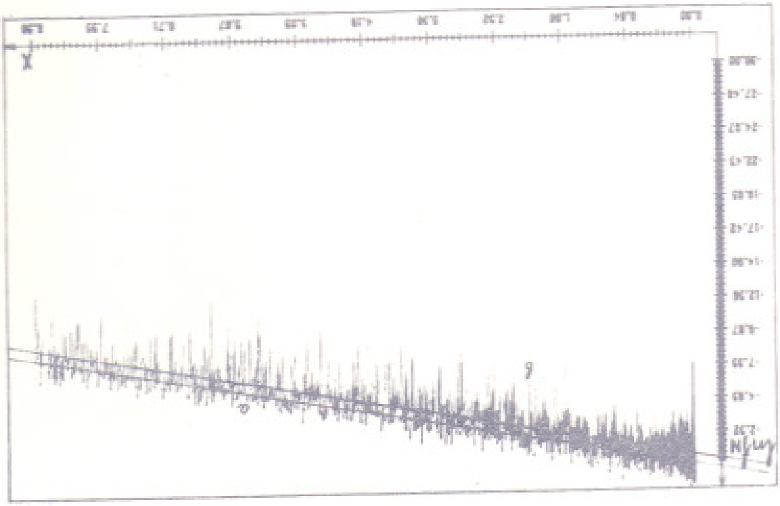


Fig. 5A

Fig. 6B



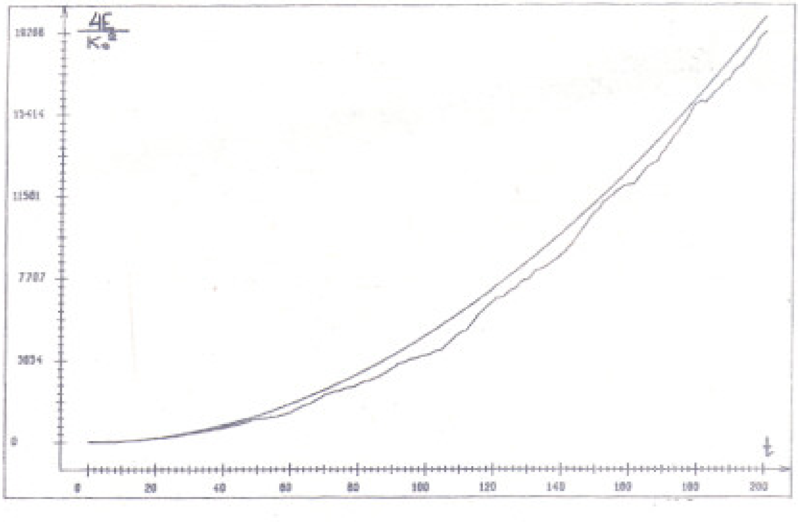


Fig. 6A