tal shift of the curve 4. However, i_0 can be computed exactly by the nonlinear theory of Ref. 7, and this yields a unique criterion for the selection of α.

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Quasiclassical approximation for stochastic quantum systems

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1. In recent years there has been an increasing interest in the dynamics of nonlinear quantum systems that are stochastic in the classical limit $\hbar = 0$ (in what follows, these systems will be called stochastic quantum systems) (Refs. 1-7). This is due mainly to the new possibilities of experimental study of the behavior of atoms and molecules in a laser-radiation field.[8] One of the methods of investigation of such systems is the quasiclassical approximation (see Refs. 1, 3, 4, and 7). It is also known that in nonlinear systems corrections to the quasiclassical approximation increase with time,[10] and after some time τ^* the quasiclassical approximation becomes inapplicable. For integrable systems, these times are proportional to some characteristic quantum number of the problem ($\tau^* \propto n_{\text{char}} \propto 1/\hbar$). This follows from the Ehrenfest theorem, and is a result of the fact that, until the packet spreads out, it moves along classical trajectories. For the stochastic quantum systems, the problem of times for which the WKB method is valid is more complicated because of the local instability of the classical trajectories. This instability leads to an exponentially fast spreading of the quasiclassical packet during the time $\tau_0 \propto \ln n_{\text{char}} \propto \ln(1/\hbar)$. In the present work we use the results of Maslov[14,15] to obtain a general condition of applicability of the quasiclassical approximation to stochastic quantum systems. For simple models we find the times for which the deviations from the classical values are small.

2. Suppose a classical system is described by the Hamiltonian $H = H_0(I) + \varepsilon V(I, \theta, t)$, where I and θ are the action and angle of the unperturbed problem, $\varepsilon \ll 1$, and the condition of moderate nonlinearity is satisfied.[11,13] In this case, in the study of the quantum corrections it is sufficient to limit oneself to terms up to $(\Delta I)^2$ in the expansion of H about the initial I_0. For $I_0/\hbar \gg 1$, the standard quantization[1,10] leads to the Hamiltonian

$$\hat{H} = \omega\hat{I} + \gamma\hat{I}^2 + \epsilon[V(I_0, \theta, t) + \tfrac{1}{2}(\hat{I}V_1(\theta, t) + V_1(\theta, t)\hat{I}) + \tfrac{1}{2}\hat{I}V_2(\theta, t)\hat{I}], \quad (1)$$

where

$$\omega = \frac{dH_0}{dI}\bigg|_{I=I_0}, \quad \gamma = \frac{1}{2}\frac{d\omega}{dI}\bigg|_{I=I_0}, \quad V_1 = \frac{dV}{dI}\bigg|_{I=I_0}, \quad V_2 = \frac{d^2V}{dI^2}\bigg|_{I=I_0}, \quad \hat{I} = -i\hbar\frac{\partial}{\partial\theta}.$$

Following Refs. 14 and 15 we obtain the quasiclassical approximation for the wave function which satisfies the Schrödinger equation with Hamiltonian (1) and initial condition $\psi(\theta, t = 0) = \varphi_0(\theta)\exp(iS(\theta)/\hbar)$:

$$\psi(\theta, t) = \sum_{k=1}^{N} |J_k|^{-1/2} \exp\left(\frac{i}{\hbar}S_k(\theta, t) - i\frac{\pi}{2}\mu_k\right) \times \left\{\sum_{m=0}^{\infty} [\hat{L}_k^m \varphi_0(\theta_0)]\big|_{\theta_0 = \theta_0^k(\theta, k)}\right\} + O(\hbar^\infty), \quad (2)$$

where the summation over k is carried out over all classical trajectories which arrive at the point θ at time t and satisfy the initial conditions

$$\theta_0(\theta, t) = \theta_0^k, \quad I_0(\theta_0) = \frac{\partial S}{\partial\theta_0}\bigg|_{\theta_0=\theta_0^k}, \quad J_k = \frac{\partial\theta(\theta_0, t)}{\partial\theta_0}\bigg|_{\theta_0=\theta_0^k},$$

where $S_k(\theta, t)$ is the action along the classical trajectory which connects θ_0^k and θ, and μ_k is the Morse index. The sum over m is essentially an expansion in powers of $\hbar$. The quantum corrections are small if

$$\hat{L}_k\varphi_0(\theta_0) = i\hbar\int_0^t \{(\gamma + \frac{\epsilon}{2}V_2)|J_k|^{1/2}(J_k^{-1}\frac{\partial}{\partial\theta_0})^2(|J_k|^{-1/2}\varphi_0) + \frac{\epsilon}{2}|J_k|^{-3/2}\frac{\partial V_2}{\partial\theta_0}\frac{\partial}{\partial\theta_0}(|J_k|^{-1/2}\varphi_0)\}dt \ll \varphi_0(\theta_0). \quad (3)$$

Since the classical system is stochastic, the Jacobian J_k and the number N of terms in the sum increase exponentially with time: $J_k, N \sim \exp(ht)$, where h is the KS entropy;[11-13] this leads to exponentially fast spreading of the initial packet. Nonetheless, a detailed analysis[9] shows that the condition for the applicability of the quasiclassical expansion (2) is (3). As usually,[14] the term with $m = 0$ gives the classical value for the averages (the contribution of the interference terms is small because of the random nature of S_k, and also because of the absence of a saddle point in these terms[14]). The successive terms with $m \neq 0$ give quantum corrections which will be small while (3) is

TABLE I

kT	$k/\hbar$	α	t^*/T	δ	$\langle\delta\rangle$	$\sigma_\delta/\langle\delta\rangle$
5–36	5–80	0	5–320	0.14–0.50	0.27	0,32
5–10	5–10	0,1–0,35	40–500	0,53–1,7	1.1	0,31

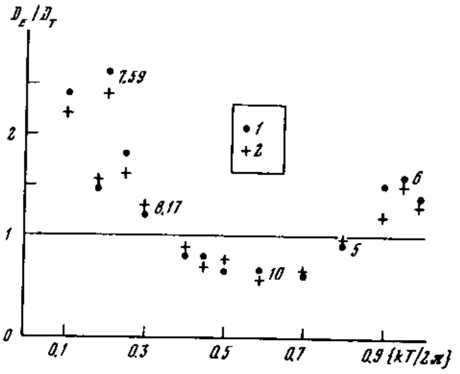

FIG. 1. Dependence of the ratio of the experimental diffusion coefficient D_E to the theoretical value $D_T = k^2/4$ on the fractional part $\left\{\frac{kT}{2\pi}\right\}$; 1) for the classical system; 2) for the quantum system with $k/\hbar \approx 40$. The numbers by some points give the values of kT.

satisfied. We note that this is not valid for the correlators which decay exponentially in the classical problem, since in the time τ_0 they become $\sim \hbar$, and so the quantum corrections, which are small in absolute value, are relatively large.

From (3) we obtain the condition for applicability of the quasiclassical approximation:

$$|\delta_1^{(k)}| = \left|i\hbar\int_0^t\left\{\left(\gamma+\frac{\epsilon}{2}V_2\right)\left[\frac{5}{4}J_k^{-4}\left(\frac{\partial J_k}{\partial\theta_0}\right)^2 - \frac{1}{2}J_k^{-3}\frac{\partial^2 J_k}{\partial\theta_0^2}\right] - \frac{\epsilon}{4}J_k^{-3}\frac{\partial V_2}{\partial\theta_0}\frac{\partial J_k}{\partial\theta_0}\right\}dt\right| \ll 1. \tag{4}$$

The integral over time in (4) should be understood in the sense of a difference of primitive functions at times t and 0, since at intermediate times J_k can be zero (corresponding to passage through a caustic), and the result of the integration does not have a definite sign. Since $J_k \sim \exp(ht)$, $\partial J_k/\partial\theta_0 \sim \exp(2ht)$, etc., δ_1 increases no faster than linearly with time, and the quasiclassical approximation is therefore valid for times $t_0 \sim 1/\hbar$.

3. Consider a system with $H = H_0(I) + \varepsilon V(I, \theta)g(t)$, where $g(t)$ has the form of impulses acting for a time T_0 and succeeding each other a time interval T ($T \gg T_0$). Suppose that the change of the action during the time of the impulse is ΔI, and the criterion of stochasticity is satisfied, i.e., $\mathcal{K} \approx \gamma T \Delta I \gg 1$.[11–13] The integral in (4) can be decomposed into a sum of integrals over the intervals $T + T_0$. Noting that the terms in the sum are statistically independent because of the stochastic nature of the classical system, and

$$J(nT+T_0+\tau) \sim J(nT+T_0)\left(1+\mathcal{K}\frac{\tau}{T}\right),\quad \frac{\partial J}{\partial\theta_0} \sim J^2(nT+T_0)\left(1+\mathcal{K}\frac{\tau}{T}\right),$$

etc., where n is the dimensionless time measured by the number of impulses, we obtain that, on average, $\delta_1^{(k)}$ increases according to the law

$$\langle|\delta_1^{(k)}|^2\rangle \sim \hbar^2\sum_{j=0}^{n}\left(\frac{1}{\overline{\Delta I_k}(j)} + \gamma^{(k)}T_0\right)^2, \tag{5}$$

where $\overline{\Delta I_k}(j) = \langle(\Delta I_k(j))^2\rangle^{1/2}$ is the change of the action during an impulse, averaged over the random phase θ. Since $\Delta I \sim \varepsilon$, the terms εV_2 in (4) can be neglected.

We now consider a rotator with $H = I^2/2 + k\cos\theta \cdot \delta_T(t)$, where k is a parameter which characterizes the magnitude of the perturbation, and $\delta_T(t) = \sum_{n=-\infty}^{\infty}\delta(t - nT)$. In the classical system for $kT > 1$, the energy of the rotator increases according to the diffusion law $E = \langle I^2/2\rangle = k^2n/4$ (see Refs. 11–13). The numerical experiments of Ref. 2 showed that, in a quantum system the energy increases diffusively with rate close to the expected rate, but only during a certain time t^*. For $t > t^*$, the rate of diffusion decreases substantially. It follows from (5) and the condition $\delta_1 \sim 1$ that the quantum corrections are small for $t \lesssim t_0$, where $t_0 \sim T(k/\hbar)^2$. Thus, it is natural to expect that for $t \gtrsim t_0$ the characteristics of the quantum problem (e.g., the energy of the rotator) will deviate appreciably from their classical values. Hence one can give an estimate of the time t^* after which the diffusion in energy begins to slow down, as observed in Ref. 2: $t^* \sim t_0 \sim T(k/\hbar)^2$.

Suppose now that k increases with time: $k(t) = k \cdot (t/T)^\alpha$. This roughly corresponds to the case $k = k(I)$. Then for $0 \leq \alpha \leq 0.5$, the quasiclassical approximation is valid for

$$t \lesssim t_0 \sim T[(k/\hbar)^2(1-2\alpha)]^{1/(1-2\alpha)} \tag{6}$$

and during this time the energy increases in the same way as in the classical system:

$$E = \frac{k^2}{4(1+2\alpha)}n^{1+2\alpha} + E(0). \tag{7}$$

For $\alpha > 0.5$, the quantum corrections are small at all times, and (7) is valid at all times.

When the classical system is a current, and when its dynamics cannot be reduced to the action of impulses, we have $\partial J/\partial\theta_0 \sim J^2$, $\partial^2 J/\partial\theta_0^2 \sim J^3$, etc. The integral in (4) can then be decomposed into a sum of integrals over the time intervals $\Delta t \sim \tau_e = 1/h$, which are now statistically independent, and we thus obtain

$$|\delta_1^{(k)}|^2 = \hbar^2\int_0^t(\gamma^{(k)}/h)^2 h\,dt \ll 1. \tag{8}$$

For $\gamma = \text{const}$, (8) gives $|\delta_1| \sim (\hbar\gamma/h)(ht)^{1/2}$. For example, if

$$H = \gamma\frac{I^2}{2} + k\sum_{m=M}^{M}\cos(\theta + m\Omega t + \varphi_m),$$

where φ_m is a set of random phases and $M \gg s = (k\gamma)^{1/2}$. $\Omega^{-1} \gg 1$, then $h \sim \Omega s^{4/3}$ (Ref. 11), and $t_0 \sim h/(\hbar^2\gamma^2)$.

4. To check the results obtained we have carried out a numerical study of the rotator model with k = const and $k(t) = k(t/T)^{\alpha}$. For the system with k = const we have compared some sensitive characteristics of the classical and quantum problems such as the dependence of the diffusion coefficient D = DE/dt on the parameter kT at times smaller than t* (for the classical system this problem was studied in Ref. 13). For identical initial conditions (in the classical system, the line $p(\theta) = 0$, $0 \leq \theta \leq 2\pi$, and in the quantum problem, the ground state with E = 0 and $k/\hbar \approx 40$) the experimental data for D(kT) are given in Fig. 1. The small difference is at the level of the quantum corrections.

From the experimental data we have determined the time during which the energy increases according to a law close to the classical law. For t* we have taken the time t after which the energy of the quantum rotator differed by 25% from the energy in the classical case. To check the functional dependence (6) we have evaluated the quantity

$$\delta = \left[\frac{(t^*/T)^{1-2\alpha}}{(k/\hbar)^2(1-2\alpha)}\right]^{1/2} = \text{const}.$$

The experimental results for the average value $\langle\delta\rangle$, the standard deviation σ_{δ}, and the ranges of variation of the parameters are given in Table I. The results of the experiments show that, in agreement with the predictions, the time t* increases sharply with increasing k/h and α.

5. The investigations carried out in the present work show that the quasiclassical approximation for stochastic quantum systems is valid at times $t_0 \propto n_{char} \propto 1/\hbar$. For concrete systems, an estimate for t_0 can be obtained from (4). It is interesting to note that in some cases (6) [k = k(t), $\alpha > 0.5$] the quasiclassical approximation is applicable at all times. Knowledge of the time t_0 over which the quasiclassical approximation is applicable can be useful in the study of the excitation of nonlinear quantum systems by an external variable field.

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