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ABOUT DYNAMICAL STOCHASTICITY IN NONLINEAR QUANTUM SYSTEMS

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A b s t r a c t

The properties of nonlinear quantum systems stochastical in classical limit are investigated. On a example of concrete model it is shown that for a quantum system in difference from corresponding classical one KS - entropy is zero and correlations decay not in exponential but in degree way.

In recent years the interest to the dynamics of nonlinear quantum systems which in classical limit are regarded as K - systems 8, 9 has substantially grown 1-7 . In present paper it is shown, on the example of simple model, that such quantum systems haven't typical properties of classical stochastic systems: different from zero KS - entropy 10-11 and exponential decay of correlations.

Let's consider a model of rotator in external field with the Hamiltonian:

$$\hat{H} = -\frac{\hbar^2}{2J}\frac{\partial^2}{\partial\theta^2} + \tilde{k}(t)\cos\theta\,\delta_{\tilde{T}}(t) \qquad (1)$$

where $\tilde{k}(t)$ is the perturbation parameter, $\delta_{\tilde{T}}(t) = \sum_{n=-\infty}^{\infty}\delta(t-n\tilde{T})$ is the periodic delta function of period ("kicks"), J is the moment of inertia of the rotator, θ is the angular variable. In the following J = 1.

The corresponding classical problem with k = const was discussed in details in 12 , where it was shown that for $\tilde{k}\tilde{T} \gg 1$, the rotator energy increases according to the diffusion law:

$$E(t) = \frac{\tilde{k}^2}{4}t + E(0) \qquad (2)$$

Here and in the following t is measured in number of "kicks". Almost for all initial conditions, except small islands of stability (in case $\tilde{k}\tilde{T} \gg 1$), the distance between initially close trajectories grows exponentially: $d = d_0\exp(ht)$, where $d = \sqrt{(\tilde{T}\Delta p)^2 + (\Delta\theta)^2}$ and $h \approx \ln(\tilde{k}\tilde{T}/2)$ (for $\tilde{k}\tilde{T} > 4$) is KS-entropy 12 .

In following we will use the Heisenberg equations for operators which after integration on period T are transformed in operator's mapping:

$$\hat{p}_{t+1} = \hat{p}_t + \tilde{k}(t)\sin\hat{\theta}_t \qquad (3)$$
$$\hat{\theta}_{t+1} = \hat{\theta}_t + \tilde{T}\,\hat{p}_{t+1}$$

where $\hat{p}_t$, $\hat{\theta}_t$ satisfy the commutation rule: $[\hat{p}_t, \hat{\theta}_t] = -i\hbar$.

When $\hbar = 0$ (3) turns to the standart map for the classical rotator 12 .

To analyze the obtained map (3) we represent $\hat{p}_t, \hat{\theta}_t$ in a normal form in respect to the initial operators $\hat{p}_0, \hat{\theta}_0$ (for example, let's all $\hat{p}_0$ to be in the right part). After this procedure one can obtain in a simple way the projection of these operators on the space of initial states. Such investigation method was used in 3 in quasiclassical approximation.

Using known relation

$$\exp(\hat{a} + \hat{b}) = \exp\left(\hat{b}\,\frac{e^c - 1}{c}\right)\exp(\hat{a}) \tag{4}$$

for operators with the commutation rule $[\hat{a}, \hat{b}] = c\hat{b}$ we obtain

$$\hat{p}_1 = \hat{p}_0 + \tilde{k}(0)\sin\hat{\theta}_0\,, \quad \hat{p}_2 = \hat{p}_1 + \Delta\hat{p}_2$$

$$\Delta\hat{p}_2 = \frac{\tilde{k}(1)}{2i}\left\{\sum_{m_0=-\infty}^{\infty} J_{m_0}(k_{1,0})\, e^{i\frac{T}{2}(m_0+1)} e^{i(m_0+1)\hat{\theta}_0} e^{i\hat{p}_0\tilde{T}} - c.c.\right\} \tag{5}$$

where $k_{1,0} = 2k(0)\sin T/2\,, \; k(0) = \tilde{k}(0)/\hbar\,, \; T = \hbar\tilde{T}$,

$J_{m_0}(k_{1,0})$ Bessel function, c.c. - complex conjugated term. From (3), (5) we get that the normal form $\Delta\hat{p}_3 = \tilde{k}(2)\sin\hat{\theta}_2$ in respect to $\hat{p}_1, \hat{\theta}_1$ can be obtained from $\Delta\hat{p}_2$ by changing of indexes $1 \to 2$, $0 \to 1$. Using (4) one can find the normal form $\Delta\hat{p}_3$ in respect to $\hat{p}_0, \hat{\theta}_0$. An arbitrary $\Delta\hat{p}_{t+1} = \tilde{k}(t)\sin\hat{\theta}_t$ is obtained from $\Delta\hat{p}_t$ by recurrence method. So if $\hat{p}_t, \hat{\theta}_t$ are presented in the normal form, then

$$\hat{p}_{t+1} = \hat{p}_t + \Delta\hat{p}_{t+1}$$

$$\hat{\theta}_{t+1} = \hat{\theta}_t + \tilde{T}\hat{p}_{t+1}$$

(6)

$$\Delta\hat{p}_{t+1} = \frac{\tilde{k}(t)}{2i}\Big\{\sum_{m_0, m_1, \ldots, m_{t-1} = -\infty}^{\infty} J_{m_0}(k_{1,t-1})\, J_{m_1}(k_{2,t-2}) \times \ldots$$

$$\ldots \times J_{m_{t-1}}(k_{t,0}) \cdot \exp(i\varphi_{m_0,\ldots,m_{t-1}}) \times \tag{6}$$

$$\times \exp(i\alpha_{m_0,\ldots,m_{t-1}}\hat{\theta}_0) \cdot \exp(i\beta_{m_0,\ldots,m_{t-1}}\tilde{T}\hat{p}_0) - c.c.\Big\}$$

where

$$\varphi_{m_0} = \frac{T}{2}(1 + m_0)\,,$$

$$\alpha_{m_0} = m_0 + 1\,, \quad \beta_{m_0} = 1$$

$$\varphi_{m_0,\ldots,m_n} = \varphi_{m_0,\ldots,m_{n-1}} + \frac{T}{2} m_n(\alpha_{m_0,\ldots,m_{n-1}} + \beta_{m_0,\ldots,m_{n-1}}) + \frac{T}{2}\alpha^2_{m_0,\ldots,m_{n-1}}\,,$$

$$\alpha_{m_0,\ldots,m_n} = \alpha_{m_0,\ldots,m_{n-1}} + m_n\,, \quad \beta_{m_0,\ldots,m_n} = \alpha_{m_0,\ldots,m_{n-1}} + \beta_{m_0,\ldots,m_{n-1}}\,,$$

$$k_{n,t} = 2k(t)\sin\left(\frac{T}{2}\beta_{m_0,\ldots,m_n}\right)$$

To investigate the obtained representation we project (6) on the basis of initial states $\Psi_n(\theta_0) = (2\pi)^{-1/2} e^{in\theta_0}$. Then (6) becomes C-number mapping (with $\hbar n$ instead $\hat{p}_0$), which may be analyzed as mapping describing dynamics of some classical system with the average values p_t, θ_t coinciding with quantum $\left(\langle p_t \rangle = \frac{1}{2\pi}\int_0^{2\pi} p_t(n, \theta_0)\, d\theta_0\right)$ one. For $\hbar \to 0$ this system turns in classical system with standart map ((3) with $\hbar = 0$). So, to understand the properties of quantum system one can study the classical system described by mapping (6) where p_0, θ_0 are C-numbers. It should be noted, that obtained mapping does not preserve the Jacobian $J = \frac{\partial(p_t, \theta_t)}{\partial(p_0, \theta_0)}$, which oscillates with time that indicates the presence of some "decay" with changing sign.

Let's analyse the case with k(t) = k = const. In classic

($\hbar = 0$) $m_0 \sim kT$, $m_1 \sim m_0 kT \sim (kT)^2$ and so on. Therefore $\alpha_{m_0,\dots,m_{t-1}} \sim (kT)^t$, $\frac{\partial \theta_t}{\partial \theta_0} \sim \alpha_{m_0,\dots,m_{t-1}} \sim (kT)^t$ and close trajectories diverge exponentially. For $\hbar \neq 0$ and $t < t_s$

$$t_s \sim \frac{\ln(\tilde{k}/\hbar)}{\ln(\tilde{k}T)} \tag{7}$$

that is if $T\beta_{m_0,\dots,m_{t_s}} \ll 1$ sine in Bessel function in (6) may be substituted for an argument and then the local instability of close trajectories takes place. For $t > t_s$ one can use the fact that in (6) $|m_n| \lesssim 2k$ (in opposite case $J_{m_n}(k)$ is exponentially small) and therefore $|\alpha_{m_0,\dots,m_{t-1}}| \lesssim 2kt$, $|\beta_{m_0,\dots,m_{t-1}}| \lesssim 2kt^2$. Then for $t > t_s$ $d/d_0 \sim (|\beta_{m_0,\dots,m_t}| + |\alpha_{m_0,\dots,m_t}|) \lesssim 2kt^2$ and entropy h decreases with time as

$$h \sim \frac{\ln k + 2\ln t}{t} \tag{8}$$

Thus the entropy is zero for the system (1) and this means that (1) is not the K-system, although for the corresponding classical system $h \approx \ln(\frac{kT}{2}) > 0$ ($kT > 4$) 12 . Note that for $k(t) = k(0)t^{\alpha}$ the entropy goes to zero in a similar way (8) (another constant before $\ln t$). And only for $k(t) \sim \exp(\mu t)$ (actually this case is not of physical interest) the entropy of quantum system is greater than zero: $h = \mu$.

Now we consider the dependence of correlations from time for quantum system (1): $R(t,n,q) = \frac{1}{2}\langle n| e^{-iq\hat\theta_0} e^{i\hat\theta_t} + e^{i\hat\theta_t} e^{-iq\hat\theta_0} |n\rangle$, where the brackets $\langle n| \dots |n\rangle$ imply the averaging over initial state $\Psi_n(\theta_0) = (2\pi)^{-1/2}\exp(in\theta_0)$. From (6) we get the expression for $R(t,n,q)$ (k = const):

$$R(t,n,q) = \frac{1}{2}\sum_{m_0,\dots,m_{t-1}} J_{m_0}(k_1) J_{m_1}(k_2) \dots J_{m_{t-1}}(k_t) \cdot \exp(i\varphi_{m_0,\dots,m_{t-1}}) \exp(i\beta_{m_0,\dots,m_{t-1}} Tn) \cdot$$
$$\cdot (1 + \exp(-i\beta_{m_0,\dots,m_{t-1}} Tq))\, \delta_{\alpha_{m_0,\dots,m_{t-1}},q} \tag{9}$$

For $t < t_s$ quantum corrections are negligible and the classical value for R may be used. Then $R(t,n,q) \sim e^{-\sigma t}$, where $\sigma \sim \frac{1}{2}\ln(kT)$ 3, 13 . For $t \sim t_s$ (7) correlations decay to the value $R(t_s,n,q) \sim k^{-1/2}$ and precise expression (9) must be used for the following calculations that is difficult problem. So in following only rough estimate and low limit for $|R(t,n,q)|$ will be obtained.

For $t \gg t_s$ almost for all m_j ($j > t_s$) $k_j \sim k$. Consider that φ and βT are distributed in the interval equipartially (then $\sum_{m_j=-\infty}^{\infty} J_{m_j}(k) e^{i\varphi_{m_j}} \approx 1$) one can obtain: $R(t,n,q) \sim R(t_s,n,q) \sim k^{-1/2}$, $t > t_s$. To find the maximal speed of decay $R(t,n,q)$ let's use the condition of unitary for operator $e^{i\hat\theta_t}$ and the condition $\langle n|\exp(-i\hat\theta_t)\exp(i\hat\theta_t)|n\rangle = 1$, from which one get $\exp(i\hat\theta_t)|n\rangle = (\sum_{m=-\infty}^{\infty} A_m^{(n)} e^{im\theta_0})|n\rangle$ and $\sum_{m=-\infty}^{\infty}|A_m^{(n)}|^2 = 1$. But from (6) it follows (due to $\alpha_{m_0,\dots,m_{t-1}} \sim kt$), that the sum by m has $m_{max} \sim kt$ terms ($A_m^{(n)}$ with $|m| > m_{max}$ are exponentially small). Assuming that all A_m with $|m| < m_{max}$ are of one order (on the contrary there are q with $|q| < m_{max}$ for which the correlations will decay slower than (10)) and using precise equation $R(t,n,q) = \frac{1}{2}(A_q^{(n)} + A_q^{(n-q)})$ one gets for $t \gg t_s$

$$|R(t,n,q)| \gtrsim \frac{1}{\sqrt{kt}} \tag{10}$$

In the case $k(t) = kt^{\alpha}$ we must change $t \to t^{1+\alpha}/(1+\alpha)$.

Although after $t \sim t_s$ the classical value of correlations will differ from quantum one (the comparison is made in quasiclassical region), the absolute value of correlations will be small $R \sim \sqrt{\hbar/k}$. Therefore in quantum case the characteristics, which don't decrease in the classical system (for example, the energy of rotator $E = \langle n | \frac{1}{2} p_t^2 | n \rangle$), will differ from their classical sizes only on small value $\sim (\hbar/\tilde{k})^{1/2}$ during time $t_0 \ll 1/\hbar$ (the number of correlations grows with time according to the degree law). The more precise estimate for t_0 7 gives ($\alpha \leq 1/2$)

$$t_0 \sim [(\tilde{k}/\hbar)^2 (1-2\alpha)]^{1/(1-2\alpha)} \quad (11)$$

For $\alpha > 1/2$ the quantum corrections are small during all the time.

It's interesting to note that degree decay of correlations takes place due to degree increasing of number of harmonics θ in U with time (U is the operator of evolution (1) or, to say in other words, due to degree grow of number of excited nonperturbed levels (one kick occupies $\approx 2k$ levels). In view of this fact the number of harmonics of θ in $\hat{p}_t = U^+(-i\hbar \frac{\partial}{\partial \theta}) U$ also grows in degree way that leades to $h = 0$ and not exponential decay of correlations. Because this property of U takes place really for all types of perturbations it's naturally to expect that and other quantum systems, stochastical in classical limit will have zero KS-entropy and degree decay of correlations. This result showes that the generalization of conception of Kolmogorov's entropy for quantum systems 14, 15 apparently has no interest for wide class of physical systems.

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