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THE MODULATION DIFFUSION IN NONLINEAR OSCILLATOR SYSTEMS

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NONLINEAR OSCILLATOR SYSTEMS

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A b s t r a c t

A fairly general phenomenon of the modulation diffusion in a many-dimensional nonlinear oscillator system is demonstrated numerically and treated analytically using a simple model of two weakly coupled nonlinear oscillators one of which is driven by a quasiperiodic frequency modulated perturbation. The modulation splits up the driving resonance into a multiplet which forms, under appropriate conditions, a narrow stochastic layer. A far-reaching diffusion is spreading along this layer due to the coupling between oscillators.

The modulation diffusion is similar in mechanism to the Arnold diffusion along the stochastic layer of a single nonlinear resonance. Both are comparable in rate and expected to be dangerous for the motion stability of heavy particles in colliding beam facilities.

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1. Introduction

As is well established by now the behaviour of classical (non-quantal) dynamical systems may exhibit, under appropriate conditions, a broad variety of statistical properties up to the true randomness (see, e.g. Refs. /1-5/). In the latter limiting case the motion admits a simple statistical description, and is essentially the relaxation to the microcanonical distribution, for a closed (conservative) system, or some unbounded diffusion in phase space for the Hamiltonian system under an external regular perturbation. This sort of intrinsic stochasticity may be contrasted with the opposite limiting case of dynamical evolution - the quasiperiodic (regular) motion - whose structure is also fairly simple.

In practical applications, however, one encounters, as a rule, a much more complicated intermediate situation of the so-called divided phase space /5/ which is generally a highly intricate mosaic of regions with both the regular as well as stochastic motions. A striking example is the so-called Arnold diffusion which propagates over an everywhere dense set of very narrow stochastic layers of nonlinear resonances (Fig. 1). This peculiar example of a subtle Hamiltonian dynamics was already discussed at one of the previous International Conference on Nonlinear Oscillations /6/ (see also Refs. /5,10,11/.

Here we are going to discuss another diffusion mechanism in a many-dimensional oscillator system /7,8/ which has been called the modulation diffusion, and which is somewhat similar to the Arnold diffusion. Particularly, both require more than two degrees of freedom from topological arguments. The principal difference between the two mechanisms lies in the nature of stochastic layers supporting the diffusion. In case of Arnold diffusion the layers are formed on place of the unperturbed separatrices of nonlinear resonances (Fig. 1) and, typically, are exponentially narrow but inherent, or universal in that they persist under arbitrarily weak perturbation /5,9/.

The modulation stochastic layer, on the other hand, is the result of resonance overlap within a multiplet produced by any low frequency modulation in the system, either internal or external. The layer width depends in this case not so much on the perturbation but rather on the modulation factor and is typically much broader as compared to a separatrix stochastic layer (Fig. 1). On the other hand, a finite perturbation is now required to provide the overlap of multiplet resonances. The rate of diffusion within a layer turns out to be comparable in both cases, yet due to a substantially bigger width of modulation stochastic layers the latter seem to be much more dangerous.

Both the modulation as well as Arnold diffusion may play an important role in some applications to the dynamical systems with negligible dissipation. An important example, which is actually the main motivation for the present work, is the problem of particle motion stability in a storage ring of the proton-antiproton colliding beam facility, a huge project which is sufficiently expensive to justify extensive studies of even such peculiar and subtle effects as the Arnold and modulation diffusion.

2. Motion spectrum in a modulation stochastic layer

The spectral properties of motion in a stochastic layer is the central problem in evaluation of the diffusion rate as we shall see below. For a sufficiently narrow separatrix stochastic layer ($\Delta\omega_s \ll \Delta\omega$, see Fig. 1) this problem turns out to be suprisingly simple since the motion within such a layer is close to that along the unperturbed separatrix. Yet, for a modulation stochastic layer that simplification is no longer the case, and we are confronted with an interesting problem in nonlinear dynamics.

We have studied into this problem /8/ via numerical simulation on a simple model specified by the Hamiltonian:

$$H(\varphi, p, t) = \frac{p^2}{2} + k \cdot \cos(\varphi + \lambda \cdot \cos \omega_M t) \qquad (2.1)$$

Here ω_M is the modulation frequency while modulation factor λ determines effective number of lines in the multiplet.

Under condition (see Refs. /5,12/):

$$K_M \approx 2.5\, s^2 \sim 23 \frac{k}{\sqrt{\lambda}\,\omega_M^2} > 1 \qquad (2.2)$$

where s is the resonance overlap parameter, a solid stochastic component, or modulation stochastic layer, of width $2\Delta\omega \approx 2\lambda\omega_M$ $(s \lesssim \lambda)$ is formed within the multiplet. For diffusion in the layer to be fast one more condition has to be met /12/

$$V = \frac{|\dot{\omega}_r|}{\Omega_\phi^2} = \frac{\lambda\omega_M^2}{k} > 1 \qquad (2.3)$$

The physical meaning of this condition relates to a different representation of the perturbation in Eq. (2.1), namely, we may consider not a multiplet of stationary resonances $\dot{\varphi} = p = m\,\omega_M$ (m integer) but rather a single slowly moving resonance $\dot{\varphi} = \omega_r(t) = \lambda\omega_M \cdot \mathrm{Sin}\, \omega_M t$ with phase oscillation frequency $\Omega_\phi = \sqrt{k}$. Then multiple crossing of this resonance takes place, and for diffusion to be fast the crossing needs to be fast also which depends on dimensionless speed of crossing V. In case of the slow crossing ($V < 1$) the diffusion rate drops rapidly with V, and the time interval t_f required for trajectory to reach across the layer (the layer fill-up time) becomes too large. Normally ($V > 1$), $t_f \sim (\Delta\omega)^3/k^2$ /12/.

We consider now an auxiliary dynamical variably z representing another degree of freedom and obeying the evolution equation

$$\dot{z} = \varepsilon \cdot \mathrm{Sin}(\varphi - \omega t) \qquad (2.4)$$

where $\varphi(t)$ is determined by the motion of system (2.1); ε is a perturbation parameter and ω the frequency detune between the two degrees of freedom. For a random $\varphi(t)$, due to stochastic motion in the layer, the variable $z(t)$ describes also a random process with the diffusion rate

$$D(\omega) \equiv \lim_{t\to\infty} \frac{\overline{(\Delta z)_t^2}}{t}; \quad (\Delta z)_t = \varepsilon \int_0^t \mathrm{Sin}(\varphi(t') - \omega t')\,dt' \qquad (2.5)$$

Note that $D(\omega)$ is proportional to a Fourier component of the correlation function for Sin $\varphi(t)$.

A numerical example of the dependence $D(\omega)$ within a fairly big range of almost $30\,(!)$ orders of magnitude is shown in Fig. 2 in dimensionless variables $\omega/\Delta\omega$ and $D_R = D\cdot\Delta\omega/\varepsilon^2$. There are two qualitatively different diffusion regions:

i) a resonance "plateau" ($|\omega| \lesssim \Delta\omega$) with the highest and roughly constant diffusion rate due to resonance at $\dot{\varphi} = \omega$ ($|\dot{\varphi}| \lesssim \lambda\omega_M$ within the layer);

ii) the "non-resonant" region ($|\omega| \gtrsim \Delta\omega$) of exponentially slow diffusion. In the latter case the exact first order resonance $\dot{\varphi} = \omega$ is never reached, and the diffusion is caused here by second order resonances due to the motion high frequency "tail" in modulation stochastic layer. In many applications that slow diffusion under a high frequency perturbation can be completely neglected, as follows also from the simple averaging method (see, e.g.,/13/). Fig. 2 shows that the accuracy of this method is fairly high. However, we are concerned here just with those exceptional cases when even such tiny effects may be of importance.

The diffusion rate in question can be roughly described by the expressions /8/:

$$D_R \approx \begin{cases} \frac{\pi}{2}; & |\omega| < \Delta\omega \approx \lambda\omega_M \\ \frac{\ln s}{2\lambda}\cdot e^{-\alpha\left(\left|\frac{\omega}{\Delta\omega}\right| - 1\right)}; & |\omega| > \Delta\omega \end{cases} \tag{2.6}$$

The constant rate ($\pi/2$) on the plateau is readily obtained from the normalization condition /8/

$$\int_{-\infty}^{\infty} D_R\left(\frac{\omega}{\Delta\omega}\right)\frac{d\omega}{\Delta\omega} = \pi$$

assuming that main contribution to D_R comes from the resonance domain $\dot{\varphi} \approx \omega$. A more interesting second expression in Eq. (2.6), shown in Fig. 2 by straight line, is semiempirical. The average numerical value of the factor in exponent $\langle\alpha\rangle = 6.21 \pm 0.17 \approx 2\pi$. Note, that the exponent in Eq. (2.6) is close to that for the Arnold diffusion if in the latter case one understands $2\Delta\omega$ as the width of the resonance in frequency (see Fig. 1) that is the width of frequency band occupied by the resonance.

The coefficient $(\ln s/2\lambda)$ in Eq. (2.6) has been obtained from the assumption that the rate γ of correlation decay $R(\tau)$ is asymptotically proportional to the KS-entropy h in modulation stochastic layer

$$R(\tau) \underset{\tau\to\infty}{\longrightarrow} e^{-\gamma\tau}; \quad \gamma = c\cdot h \approx c\cdot\frac{\omega_M}{\pi}\ln s \tag{2.7}$$

The last expression for h was well confirmed by numerical experiments, and numerical value of the factor $c \approx 1/2$ (see Ref. /8/ for details).

As seen in Fig. 2 the estimate (2.6) does agree with numerical data in order of magnitude, although one may notice that, besides big fluctuations, the true dependence $D(\omega)$ is actually more complicated (see also Fig. 4 in Ref. /8/).

Since the dynamics of phase $\varphi(t)$ is very complicated and largely unknown (especially at the layer edge, see Ref. /9/) it is very important to observe the diffusion in z on a time-scale much in excess of the layer fill-up time t_f (see above, is section). For data in Fig. 2 the ratio $t_m/t_f \sim 300$.

3. Numerical techniques

The main technical difficulty encountered in numerical experiments was related to a very low diffusion rate to be computed (see Fig. 2). To suppress the "background" (mainly due to big oscillations of frequency $\sim\omega$) we applied a special averaging of the diffusing quantity $v(t)$ ($= z(t)$ for Eq.(2.5)).

$$v(t) \to \overline{v_i} = \int_{t_i-\frac{T}{2}}^{t_i+\frac{T}{2}} v(t)\cdot g\left(\frac{t-t_i}{T/2}\right)\frac{dt}{T/2} \tag{3.1}$$

Here T is the averaging period (typically 0.01 through 0.1 of the total motion time t_m), and $g(\tau)$ ($|\tau|\leq 1$) a normalized weighting function. In the simplest case of $g(\tau) = 1/2$ the numerical techniques for computing a low diffusion rate was described in detail in Refs. /5,11/. Since the suppression factor for an oscillation in $v(t)$ at frequency ω , that is a relative decrease in amplitude of this oscillation as a result of averaging, is proportional to the Fourier component of $g(\tau)$, a smoother $g(\tau)$ seems to be more preferable. A convinient type of $g(\tau)$, used in the present work, is as follows

$$g(\tau)=\frac{(2n+1)!!}{2^{n+1}\,n!}(1-\tau^2)^n \approx \sqrt{\frac{2n+3}{2\pi e}}\cdot\left(\frac{2n+3}{2n+2}\right)^{n+\frac{1}{2}}\cdot(1-\tau^2)^n \quad (3.2)$$

resulting in the suppression factor

$$S(\omega)=2^{2n+1}\Gamma(n+\tfrac{3}{2})\frac{J_{n+\frac{1}{2}}(\frac{\omega T}{2})}{(\omega T)^{n+\frac{1}{2}}} \rightarrow \sqrt{\frac{2}{e}}\left(\frac{4n+6}{e\omega T}\right)^{n+1}\cdot \sin\left(\frac{\omega T-\pi n}{2}\right) \quad (3.3)$$

Here Γ, J_ν are gamma and Bessel functions, respectively; e = 2.71 ..., and the last expression in Eq. (3.3) gives asymptotic behaviour of S for $\omega T \gtrsim n$ while in the opposite limit $(\omega T \lesssim 4\sqrt{n})$ factor $S \approx 1$. The latter ensures that averaging does not distort the true diffusion which is determined by a low frequency band $|\omega| \lesssim 1/t_m \ll 4\sqrt{n}/T$. Curiously, the approximate expressions in Eqs. (3.2) and (3.3) hold to the accuracy of a few per cent even for n = 0.

Thus, a 'smooth' averaging does greatly improve suppression of the background although it increases, at the same time, the boundary frequency of efficient suppression ($\omega T \gtrsim n$). The numerical data in Fig. 2 have been obtained using weighting function (3.2) with n = 6. Residual background in this case (3 lowest points in Fig. 2) is determined by round-off errors (56 bit mantissa).

Let us mention also that well above the background the measured diffusion rate is only weakly dependent (within a factor of 2) on either averaging period or the total motion time. We keep using that check-up to be sure of a diffusive nature of the motion.

Another peculiarity of our numerical techniques is a special procedure for numerical integration of motion equations /6,11/ which may be called a Hamiltonian, or canonical procedure since, being approximate to the Hamiltonian equations, it <u>exactly</u> conserves, nevertheless, the phase density, and decrease, in this way, the accumulation of numerical errors.

4. A simple example of modulation diffusion

For initial studies of the modulation diffusion we have chosen a simple model described by the Hamiltonian

$$H=\left(\frac{p_1^2}{2}+\frac{x_1^4}{4}\right)+\left(\frac{p_2^2}{2}+\frac{x_2^4}{4}\right)-\mu x_1 x_2-\varepsilon x_1 \cos(\Omega t+\lambda \sin \omega_M t) \qquad (4.1)$$

The model represents two nonlinear oscillators coupled by a small linear (in force) perturbation with parameter $\mu \ll a_i^2$ (a_i the oscillation amplitudes, i = 1,2) and driven by a frequency modulated perturbation with the mean frequency Ω and a small parameter $\varepsilon \ll a_1^3$. Without modulation (λ = 0) the dynamics of a close model was studied in some detail in Refs. /6,5,11/, mainly, in respect to Arnold diffusion in the stochastic layer of coupling resonance $\omega_1 = \omega_2$ ($\omega_i = \beta a_i$ ($\beta \approx$ 0.85) are the unperturbed frequencies ($\mu = \varepsilon = 0$)). Here we consider a different problem, namely, diffusion in a modulation stochastic layer of driving resonance $\omega_1 = \Omega$.

The unperturbed motion is described by (see, e.g., Ref./5/):

$$\frac{x_i(t)}{a_i} \approx \sum_{q=0}^{\infty} \frac{\cos[(2q+1)\omega_i t]}{23^q} \approx a_i \cos\theta_i \qquad (4.2)$$

The driving perturbation (ε) results in formation of a modulation stochastic layer with a short fill-up time provided (comp. Eqs. (2.2) and (2.3)):

$$K_M \approx 11\beta^2 \frac{\varepsilon}{\sqrt{\lambda}\, a_1 \omega_M^2} > 1$$

$$V \approx \frac{2}{\beta^2} \cdot \frac{\lambda a_1 \omega_M^2}{\varepsilon} \approx 22 \frac{\sqrt{\lambda}}{K_M} > 1 \qquad (4.3)$$

The coupling term (μ) leads then to a diffusion along the driving resonance $\omega_1 = \Omega$.

Evaluation of this diffusion rate in a_2 (or ω_2) can be performed (see Ref. /8/) using the equation

$$\dot{\omega}_2 = \dot{I}_2 \frac{d\omega_2}{dI_2} \approx \frac{\beta^2}{2}\mu \frac{a_1}{a_2} \, Sin\,(\theta_1 - \omega_2 t) \tag{4.4}$$

where we have set, approximately, $\theta_2 \approx \omega_2 t$, taken only the first term in series (4.2) for X_2 (see below) and neglected the nonresonant term $Sin\,(\theta_1 + \omega_2 t)$. Now, if we put $\theta_1 \approx \omega_1 t + \varphi$ the latter Eq. (4.4) becomes of the form of Eq. (2.4) with the detune $\omega = \omega_2 - \omega_1$ while $\varphi(t)$ is determined by the motion in modulation stochastic layer around resonance $\omega_1 = \Omega$. Provided μ is sufficiently small ($\mu \ll \varepsilon / a_2$) the latter motion is nearly independent from the diffusion in ω_2, and we apply the results of Section 2 (see Eq. (2.6)) to arrive at

$$D^{(1)}_{\omega_2} \approx \begin{cases} \frac{1}{5} \cdot \frac{\mu^2}{\lambda \omega_M} \cdot \left(\frac{\omega_1}{\omega_2}\right)^2; & |\omega_1 - \omega_2| < \lambda \omega_M \\ \frac{\mu^2}{16} \cdot \frac{\ln S}{\lambda^2 \omega_M} \cdot \left(\frac{\omega_1}{\omega_2}\right)^2 \cdot e^{-2\pi \left[\frac{\omega_1}{\lambda \omega_M}\left(1 - \frac{\omega_2}{\omega_1}\right) - 1\right]}; & |\omega_1 - \omega_2| > \lambda \omega_M \end{cases} \tag{4.5}$$

where the superscript indicates that only the first term in Eq. (4.2) is kept.

The diffusion along resonance $\omega_1 = \Omega$ driven by the coupling can be regarded also as the "stochasticity pumping" from one degree of freedom into another. That graphic picture of the diffusion in a stochastic layer has been developed in Ref. /10/, and had been mentioned briefly in Ref. /14/. The so-called "thick layer diffusion" studied in Ref. /10/ is similar in mechanism to the modulation diffusion on the plateau (first expression in Eq. (4.5)).

An example of the dependence $D_{\omega_2}(\omega_1 / \omega_2)$ as revealed by a series of preliminary numerical experiments is shown

in Fig. 3 (circles). The first 3 numerical points do resemble the exponential dependence in Eq. (4.5). Yet, for a larger ω_1 / ω_2 the diffusion rate behaves in a much more complicated way. We have guessed that it can be explained by the influence of higher coupling resonances ($\omega_1 = m\,\omega_2$; $m = 2q + 1$) corresponding to harmonics of the unperturbed motion (4.2) even though they appear, at the first glance, negligible.

The diffusion rate caused by m-th harmonic is readily obtained from Eqs. (4.2) and (4.5)

$$D_{\omega_2}^{(m)}\left(\frac{\omega_1}{\omega_2}\right) = D_{\omega_2}^{(1)}\left(\frac{\omega_1}{m\omega_2}\right) \cdot \frac{m^4}{23^{m-1}} \tag{4.6}$$

Assuming the effect of different coupling resonances to be independent and summing up $D_{\omega_2}^{(m)}$ we arrive at the theoretical dependence $D_{\omega_2}(\omega_1/\omega_2)$ plotted in Fig. 3 by solid line. The first 3 resonances are clearly seen with their plateaus and exponential "tails". Accordance between theory and numerical data remains fairly good up to $\omega_1/\omega_2 \approx 9$, and still persists, in order of magnitude, even up to $\omega_1/\omega_2 \approx 15$. Here, apparently, the approximation $x_2 \approx a_2 \cos \omega_2 t$, used above, does no longer hold since

$$\frac{\mu a_1 a_2}{a_2^4/4} = \frac{4\mu}{a_1^2}\left(\frac{\omega_1}{\omega_2}\right)^3 \approx 0.3$$

The nature of a would-be plateau at $(\omega_1/\omega_2) > 15$ remains thus far unknown, and requires further studies. Let us mention, however, that in this region the measured diffusion rate off the layer ($\omega_1 - \Omega \approx 3\lambda\omega_M$) drops by, at least, 4 orders of magnitude. It would mean that the background level is much lower as compared to the diffusion rate even at $(\omega_1/\omega_2) > 15$. For a typical $t_m = 10^6$ the ratio $t_m/t_f \sim 30$ is fairly big (see Section 2), and over the whole range $1 \leqslant (\omega_1/\omega_2) \leqslant 21$ the diffusion rate is nearly independent (within a factor of 2) on either T or t_m. On the other hand, the change of ω_1/ω_2 due to diffusion is less than 0.05 for $t_m = 10^6$, so Fig. 3 represents a truely local dependence $D(\omega_1/\omega_2)$.

In conclusion we would like to emphasize importance of the modulation diffusion for the dynamics of many-dimensional Hamiltonian systems.

R e f e r e n c e s

1. V.I.Arnold and A.Avez, Ergodic Problems of Classical Mechanics (Benjamin, 1968).
2. I.P.Kornfeld, Ya.G.Sinai, S.V.Fomin, Ergodic Theory (Moskva, Nauka, 1980) (in Russian).
3. V.M.Alekseev and M.V.Jakobson, Symbolic Dynamics and Hyperbolic Dynamic Systems, Physics Reports (to appear).
4. J.Ford, A Picture Book of Stochasticity, A.I.P. Conf. Proc. No 46, 1978, p. 121.
5. B.V.Chirikov, Physics Reports 52 (1979) 263.
6. G.V.Gadiyak, F.M.Izrailev, B.V.Chirikov, Proc. 7th Intern. Conf. on Nonlinear Oscillations (Berlin, 1975) Vol. II, 1, p. 315.
7. B.V.Chirikov, Adiabatic Invariants and Stochasticity in Magnetic Confinement Systems, Proc. Int. Conf. on Plasma Physics (Nagoya, 1980), Vol. II, p. 176.
8. B.V.Chirikov, F.M.Izrailev, D.L.Shepelyansky, Dynamical Stochasticity in Classical Mechanics, Preprint 80-209, Institute of Nuclear Physics, Novosibirsk, 1980.
9. B.V.Chirikov, D.L.Shepelyansky, Statistics of Poincaré Recurrences, and the Structure of Stochastic Layer of Nonlinear Resonance, Report at this conference.
10. J.L.Tennyson, M.A.Lieberman, A.J.Lichtenberg, in Nonlinear Dynamics and the Beam-Beam Interaction, A.I.P. Conf. Proc. No 57, 1979, p. 272.
11. B.V.Chirikov, J. Ford, F.Vivaldi, ibid, p. 323.
12. B.V.Chirikov, D.L.Shepelyansky, Diffusion due to Multiple Crossing of a Nonlinear Resonance, Preprint 80-211, Institute of Nuclear Physics, Novosibirsk, 1980.
13. N.N.Bogolyubov, Yu.A.Mitropolsky, Asymptotic Methods in the Theory of Nonlinear Oscillations (Dehli, 1961).
14. C.Froeschle, Astroph. and Space Sci. 14 (1971) 110.

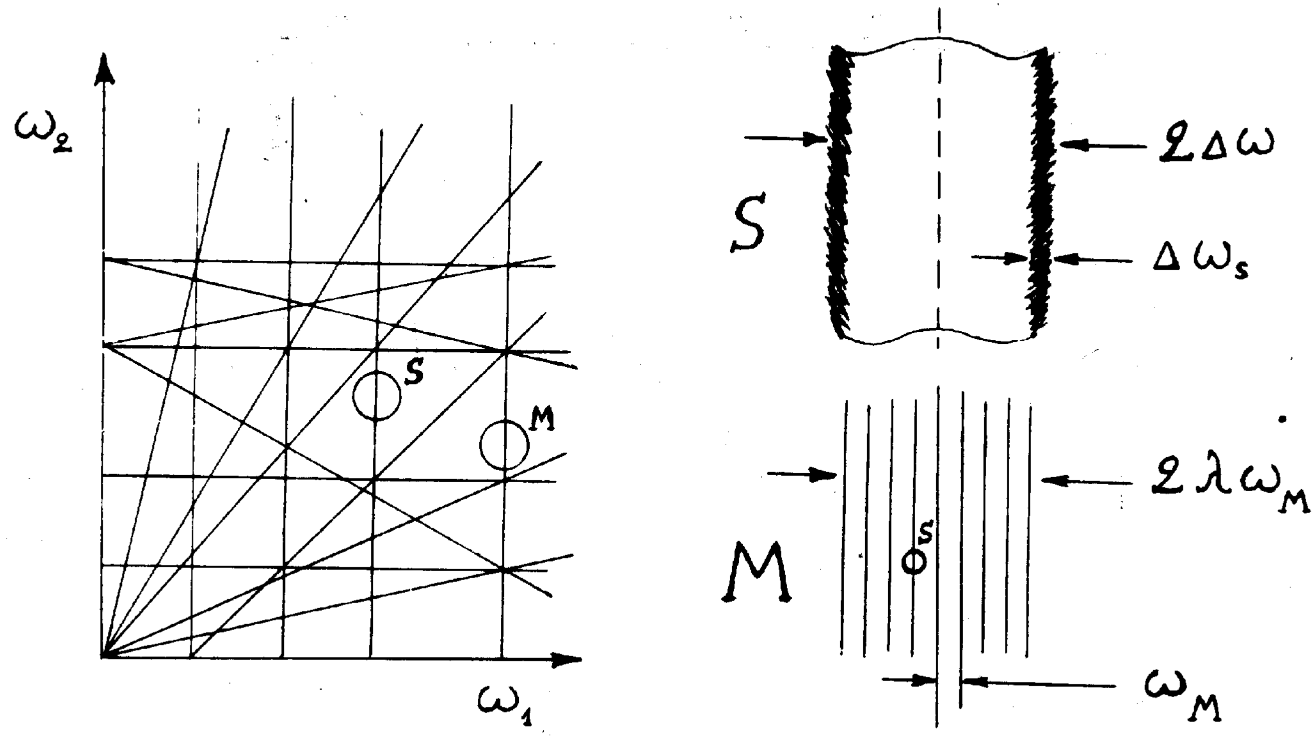

Fig. 1. Outline of a resonance structure: ω_1, ω_2 are the basic unperturbed frequencies; $\lambda \gg 1$, $\omega_M \ll \omega_{1,2}$ modulation factor and frequency. S is enlarged section of a driving resonance without modulation: $\Delta\omega$ and $\Delta\omega_s \sim \Delta\omega \cdot exp(-\pi\omega/\Delta\omega)$ the width of the resonance and its stochastic layer (shaded), ω being detune (see text); M is the same for a driving resonance split up into multiplet under modulation.

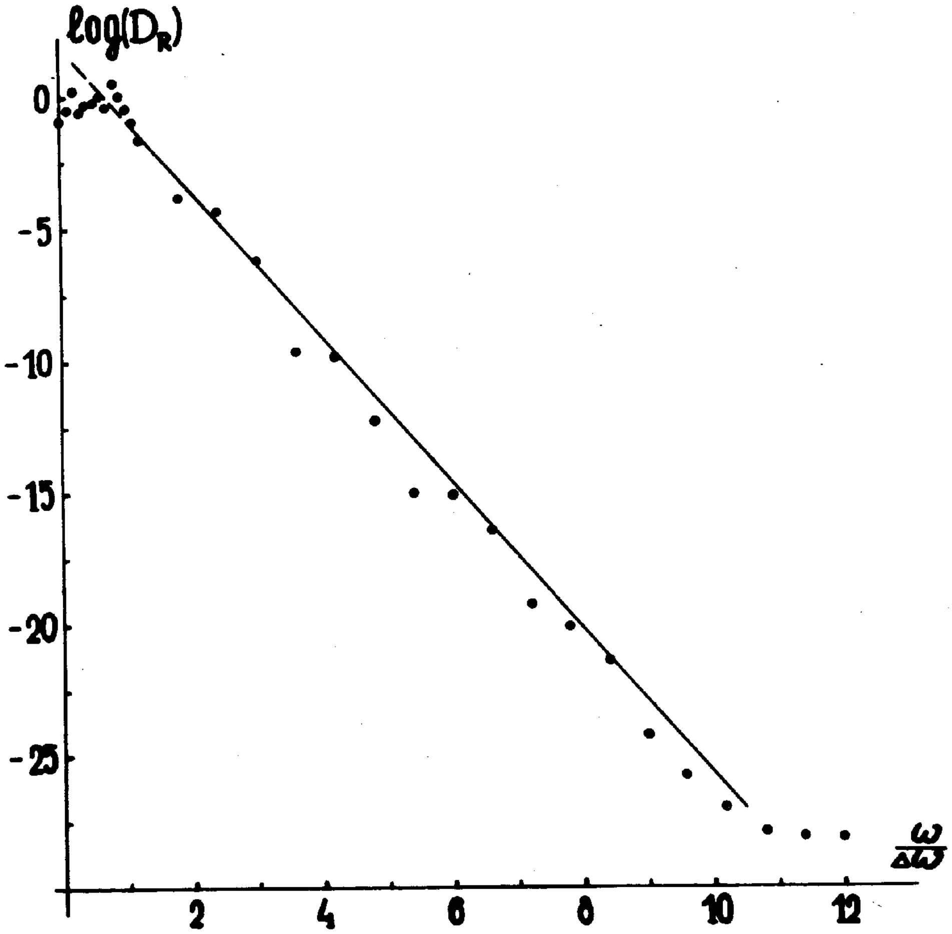

Fig. 2. Diffusion rate $D(\omega)$ along a modulation stochastic layer for model (2.1): $D_R = D \cdot \Delta\omega / \varepsilon^2$; $\lambda = 10$; $\omega_M = 0.01$; $\Delta\omega = 0.13$ (empirical); $k = 5 \times 10^{-4}$; $t_m = 10^6$; $s \approx 4$; $K_M \approx 40$; $V \approx 2$. The averaging period $T = 10^5$; $n = 6$ (3.2). The logarithm here and below is decimal.

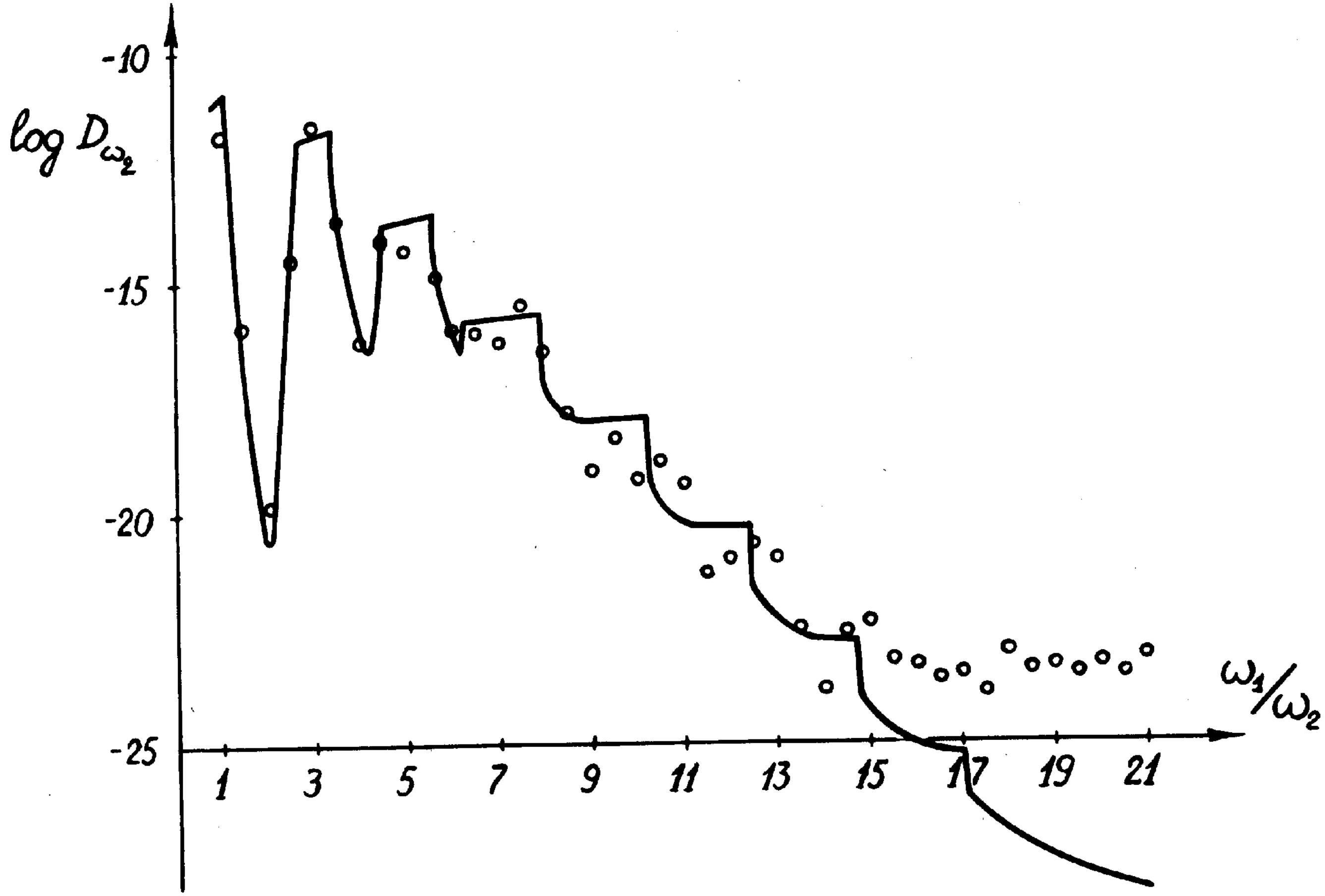

Fig. 3. An example of modulation diffusion in model (4.1): circles are numerical data; solid line shows the theory (4.5,6); $a_1 = 0.2$; $\mu = 10^{-6}$; $\varepsilon = 10^{-5}$; $\Omega = 0.169$; $\lambda = 10$; $\omega_M = 0.002$; $t_m = 10^6$; $s \approx 3.5$; $K_M \approx 30$; $V \approx 2$; $T = 10^5$; $n = 4$ (3.2).