Stochastic oscillations of classical Yang-Mills fields

B. V. Chirikov and D. L. Shepelyanskiĭ

Institute of Nuclear Physics, Academy of Sciences of the USSR, Siberian Branch

(Submitted 17 June 1981)

Pis'ma Zh. Eksp. Teor. Fiz. **34**, No. 4, 171–175 (20 August 1981)

Numerical experiments, which show that the uniform Yang-Mills fields have stochastic oscillations, are discussed. There is evidence, moreover, that the white and almost white colors are stable.

PACS numbers: 03.50.Kk

The dynamics of classical, spatially uniform Yang-Mills (YM) fields have been investigated in Ref. 1. Although this is a special case, it reduces, as shown in Ref. 1, to a simple Hamiltonian system with $N=9$ degrees of freedom. The dynamics of this system for the special case $N=2$ have also been investigated numerically in Ref. 1 and it was assumed there that its motion is stochastic. The numerical experiments for $N=2$ and 3 performed by us showed that the color in this case fluctuates stochastically, i.e., the classical YM equations appear to be nonintegrable at least in this special case.

Specifically, we have investigated the dynamics of a system with the Hamiltonian ($N=3$)

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + x^2y^2 + x^2z^2 + y^2z^2), \tag{1}$$

where x, y, z are proportional to the vector-potential components, and $\dot{x}$, $\dot{y}$, and $\dot{z}$ are proportional to the chromoelectric-field components.[1] The exponential local motion instability, which is characterized by the KS entropy h (Krylov-Kolmogorov-Sinaĭ entropy), was used as the stochasticity criterion (see, for example, Refs. 2 and 3).

 0021-3640/81/16163-04$00.60

To calculate the local stability numerically, along with the equations of motion of the system (1)

$$\ddot{x} = -x(y^2 + z^2); \ \ddot{y} = -y(x^2 + z^2); \ \ddot{z} = -z(x^2 + y^2) \tag{2}$$

we have simultaneously integrated the linearized adjoint equations

$$\ddot{\xi} = -(y^2 + z^2)\,\xi - 2x(y\eta + z\zeta); \ \ddot{\eta} = \ldots; \ \ddot{\zeta} = \ldots, \tag{3}$$

which can be derived by substituting in Eq. (2) the expressions $x+\xi, y+\eta$, and $z+\zeta$ and which characterize the behavior of a beam of adjacent trajectories in the linear approximation. The $x(t), y(t)$, and $z(t)$ functions in Eq. (3) are determined by Eqs. (2). The energy conservation in the calculation was better than 1%.

The stability of motion of the system (2) is characterized by the Lyapunov indices (LI) λ_i of the system (3) (see, for example, Refs. 3 and 4). The easiest to calculate is the maximum LI

$$\lambda_m = \lim_{t \to \infty} \left(\frac{\ln \rho(t)}{t} \right); \ \rho^2 = \xi^2 + \eta^2 + \zeta^2 + \dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2 \tag{4}$$

and $\rho(0) = 1$. The following inequalities are valid for the KS entropy:

$$\lambda_m \leqslant h \leqslant \lambda_m (N-1), \tag{5}$$

of which the most important one is the inequality on the left-hand side, which shows that at $\lambda_m > 0$ the KS entropy h is >0 and hence the motion has a stochastic component (see below). In the case of an integrable system (quasiperiodic motion) the "distance" between the adjacent trajectories $\rho(t) \propto t$ (on the average) (see, for example, Ref. 5), and $\lambda_m = 0$. The function $\Lambda(t) = [\ln \rho(t)]/t$ is shown in Fig. 1. $\Lambda(t_{max})$ was assumed to be the Lyapunov index, where t_{max} is the total time of motion.

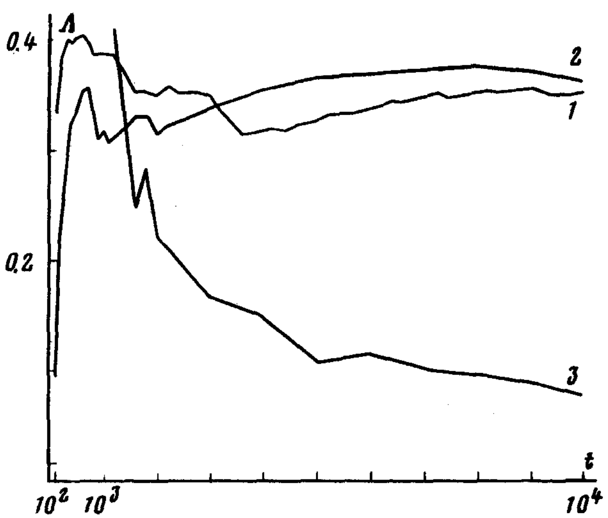

FIG. 1. Lyapunov index for the system (1). 1—Typical trajectory of the stochastic component ($\lambda_r = 0.35$); 2—trajectory with initial conditions near the white oscillations [$(r_{max}/u_0) \sim 0.1$; $\lambda_r = 0.37$]; 3—stable trajectory ($r_{max}/u_0 \sim 10^{-2}$; $\lambda_r < 4 \times 10^{-4}$, the scale is increased 200 fold).

Because the Hamiltonian (1) is uniform, $\lambda_m \propto H^{1/4}$. The average value of the dimensionless LI $\lambda_r = \lambda_m/H^{1/4}$ along the 22 trajectories with different, randomly selected initial conditions is $\langle\lambda_r\rangle \approx 0.38$ ($t_{max} = 10^2 - 10^3$, $H = 1$). The standard spread of the specific values of λ_r turned out to be small (0.04), indicating that all the trajectories belong to the same stochastic component. This does not rule out the existence of other independent stochastic or even stable components (see below); however, it is unlikely that they can replace an appreciable part of the energy surface. In any event, we were able to detect only one very small, stable region in the neighborhood of the in-phase oscillations [$x(t) = y(t) = z(t)$ are the white oscillations (WO) (Ref. 6)].

In the small neighborhood of this periodic solution the Hamiltonian (1) can be conveniently represented in the form

$$H = \frac{3\dot{u}^2}{2} + \frac{3u^4}{2} + \frac{6\dot{q}_1^2 + 2\dot{q}_2^2}{2} + 6\,uq_1(q_1^2 - q_2^2) + \frac{(3q_1^2 + q_2^2)^2}{2}, \tag{6}$$

where $x = u + q_1 + q_2$, $y = u + q_1 - q_2$, and $z = u - 2q_1$. At $q_1 = q_2 = 0$ the white oscillation along the u coordinate is described by the expression $u(t) \approx u_0 \cos\omega t$; $\omega \approx 1.20\, u_0$ (Ref. 3) (see Ref. 6). At $q_1, q_2 \ll u_0$ the influence of transverse motion (q_1, q_2) (TM) on $u(t)$ can be disregarded, assuming that it is given. A characteristic feature of transverse motion is the absence of quadratic (with respect to q_1 and q_2) terms in the potential energy (6). The white oscillations are therefore stable in the linear approximation. If $u =$ const, then the cubic terms in (6) will lead to an unstable motion. The rapid oscillations in u, however, lead to a dynamic stability of the transverse motion (Kapitsa's pendulum; see, for example, Ref. 7). The effective average Hamiltonian of transverse motion is

$$\langle H_\perp \rangle \approx 3\dot{q}_1^2 + \dot{q}_2^2 + \frac{1}{2}\left(1 + \frac{3u_0^2}{\omega^2}\right)\left(3q_1^2 + q_2^2\right)^2. \tag{7}$$

It is important that this system has an additional integral of the motion because of axial symmetry in the variables $\sqrt{3}\, q_1$ and q_2. According to the numerical results, the size of the stable region $r_{max}/u_0 \sim 10^{-2}$ ($r^2 = 3q_1^2 + q_2^2$, $t_{max} = 10^5$, or $\sim 10^3$ transverse oscillations).

At $N = 2$ the in-phase oscillations of the two colors [$x(t) = y(t)$, $z = 0$] turn out to be unstable in the linear approximation, since the transverse-motion Hamiltonian contains a defocusing quadratic term whose effect exceeds the dynamic focusing. The numerical values of λ_r for $t_{max} \sim 10^3$ turn out to be of the same order of magnitude as those for $N = 3$, and the stochastic component apparently includes the entire energy surface. It should be noted, however, that the ergodic dimension on the energy surface diverges as $|x|$ and $|y| \to \infty$, and the distribution function does not reach an equilibrium state in spite of stochastic motion. Note that for $N = 3$ this turns out to be a finite dimension in spite of infinite motion.

Thus the oscillations of the classical, spatially homogeneous YM fields turn out to be stochastic at least in the special case $N = 2$ or 3. An example of oscillations of one of the components of chromoelectric field is shown in Fig. 2. It is highly unlikely that the stochastic component vanishes completely in a more complex system

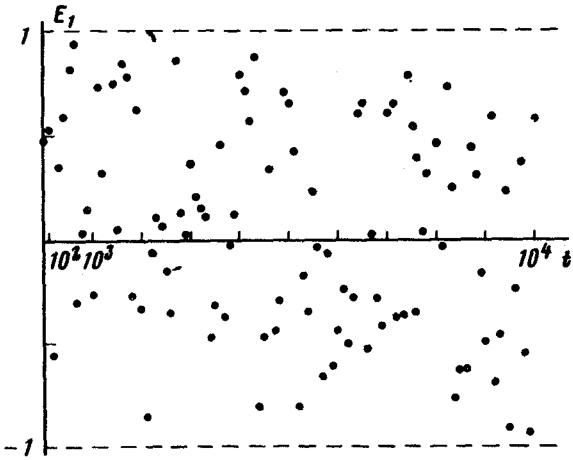

FIG. 2. Stochastic oscillations of the YM fields. The dark circles represent the instantaneous values of the component E_1 ($|E_1| \leqslant E^{\max} = 1$) of the chromoelectric field ($N = 3$).

($3 < N \leqslant 9$) and especially in the general case of spatially inhomogeneous YM fields.

We take this opportunity to express our sincere gratitude to V. E. Zakharov, Yu. F. Pinelis, and I. B. Khriplovich for useful discussions.

1. S. G. Matinyan, G. K. Savvidi, and N. G. Ter-Arutyunyan-Savvidi, Zh. Eksp. Teor. Fiz. **80**, 830 (1981).
2. I. P. Kornfel'd, Ya. G. Sinaĭ, and S. V. Fomin, Ergodicheskaya teoriya (Ergodic Theory), Nauka, Moscow, 1980.
3. B. V. Chirikov, Physics Reports **52**, 263 (1979).
4. Ya. B. Pesin, Dokl. Akad. Nauk **226**, 774 (1976).
5. G. Casati and B. V. Chirikov, J. Ford. Phys. Lett. **77A**, 91 (1980).
6. G. Z. Baseyan, S. G. Matinyan, and G. K. Savvidi, Pis'ma Zh. Eksp. Teor. Fiz. **29**, 641 (1979) [JETP Lett. **29**, 587 (1979)].
7. L. D. Landau and E. M. Lifshitz, Mekhanika (Mechanics), Nauka, Moscow, 1973.

Translated by S. J. Amoretty
Edited by Robert T. Beyer