PHYSICAL REVIEW LETTERS

VOLUME 57 | 13 OCTOBER 1986 | NUMBER 15

Chaos and Interaction of Atoms with Self-Consistent Fields in the Case of Small Coupling Constants

D. L. Shepelyansky
Institute of Nuclear Physics, 630090 Novosibirsk, U.S.S.R.
(Received 19 June 1986)

The interaction of three-level atoms with a two-mode classical electromagnetic field is considered in the case when transitions between all levels are allowed. It is found that for exact resonance with field frequencies the dynamics is chaotic in the rotating-wave approximation, i.e., for an arbitrarily small atomic density. The possibility of experimental observation of this phenomenon for Rydberg atoms is discussed.

PACS numbers: 05.45.+b, 31.60.+b, 42.50.−p

After the pioneering work of Jaynes and Cummings,[1] the problem of a collection of two-level atoms interacting with a self-consistent field in a resonator attracted the attention of many physicists (see, e.g., Allen and co-workers[2] and references therein). To analyze the interaction the well-known rotating-wave approximation (RWA) is usually used. The validity of the RWA is based on the fact that for ordinary density of atoms ρ the dimensionless coupling constant is small,

$$\Lambda = (16\pi\rho d^2/\hbar\omega)^{1/2} << 1, \tag{1}$$

and the nonresonant terms may be neglected (here d is the dipole moment and ω is the transition frequency). In this case the motion is integrable.[1,2] For $\Lambda \sim 1$ the influence of nonresonant terms becomes significant and leads to the failure of the RWA and chaos.[3-5] The review of works in this direction is given by Ackerhalt, Milonni, and Shin.[6] However, it is important to note that the realization of this interesting regime for optic frequencies requires extremely high density, $\rho \sim 10^{21}$ cm^{-3}, which makes its experimental observation very difficult.

In this paper, on the basis of the RWA, we consider a model of an atom in which chaos exists for $\Lambda \to 0$. In the model the atom has three approximately equidistant levels. The transition matrix elements between all three levels are different from zero and we assume that $V_{12} = V_{23} = d$, $V_{13} = d_1 \neq 0$. Such a system may be considered as a model of the hydrogen atom excited in the states with magnetic and parabolic quantum numbers $m = 0$ and $n_1 >> n_2 \sim 1$. As these states are very extended along the field direction we obtain a one-dimensional atom.[7] If the main quantum number $n >> 1$, then the spectrum is close to equidistant and its three levels give the suggested model with $d \approx 0.325 n^2$ and $d_1/d \approx 0.344$. Here we use atomic units and numerical factors taken from Shepelyansky[7] and Goreslavsky, Delone, and Krainov.[8] The essential new element is the possibility of direct transition $1 \to 3$ which is comparable with the transitions $1 \to 2$ and $2 \to 3$. This leads to an effective excitation of two modes of the field if the resonator frequencies are close to the transition frequencies.

The interaction of three-level atoms with a two-mode electric field in the RWA is described by the equations[4,6]

$$\ddot{\epsilon}_1 + \omega_1^2\epsilon_1 = 4\pi\omega_1^2\rho d(C_1^*C_2 + C_1C_2^* + C_2^*C_3 + C_2C_3^*), \quad \ddot{\epsilon}_2 + \omega_2^2\epsilon_2 = 4\pi\omega_2^2\rho d_1(C_1^*C_3 + C_1C_3^*),$$
$$i\dot{C}_1 = -(\epsilon_1 dC_2 + \epsilon_2 d_1 C_3), \quad i\dot{C}_2 = \omega C_2 - \epsilon_1 d(C_1 + C_3), \quad i\dot{C}_3 = (2\omega + \Delta\omega)C_3 - (\epsilon_1 dC_2 + \epsilon_2 d_1 C_1), \tag{2}$$

where $\epsilon_{1,2}$ and $\omega_{1,2}$ are the field strengths and the frequencies of the modes in a resonator, $C_{1,2,3}$ are the probability amplitudes of the levels. The frequencies of transitions $1 \to 2$ and $2 \to 3$ are accordingly equal to ω and $\omega + \Delta\omega$, $\hbar = 1$. The equations (2) are obtained for the case when $\Delta\omega \ll \omega$ and $\Lambda \ll 1$. These equations may be written in the Hamiltonian form. To do this it is convenient to introduce action-phase variables:

$$C_j = (2I_j)^{1/2} e^{i\theta_j}, \quad \dot{\epsilon}_k/\omega_k + i\epsilon_k = (16\pi\rho\omega_k J_k)^{1/2} e^{i\phi_k}; \quad j = 1,2,3, \quad k = 1,2.$$

Then in the RWA we obtain the Hamiltonian

$$H = I_2 + (2+\delta) I_3 + \nu_1 J_1 + \nu_2 J_2 - \Lambda[(J_1 I_1 I_2)^{1/2} \sin(\phi_1 - \theta_2 + \theta_1) + (J_1 I_2 I_3)^{1/2} \sin(\phi_1 - \theta_3 + \theta_2) + D(J_2 I_1 I_3)^{1/2} \sin(\phi_2 - \theta_3 + \theta_1)], \quad (3)$$

where $\delta = \Delta\omega/\omega$, $\nu_k = \omega_k/\omega$, $D = (\omega_2/\omega_1)^{1/2} d_1/d$ and dimensionless time $t' = \omega t$. The system (3) has two additional integrals of motion: $H_0 = I_2 + 2I_3 + J_1 + 2J_2$ and $I_1 + I_2 + I_3 = \frac{1}{2}$. The last one corresponds to the probability conservation. After introduction of three new linearly independent phases $\Psi_1 = \phi_1 - \theta_2 + \theta_1$, $\Psi_2 = \phi_2 - 2\theta_2 + 2\theta_1$, $\chi_3 = \theta_3 - 2\theta_2 + \theta_1$, conjugate to actions J_1, J_2, I_3, and new time $\tau = -\Lambda t'$ we obtain the Hamiltonian

$$K = K_R + \Delta I_3 + \Lambda^{-1}[(\nu_1 - 1)J_1 + (\nu_2 - 2)J_2],$$

$$K_R = [J_1(\tfrac{1}{2} + I_3 + 2J_2 + J_1 - H_0)(H_0 - 2I_3 - J_1 - 2J_2)]^{1/2} \sin\Psi_1 + [J_1 I_3 (H_0 - 2I_3 - J_1 - 2J_2)]^{1/2} \sin(\Psi_1 - \chi_3) + D[J_2 I_3 (\tfrac{1}{2} + I_3 + 2J_2 + J_1 - H_0)]^{1/2} \sin(\Psi_2 - \chi_3), \quad (4)$$

where $\Delta = -\delta/\Lambda$. For exact resonance ($\Delta = 0$, $\nu_2 = 2\nu_1 = 2$) the dynamical behavior of system (4) is determined by the resonance Hamiltonian K_R and does not depend on the small coupling constant (1). Therefore, if the motion of this system is chaotic then chaos exists in (2) for arbitrarily small Λ. This beautiful phenomenon has been discovered and examined by Ford and Lunsford[9] for the problem of three interacting waves. The same effect arises for the interaction of homogeneous classical massive Yang-Mills fields.[10] Notice that the Kolmogorov-Arnol'd-Moser theorem is inapplicable to this case because of isochronism of system (3) at $\Lambda = 0$.[11]

The investigation of system (4) has been carried out by numerical simulation for $\nu_2 = 2\nu_1 = 2$. At first we consider the case of exact resonance with $\Delta = 0$. Numerical experiments have shown the existence of the chaotic component which is characterized by the maximal positive Lyapunov exponent λ_R. Its value depends on the integrals of motion H_0 and K_R and determines the maximal exponent $\lambda \approx \Lambda\lambda_R$ in system

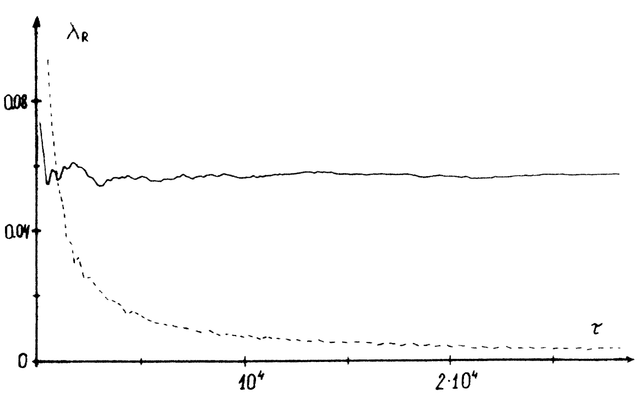

FIG. 1. The maximal Lyapunov exponent λ_R for the case of exact resonance in (4) with $K_R = 0$. Solid line corresponds to $H_0 = 0.985$ and dashed line to $H_0 \approx 0.464$. For the dashed line λ_R is multiplied by 10.

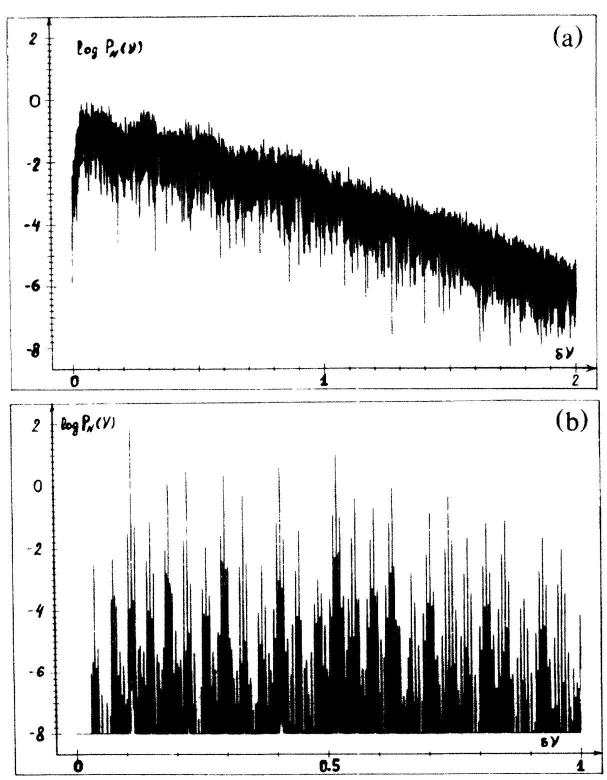

FIG. 2. The normalized power spectrum of the dipole moment $d_{13}(t)$ for the trajectories of Fig. 1: $P_N(\nu) = \omega\Lambda P(\nu)/d_1^2$, $\delta\nu = |\nu - 2\omega|/\Lambda\omega$; (a) $\lambda_R > 0$; (b) $\lambda_R = 0$. The logarithm is decimal.

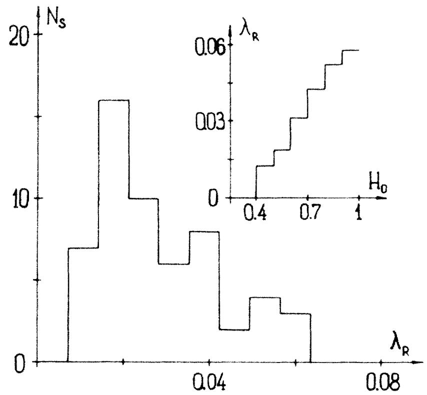

FIG. 3. The histogram of the distribution of λ_R for the system (4) with $\Delta=0$, $\nu_2=2\nu_1=2$, $D=\frac{1}{2}$, and $K_R=0$ ($J_1=J_2=0$ at $\tau=0$). N_S is the number of trajectories with λ_R in the corresponding interval. Inset: The dependence of λ_R on H_0.

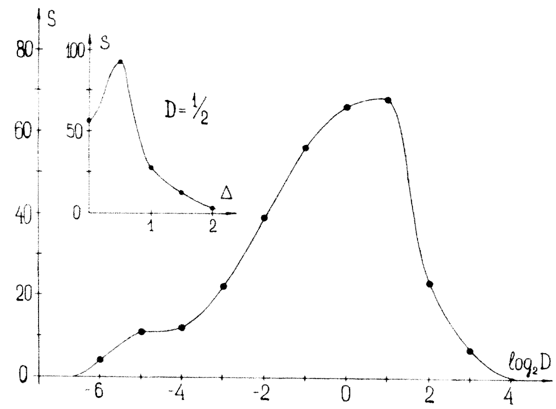

FIG. 4. The dependence of the share of the chaotic component on the parameter $D=\sqrt{2}d_1/d$ in (4) at $\Delta=0$, $\nu_2=2\nu_1=2$, initially $J_1=J_2=0$. The dependence of S on the detuning Δ is shown in the inset for $D=\frac{1}{2}$. S is measured in percent.

(2). The positivity of λ involves the positivity of Krylov-Kolmogorov-Sinai entropy $h \geqslant \lambda > 0$ and is one of the most effective numerical criteria of chaos (see, e.g., Lichtenberg and Lieberman[12]). An example of the calculation of λ_R for chaotic ($\lambda_R > 0$) and stable ($\lambda_R=0$) trajectories is shown in Fig. 1. The field in a resonator initially was equal to zero (zero-field state with $J_1=J_2=0$, $K_R=0$). The power spectrum $P(\nu)$ of dipole moment $d_{13}(t)=\frac{1}{2}d_1(C_1^*C_3+C_1C_3^*)$ for the trajectories of Fig. 1 is shown in Fig. 2. At $\lambda_R=0$ the spectrum contains only discrete lines but at $\lambda_R>0$ it becomes continuous. In the last case the main part of power is concentrated in the frequency region $\delta\nu=|\nu-2\omega|/\Lambda\omega \approx 1$. The spectrum $P(\nu)$ was obtained by taking a 16 384 fast Fourier transform.

A share of the chaotic component S was determined for the zero-field state in the following way. 100 trajectories with random values of I_j, θ_j were taken on a surface $I_1+I_2+I_3=\frac{1}{2}$ and for each of them the value of λ_R was computed. Then the number of trajectories with $\lambda_R > 0$ yields the share of chaos S in percent. For the extended states of the hydrogen atom $D=\sqrt{2}d_1/d \approx \frac{1}{2}$. In this case $S=56\%$ was obtained. The distribution of values of λ_R is shown in Fig. 3. The average value is $\langle\lambda_R\rangle \approx 0.016$. The maximum λ_R corresponds to $|C_3|^2 \approx 1$ and $H_0 \approx 1$. The dependence of λ_R on H_0 (see Fig. 3) was obtained by averaging over small interval ΔH_0 for trajectories with $\lambda_R > 0$. The relatively small scatter of values λ_R from one interval ΔH_0 ($\Delta\lambda_R/\lambda_R \sim \frac{1}{10}$) indicates the absence of an additional integral in (4). Qualitatively the same type of motion takes place also for $K_R \sim H_0 \sim 1$. However for $H_0 >> 1$ the dynamics become more stable because here the field dependence on time may be considered as a fixed one.

The motion of system (4) depends on two external parameters Δ and $D(\nu_2=2\nu_1=2)$. For $\Delta=0$ the share of chaos S is significant even for as small a ratio d_1/d as $\frac{1}{50}$ (see Fig. 4). From the numerical data obtained it follows that a significant chaotic component takes place only for $\Delta \lesssim \Delta_c = 1$. Using this value and expression (1) we can determine a critical density of highly excited atoms ρ_C above which the interaction with the self-consistent field leads to chaos. In the case of exact resonance $\nu_2=2\nu_1=2$ the critical value of the coupling constant is equal to $\Lambda_C=-\delta=3/n$ and for extended states we obtain

$$\rho_C \approx 4\times 10^{24}/n^9 \text{ cm}^{-3}. \tag{5}$$

For such a density $\rho_C(a_B n^2)^3 \sim n^{-3}$ and therefore the gas of atoms is dilute. For $n \sim 70$ the density $\rho_C \sim 10^8$ cm^{-3}. Here we need to note that for exact determination of ρ_C for atoms with $n >> 1$ it is necessary to take into account the interaction with other nearby levels which are also close to the resonance. The allowance for these levels apparently will lead to a decrease of ρ_C and one needs a separate investigation. Another interesting question is the quantization of an electromagnetic field in the region of chaos as has been done for the two-level model with $\Lambda \sim 1$ by Graham and Höhnerbach.[13]

The present high level of experiments with Rydberg atoms[14-16] allows one to excite extended states[14] and makes it quite possible to observe the described phenomenon in laboratory.

The author expresses his deep gratitude to B. V. Chirikov for attention to this work and valuable comments.

[1]E. T. Jaynes and F. W. Cummings, Proc. IEEE **51**, 89 (1963).

[2]L. Allen and J. H. Eberly, *Optical Resonance and Two-Level Atoms* (Wiley, New York, 1975); N. B. Narozhny, J. J. Sanchez-Mondragon, and J. H. Eberly, Phys. Rev. A **23**, 236 (1981); N. N. Bogolubov, Jr., V. N. Plechko, and A. S. Shumovsky, Fiz. Elem. Chastits. At. Yadra **14**, 1443 (1983) [Sov. J. Part. Nucl. **14**, 607 (1983)].

[3]P. I. Belobrov, G. M. Zaslavsky, and G. Kh. Tartakovsky, Zh. Eksp. Teor. Fiz. **71**, 1799 (1976) [Sov. Phys. JETP **44**, 945 (1976)].

[4]P. I. Belobrov, G. P. Berman, G. M. Zaslavsky, and A. P. Slivinsky, Zh. Eksp. Teor. Fiz. **76**, 1960 (1979) [Sov. Phys. JETP **49**, 993 (1979)].

[5]P. W. Milonni, J. R. Ackerhalt, and H. W. Galbraith, Phys. Rev. Lett. **50**, 966 (1983).

[6]J. R. Ackerhalt, P. W. Milonni, and M.-L. Shin, Phys. Rep. **128**, 205 (1985).

[7]D. L. Shepelyansky, in *Proceedings of the International Conference on Quantum Chaos, 1983* (Plenum, New York, 1985), p. 187.

[8]S. P. Goreslavsky, N. B. Delone, and V. P. Krainov, Zh. Eksp. Teor. Fiz. **82**, 1789 (1982) [Sov. Phys. JETP **55**, 1037 (1982)].

[9]J. Ford and G. H. Lunsford, Phys. Rev. A **1**, 59 (1970).

[10]B. V. Chirikov and D. L. Shepelyansky, Yad. Fiz. **36**, 1563 [Sov. J. Nucl. Fiz. **36**, 908 (1982)].

[11]V. I. Arnold, Usp. Mat. Nauk **18**, 91 (1963).

[12]A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer-Verlag, Berlin, 1983).

[13]R. Graham and M. Höhnerbach, Z. Phys. B **57**, 233 (1984).

[14]J. E. Bayfield and L. A. Pinnaduwage, Phys. Rev. Lett. **54**, 313 (1985).

[15]K. A. H. van Leeuwen *et al.*, Phys. Rev. Lett. **55**, 2231 (1985).

[16]P. Filipovicz, P. Meystre, G. Rempe, and H. Walter, Opt. Acta, **32**, 1105 (1985).