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LOCALIZATION OF DIFFUSIVE EXCITATION IN MULTI-LEVEL SYSTEMS

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Localization of Diffusive Excitation in Multi-Level Systems

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A B S T R A C T

The excitation of multi-level systems by a periodic field is considered in the regime of quasi-classical diffusion which takes place in the region of classical dynamical chaos. It is shown that quantum effects lead to a limitation of diffusion and to localization of quasi-energy eigenfunctions (QEE). The expression for the QEE localization length in terms of the classical diffusion rate $\left(l=\frac{D}{2}\right)$ is obtained and connection of this phenomenon with the Anderson localization in solid state problems is analysed. The localization length for photon transitions in the energy spectrum is found.

1. INTRODUCTION

In recent years a numbe: of experiments of the ionization of Rydberg (highly excited) atoms and dissociation of molecules by a strong monochromatic field have been carried out [1—5]. A characteristic peculiarity of such processes is the large number of absorbed photons $N_\phi \sim 100$ and the excitation of many unperturbed levels. Due to this the dynamics of excitation may be described in the first approximation by the classical equations of motion. Such an approach was used for molecules in Ref. [6] and for Rydberg atoms in Ref. [7]. The process of excitation obeys the diffusion law. The appearence of diffusion in the absence of any random forces is connected with the chaotic dynamics of the corresponding classical system. The nature and properties of such chaotic motion in classical mechanics is now well understood [8—10]. At the same time investigation of simple models has shown that the dynamics of classically chaotic quantum systems has a number of peculiarities (see e. g. Refs [9, 11, 12]). The most interesting one is the quantum diffusion limitation [11—16]. This limitation is the result of QEE localization (QEE decay exponentially with the serial number of unperturbed levels).

We carried out the investigation of the QEE localization mainly on the examples provided by two models. The first one is the quantum rotator model which has been investigated in Refs [11—13, 18, 20, 27—29, 32—34]. The second one is the Akulin—Dykhne model [17] which describes the general picture of excitation of a system

with irregular spectrum by a monochromatic field. It has been introduced as a model for molecular excitation in a laser field. With the help of a simple estimate based on the uncertainty relation between frequency and time, which was first used in Ref. 12, we obtain a simple expression for the QEE localization length in the rotator model (eqs (7), (13)). The generalization of this result and estimate (7) allowed to find the value of the length in the Akulin — Dykhne model (Sec. 7).

We have checked and confirmed the theoretical results for the quantum rotator by a special numerical method. The advantage of this method consists in the fact that it allows to evaluate the value of the QEE localization length without computation of the exact QEE. Indeed, we show that this evaluation can be reduced to computation of the Lyapunov exponents (LEX) in some auxiliary classical Hamiltonian system. Due to linearity of Schroedinger equation the equations of motion for this system are linear with the coefficients explicitly depending on the serial number of the unperturbed level, which plays the role of a discrete «time». For one-dimensional systems the number of equations is determined by the number of unperturbed levels effectively coupled after one period of the field. This approach to investigation of QEE was introduced in Ref. [20].

There are two simple limiting cases for the obtained system of linear equations. The first one is the case in which coefficients in the system periodically depend on discrete «time» (the serial number of the level). Then the quasi-energy spectrum is continuous and the QEE are delocalized like the Bloch eigenfunctions in a perfect crystal. The second case corresponds to a random dependence of coefficients on «time». This situation is analogous to the quantum motion in a random potential. The analogy between these two physical problems has been established in Ref. [18]. In one-dimensional random potential all eigenstates are localized that is the well-known Anderson localization [19, 30]. This corresponds to localization of all QEE [18]. In such approach the serial number of the unperturbed level plays the role of a spatial coordinate.

However, in spite of the usefulness of the analogy between the Anderson localization and the localization of dynamical chaos we need to stress two important differences between them. The first one is the absence of randomness in the dynamical system, and the second one is that QEE localization occurs in a quite different class of systems than considered is solid-state problems. In this paper we illustrate these differences mainly on the examples of the rotator

and Akulin — Dykhne models. Another example is the diffusive photoeffect in Hydrogen atom (see Refs [14 — 16]).

The contents of the paper is following. In Section 2 we describe the rotator model and LEX method, find the corresponding solid-state Hamiltonian, and give the estimate (7) for the localization length l. In Section 3 we obtain exact expressions for l in the dynamical Lloyd model and quantum standard map, and compare them with the numerical results. The localization length for the steady-state distribution $l_s(\neq l)$ is obtained in Section 4. The main results of Sections 2 — 4 have been briefly reported in Ref. [20]. In Section 5 we discuss a simple variant of the Akulin — Dykhne model which can be reduce to the Anderson model. The conditions under which the excitation of systems with many degrees of freedom may be considered within the framework of one-dimensional localization are obtained in Section 6. In Section 7 we find the QEE length for the excitation of typical multi-level systems by a monochromatic field.

2. THE QUANTUM ROTATOR MODEL

In order to investigate the motion of quantum systems which are chaotic in the classical limit we chose the generalized model of quantum rotator with Hamiltonian:

$$\hat{H} = H_0(\hat{n}) + V(\theta)\,\delta_T(t), \tag{1}$$

where $\hat{n} = -i\frac{\partial}{\partial\theta}$, $\delta_T(t)$ is the periodic delta-function with dimensionless period T, θ the phase variable, $V(\theta)$ the external perturbation, $\hbar = 1$ [11 — 13, 18]. Here $H_0(n)$ determines the energies of unperturbed levels n. The dynamics of the corresponding classical system is determined by the equations of motion with Hamiltonian (1), where n, θ are canonically conjugated action-phase variables. After integration over a period T we obtain a map:

$$\bar{n} = n - \frac{\partial V}{\partial\theta}, \quad \bar{\theta} = \theta + T\frac{\partial H_0(\bar{n})}{\partial\bar{n}} \tag{2}$$

where $\bar{n}$ and θ are the values of the variables n, θ after a period T. For strong perturbation the resonances overlap [8] and then the action grows beyond any limit according to the diffusion law:

$\langle (\Delta n)^2 \rangle = D\tau$, where τ is the number of periods. Generally, the diffusion rate D is a complicated function of the system parameters. However, in the region of a strong chaos the phases $\theta(\tau)$ are random and independent. This allows one to use the quasi-linear approximation in order to calculate D [10]. In this case the diffusion rate is equal to

$$D_{ql} = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\partial V}{\partial \theta} \right)^2 d\theta . \tag{3}$$

The quasi-classical condition is satisfied if the number of levels excited in one period is large: $D \gg 1$, and if the dimensionless parameter $T \ll 1$ ($T \propto \hbar$) [9, 11—13].

The main part of our investigations was carried out for the quantum standard map:

$$\hat{H} = \frac{\hat{n}^2}{2} + k \cos\theta \; \delta_T(t) . \tag{4}$$

The motion of the corresponding classical system is described by the standard map [8—10]:

$$\bar{p} = p + K \sin\theta, \quad \bar{\theta} = \theta + \bar{p}, \tag{5}$$

where $p = Tn$, $K = kT$. For $K \leqslant K_{cr} = 0.9716...$ [24] the change of n is finite $|\Delta n| \leqslant \sqrt{k/T}$, but for $K > K_{cr}$ $(\Delta n)^2$ grows according to the diffusion law with the rate $D = D_0(K)/T^2$, where $D_0(K)$ is the diffusion rate in p in the standard map (5). Within the chaotic component the dependence of the diffusion rate on K may be approximately described by the following expression:

$$D_0 \approx \begin{cases} \frac{K^2}{2}(1 + 2J_2(K) + 2J_2^2(K)) , & K \geqslant 4.5 \\ 0.30\,(\Delta K)^3 , & K < 4.5 \end{cases} \tag{6}$$

where $J_2(K)$ is a Bessel function, $\Delta K = K - K_{cr}$. For $K \geqslant 4.5$ the dependence of D_0 on K has the form of oscillations which decay as K grows. The limit value $D_0 = \frac{K^2}{2}$ corresponds to the quasi-linear approximation (3), when phases $\theta(\tau)$ in (5) are random and independent. For $K \to K_{cr}$ we use in (6) the empirial formula which was obtained from numerical experiments in Ref. [13]. The value of the exponent $\eta \approx 3$ in power law $D_0 \propto (\Delta K)^\eta$ is close to that given in Ref. [26].

Numerical experiments [11—13, 27, 28] with the quantum standard map have shown that in the course of time $\langle n^2 \rangle$ stops growing. This means that the external field effectively excites only a finite number of unperturbed levels ($\Delta n \sim l$). An analogous result has been obtained in Ref. [29] for the dynamical Lloyd model with $V(\theta) = 2 \arctan (E - 2k \cos\theta)$, and $H_0 = \frac{\hat{n}^2}{2}$ which had been introduced in Ref. [18]. It is natural to interpret this effect as the result of the QEE localization which is analogous to the Anderson localization in one-dimensional lattice [18, 13, 20]. For the number of excited levels and the localization length of QEE the following theoretical estimate has been obtained in Refs [12, 13]

$$l = \alpha D \sim \Delta n \sim \tau_D , \tag{7}$$

where α is an undetermined numerical constant. The derivation of (7) may be done in the following way. Let one unperturbed state contain l QEE with quasi-energies ε_i. Since all these ε_i are distributed within the interval $[0, 2\pi]$, its average spacing is equal to $\Delta\varepsilon \sim 1/l$ (here we consider the case when the unperturbed levels are uniformly distributed in this interval). If initially we excite one unperturbed state then diffusion will continue during the finite time τ_D until the discreteness of the QEE spectrum becomes effective. According to the uncertainty relation, $\tau_D \sim \frac{1}{\Delta\varepsilon} \sim l$. After this time the number of diffusively excited levels will be equal to $\Delta n \sim (D\tau_D)^{1/2} \sim l$. From this relation we obtain equation (7). The condition for its applicability is $D \gg 1$. In the case of a d-dimensional unperturbed system the number of excited levels $\Delta n \sim (D_1 \dots D_d)^{1/2} \tau^{d/2}$ and then in the absence of a degeneration of levels the condition of quantum limitation of chaos takes the form

$$\tau_D \geqslant (D_1 \dots D_d)^{1/2} \tau_D^{d/2} . \tag{8}$$

For $d = 1$ we always have localization. For $d \geqslant 3$ delocalization takes place if $D_1 \dots D_d \geqslant 1$. This condition corresponds to the Anderson criterion [30] (a small random potential provides fast diffusion and large conductance). For $d = 2$ the estimate (8) gives delocaliza-

tion for $D_1 D_2 \gg 1$. However, a more rigorous consideration shows that in this case localization always takes place but for $D_1 D_2 \gg 1$ its length is exponentially large: $\ln l \sim D_1 D_2$ (see, for example, Ref. [31] and Refs therein). For any d it follows from (8) that localization always takes place if each $D_i \ll 1$. This case corresponds to the quantum stability border [6].

Let us consider now equation for the eigenfunction with quasi-energy ε [18]:

$$u_n^- = e^{i(\varepsilon - TH_0(n))} u_n^+, \quad u^+(\theta) = e^{-iV(\theta)} u^-(\theta). \tag{9}$$

Here $u^\pm(\theta)$ are the values of the function u before and after a kick $\delta(t)$ and $u_n^\pm$ are the Fourier coefficients of $u^\pm(\theta)$. After simple transformations the equation (9) may be rewritten in the form

$$\hat{H}_{ss} u = \left\{ \cos\frac{\hat{V}}{2} \tan\left(\frac{\varepsilon}{2} - \frac{T}{2}\hat{H}_0\right) \cos\frac{\hat{V}}{2} - \frac{1}{2}\sin\hat{V} \right\} u = 0, \tag{10}$$

where $u = e^{\pm iV/2} u^\pm$. After Fourier transform

$$e^{-iV/2} = \sum_r W_r^- e^{i(r\theta + \varphi_r)}$$

it is easy to see that the Hamiltonian H_{ss} corresponds to an one-dimensional lattice with interacting sites and energy

$$E = -\sum_r W_r^- W_{-r}^- \sin\varphi_r \cos\varphi_{-r}.$$

In such an approach the quasi-energy ε determines the potential of interaction and the eigenvalue of energy E plays the role of a parameter. Moreover, the number of unperturbed levels n in the model (1) corresponds to a discrete spatial coordinate in the lattice.

Since all eigenfunctions in one-dimensional random lattice are localized [19] it is natural to expect an exponential localization of QEE in (1). If $\cos(V/2) \neq 0$ we may introduce $\bar{u} = \cos(V/2)\, u$ and devide equation (10) by $\cos(V/2)$. After that we reduce the problem to the case with Hamiltonian

$$H_{ss} = \tan\left(\frac{\varepsilon}{2} - \frac{T}{2}\hat{H}_0\right) - \tan\frac{V}{2}.$$

This procedure was implicitly used in Ref. [18]. However, it is necessary to stress that this approach leads to the appearance of a

nonphysical singularity which does not allow for an analysis of the wide class of potentials with $|V(\theta)| \geqslant \pi$.

The form of equation (10) is convenient for exploiting the analogy with solid-state problems. However, for numerical experiments it is more convenient to rewrite equations (9), (10) as follows. We introduce $\bar{u} = e^{\pm iV/2} u^\pm / g$, where $g(\theta)$ is an arbitrary real function (we will consider the case when g and V are even functions of θ). Then we obtain from (9) the equation

$$\sum_r \bar{u}_{n+r} W_r \sin(\chi_n + \varphi_r) = 0, \tag{11}$$

where

$$e^{-iV/2} g = \sum_r W_r e^{i(r\theta + \varphi_r)}, \quad \chi_n = (\varepsilon - TH_0(n))/2.$$

Let us assume that in (11) only W_r with $|r| \leqslant N$ differ from zero. Then a column of $2N$ known values of $\bar{u}_n$ determines the value of QEE for arbitrary n. The recursive computation of $\bar{u}_n$ from (11) may be considered as the motion of a dynamical system in the discrete time n. The dynamics of the $2N$-component vector is determined by a transfer matrix M_n the expression for which may be easily obtained from equation (11). It may be shown that the product of matrices M_n can be transformed by rotation into a symplectic matrix. Therefore, the dynamics in n is Hamiltonian and there are N positive and N negative Lyapunov exponents (LEX) $\gamma_i^+ = -\gamma_i^- \geqslant 0$ (see, for example, Ref. [10]). Asymptotically, the localization length of QEE is determined by the minimal positive LEX $\gamma_1 = 1/l$ [20—22]. The fact that γ_1 is different from zero leads to exponential localization of QEE and to a discrete spectrum of quasi-energies. The numerical method of computation of all LEX is described, for example, in Ref. [10]. This method allows not only to find all γ_i, but also to determine the dependence on n of the norm

$$\| u_n^{(i)} \| = \left(\sum_{m=1}^{2N} |u_{n+m}^{(i)}|^2 \right)^{1/2}$$

of an eigenvector corresponding to a given exponent. In solid-state physics the LEX method is known as the transfer matrix method. It is extensively used for investigation of localization in two- and three-dimensional solid state systems [21, 22]. For dynamical models it was first applied in Ref. [20], and then in Ref. [23].

3. LOCALIZATION OF QEE

Let us consider the rotator model with potential $V=2\arctan(E-2k\cos\theta)$ which has been introduced in Ref. [18]. Taking $g=1/\cos(V/2)$ we obtain $W_0 e^{i\varphi_0}=1-iE$, $W_{\pm1}e^{i\varphi_{\pm1}}=ik$, $W_r=0$ if $|r|>1$. Equation (11) now takes the form

$$E_n u_n + k u_{n+1} + k u_{n-1} = E u_n, \qquad (12)$$

where $E_n=\tan\chi_n$ and we dropped the bar for u_n. In the case when phases χ_n are random and independent in the interval $[0, \pi]$ equation (12) corresponds to the well known Lloyd model [18, 19]. For this model the exact expression for localization length is known (see e. g. Refs [18—20]), and for $l\gg1$ it has the form $l=\sqrt{4k^2-E^2}$. This may be used for the determination of the numerical factor in equation (7) [20]. Due to randomness of χ_n the phases $\theta(\tau)\propto\frac{\partial\chi_n}{\partial n}$ in (2) are also random and independent, and $D=D_{ql}$. The calculation of the integral in (3) for $D_{ql}\gg1$ and the comparison with l gives $\alpha=1/2$.

The same method of calculation of l can be applied also in a more complicated case when the interaction connects $2N$ sites and the potential has the form

$$V=2\arctan\Big(E-2k\sum_{m=1}^{N}\cos m\theta\Big).$$

According to (7), $l=D_{ql}/2\approx 2kN^2$ where the last equality takes place when $D_{ql}\gg N$. All values of D_{ql} were obtained by numerical calculation of the integral in (3). These values are in good agreement with the experimental values obtained by the LEX method (see Fig. 3 in Ref. [20]).

Now we consider the dynamical Lloyd model with $H_0=n^2/2$ ($N=1$). Then the phases χ_n are not random. Moreover, for rational values of $\frac{T}{4\pi}=p/q$ E_n in (12) becomes a periodic function of n, that corresponds to the case of an ideal crystal with delocalized eigenfunctions. In such a case of quantum resonance [32] the quasi-energy spectrum is continuous and consists of q zones. It was shown in Ref. [33] that the continuous component exists also for special irrational values of $T/4\pi$ which are very close to rational

numbers. However, we conjecture that for any irrational numbers $T/4\pi$ (so that $C_2 q^{-1-\varepsilon}<\left|\frac{qT}{4\pi}-p\right|<C_1 q^{-1}$ for any $\varepsilon>0$), the measure of which on the interval [0, 1] is equal to one, exponential localization will always take place with the same exponent as in the Lloyd model (12) with random phases χ_n.

This conjecture was confirmed by the numerical experiments based on the LEX method. They really showed that the localization length is the same as in the case of random phases χ_n. The parameters of the model were changed in the intervals: $0.01\leqslant k\leqslant 1000$, $10^{-5}\leqslant T\leqslant 1$, $0\leqslant E\leqslant 2$. The value of γ was determined on the interval $1\leqslant n\leqslant n_m=10^5$ and did not depend on ε and T. The relative accuracy of γ was equal to $(\Delta\gamma/\gamma)^2=2/\gamma n_m$ (see below).

The recursively obtained values of u_n may be considered to give a QEE on some interval of $n\to\infty$. For $T\ll1$ the dependence of u_n on n has steps of size $\Delta n\sim 1/Tn$. With growth of n their size decreases and for $Tn\sim1$ they disappear. An estimate for Δn may be obtained from the condition $\Delta\chi_n=\frac{1}{2}Tn\Delta n\sim1$. It is important to note that in the model under discussion the quasi-classical limit ($k\gg1$) always corresponds to the region of strong chaos where $l=D_{ql}/2$.

The LEX method has been also applyed to calculation of l in the quantum standard map (4). Taking $g=1$ we obtained in (11) $W_r=J_r(k/2)$, $\varphi_r=-\frac{\pi r}{2}$, $\chi_n=(\varepsilon-Tn^2/2)/2$, where $J_r(k/2)$ is a Bessel function. Due to fast decay of W_r for $|r|>k/2$ it is possible to use a finite number of sites $N\sim k/2$. The further increase of N is related only to exponents $\gamma\sim1$. Another check consisted in verifying the relation $\gamma_i^+ + \gamma_i^- = 0$. For a large n interval of the order $\sim10^5$ levels the sum of positive and negative exponents was small: $(\gamma_1^+ + \gamma_1^-)/\gamma_1^+ \sim 10^{-2}$.

The results of numerical experiments [20] show that in the quasi-classical region $T\leqslant1$; $5\leqslant k\leqslant75$; $1.5\leqslant K\leqslant29$ ($T/4\pi$ is a typical irrational number) the localization length is satisfactorily described by the formula

$$l=\frac{D_0(K)}{2T^2}. \qquad (13)$$

This relation holds not only in the region of strong stochasticity $K\geqslant4.5$, where the measure of the stable component is negligibly

small [8], but also in the region with $\Delta K = K - K_{cr} \ll 1$, where the diffusion rate is determined by a complicated critical structure [26] and the stable component covers approximately 50% of the whole phase space [8]. The numerical results obtained demonstrate a satisfactory agreement with formula (13) in a range of 4 orders of magnitude for the diffusion rate (Fig. 1). However, it is important to note that the relation (13) holds only in the case when the localization length is larger than the number of interacting sites $2N \approx k$. In the opposite case $\frac{D_0}{2T^2} \ll k$, the diffusion is too slow and does not lead to increasing the localization length $l \sim k$. Thus the condition of applicability of (13) has the form

$$k > k_{cr} = \frac{\varkappa K^2}{D_0} \approx \frac{3\varkappa K^2}{(\Delta K)^3} \gg 1, \quad K > K_{cr}, \tag{14}$$

where $\varkappa \approx 1.3$ is determined from numerical experiments. The inequality (14), first obtained in Ref. [13], gives the condition for the so called homogeneous (exponential) localization. In the opposite case $k \ll k_{cr}$, the localization length $l \sim N \sim k$ is comparable with the period of the resonance structure $\Delta n = 2\pi/T$ and, like the diffusion rate it is inhomogeneous, too [13]. Examples of homogeneous and inhomogeneous localization are shown in Fig. 2. In the latter case, the size of the clearly observed steps is equal to $2\pi/T$. In contrast to the dynamical Lloyd model these steps persist for arbitrary large values of n as well. For example, the dependence of $\|u_n^{(1)}\|$ on n in Fig. 2 is given in the region $n \approx 3 \cdot 10^4$. The reason for this difference is apparently related to the large number of interacting sites in the model (4). In the region of stability ($k \gg 1$, $K \ll 1$) the localization is almost homogeneous due to a small size of nonlinear resonances. Numerically, the localization length in this case is comparable with the number of interacting sites: $l \approx k/4$.

The dependence of diffusion rate D_0 on the classical parameter of chaos K (see (6)) leads to a significant change of the localization length with the parameter T even for a fixed value of k. An example of this effect is shown in Fig. 3. According to (13), the localization length repeats all oscillations of D_0. These oscillations take place for $K \geqslant 4.5$ and are given by formula (6) in which we may use the asymptotics of the Bessel function. The difference of D from the quasi-linear value is related to the influence of correlations:

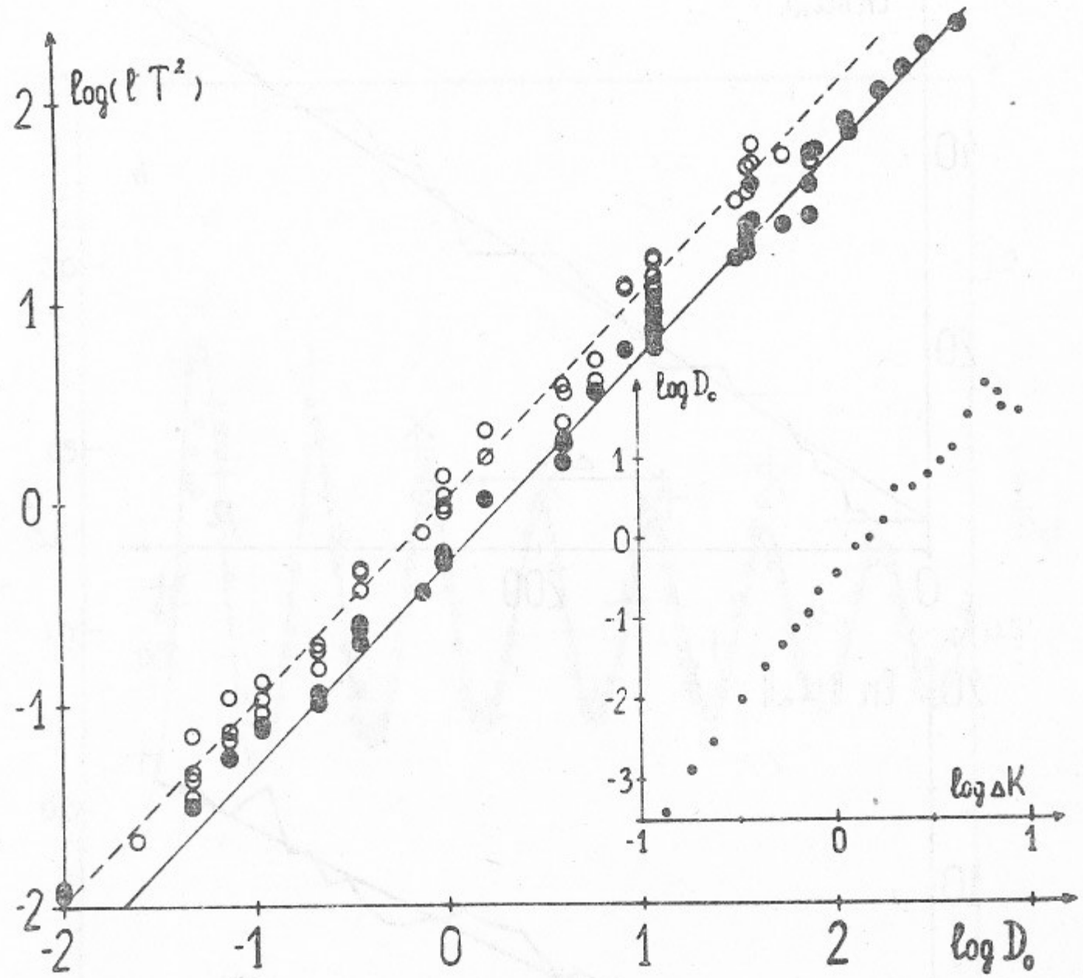

Fig. 1. Dependence of the localization length of diffusion rate D_0 in classical standard map (5). Circles represent numerical data of Ref. [13] for values of l_s obtained from steady-state distributions (15). Dashed line corresponds to average value $\langle \alpha_s \rangle = 1.04$. Points show the localization length obtained from QEE by the LEX method. Straight line shows the theoretical localization $l = D/2$. In the insert the numerical data from Ref. [13] are shown, giving dependence of D_0 on $\Delta K = K - K_{cr}$, $K_{cr} = 0.9716...$. Here and in Fig. 5 the logarithm is decimal.

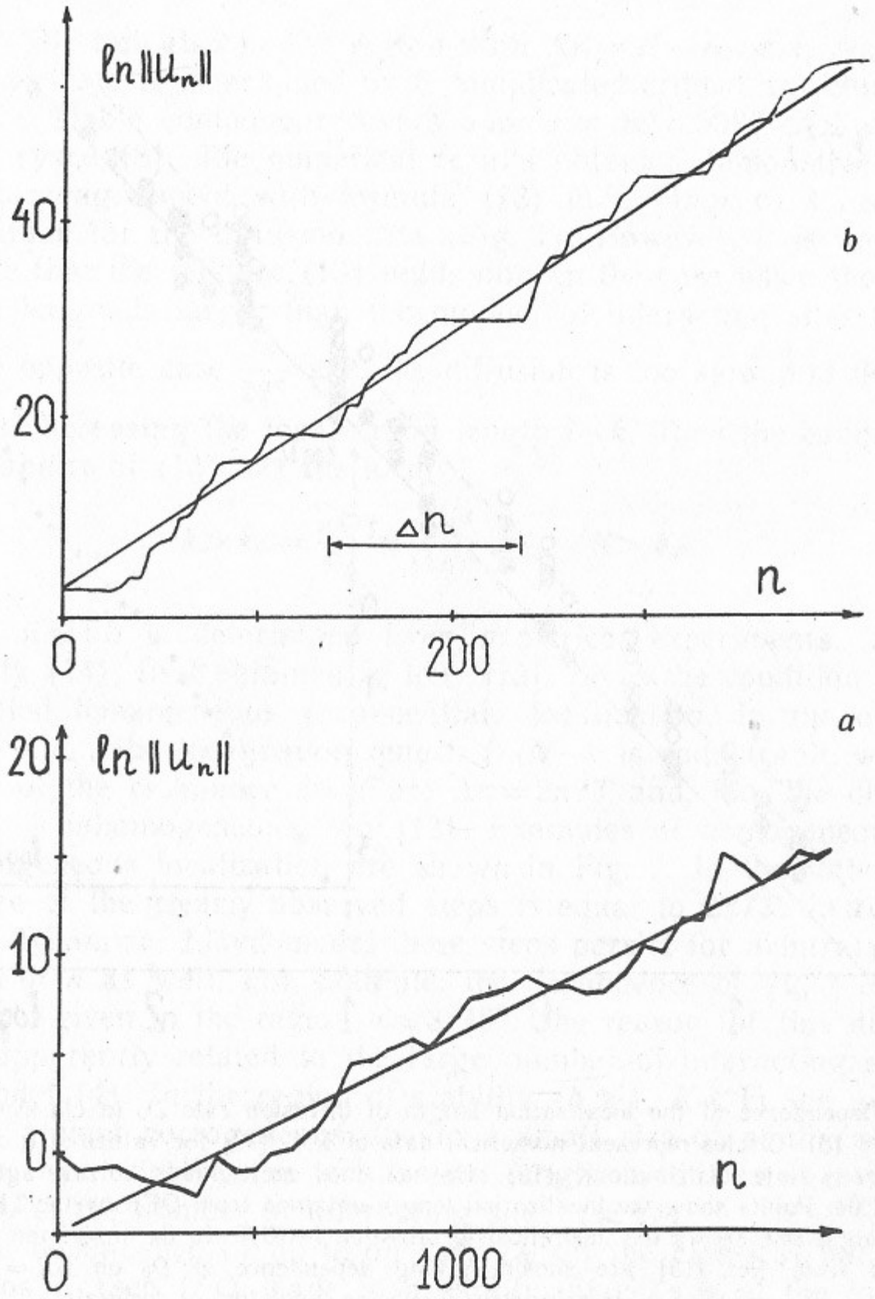

Fig. 2. Dependence on n of averaged quantity $\|u_n^{(1)}\| = (\sum_{m=1}^{2N} |u_{n+m}^{(1)}|^2)^{1/2}$ in model (4) for: a — homogeneous localization at $k=20$, $K=5$; b — inhomogeneous localization at $k=20$, $K=1.3$. Periodic oscillations with $\Delta n = \frac{2\pi}{T}$ are related to the resonance structure. Straight lines show experimental values of l obtained on interval $n_m \approx 5 \cdot 10^4$.

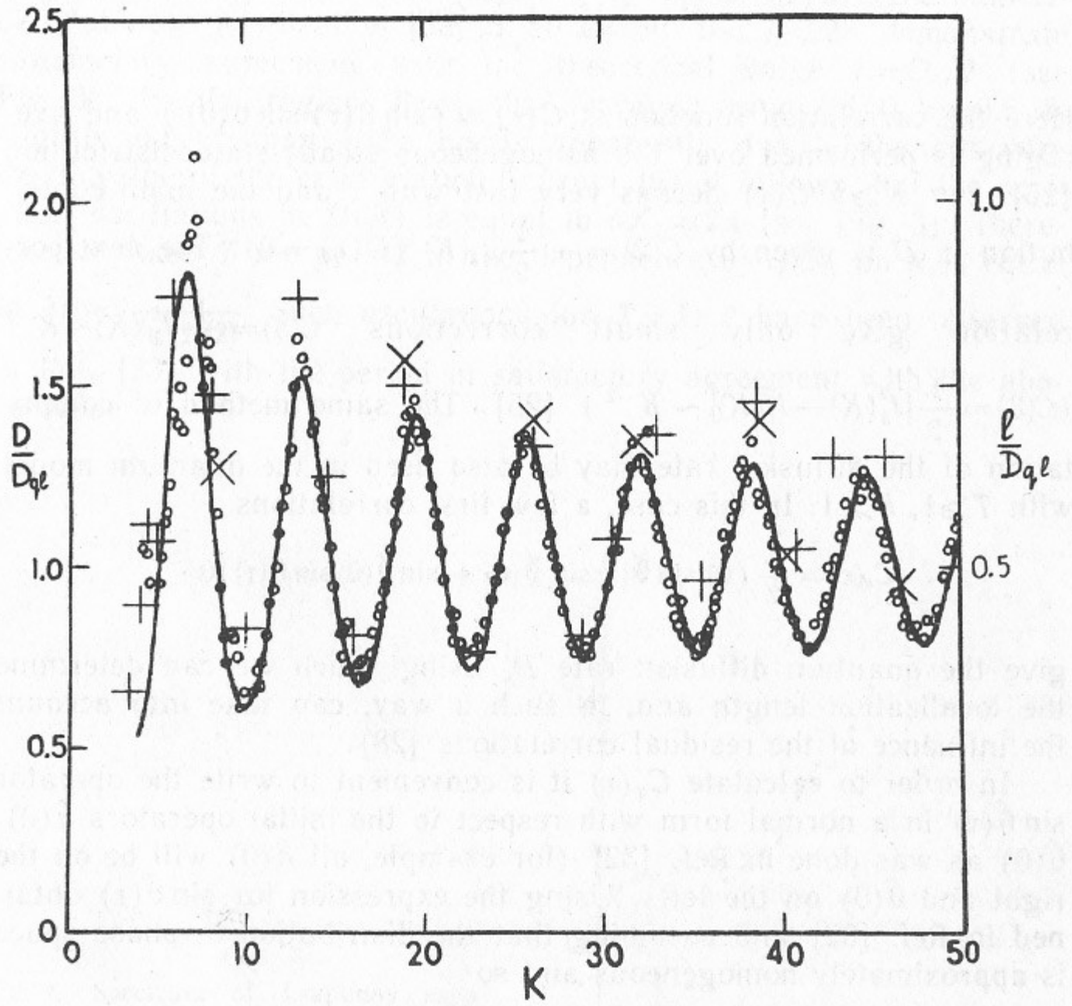

Fig. 3. Dependence of localization length in the quantum standard map on quantum parameter of chaos $K_q = 2k \sin\frac{T}{2} \to K$. Circles and the curve are numerical data and theory for classical diffusion rate $D(K)$ from Ref. [25].

$$D = D_{ql}\left[1 + 4\sum_{\tau=1}^{\infty} C(\tau)\right].$$

Here the correlation function is $C(\tau) = \langle \sin\theta(\tau)\sin\theta(0)\rangle$ and averaging is performed over the homogeneous steady-state distribution [25]. For $K \gg 5$ $C(\tau)$ decays very fast with τ and the main contribution in D is given by $C(2) = -\frac{1}{2}J_2(K)$ $(C(1)=0)$. The next correlation give only small corrections $C(4) = \frac{1}{2}J_2^2(K) \sim K^{-1}$ $(C(3) = \frac{1}{2}[J_3^2(K) - J_1^2(K)] \sim K^{-2})$ [25]. The same method of computation of the diffusion rate may be also used in the quantum model with $T \gg 1$, $k \gg 1$. In this case, a few first correlations

$$C_q(\tau) = \frac{1}{2}\langle 0|\sin\hat{\theta}(\tau)\sin\hat{\theta}(0) + \sin\hat{\theta}(0)\sin\hat{\theta}(\tau)|0\rangle$$

give the quantum diffusion rate D_q using which we can determine the localization length and, in such a way, can take into account the influence of the residual correlations [28].

In order to calculate $C_q(\tau)$ it is convenient to write the operator $\sin\hat{\theta}(\tau)$ in a normal form with respect to the initial operators $\hat{n}(0)$, $\hat{\theta}(0)$ as was done in Ref. [32] (for example, all $\hat{n}(0)$ will be on the right and $\hat{\theta}(0)$ on the left). Using the expression for $\sin\hat{\theta}(\tau)$ obtained in Ref. [32] and assuming that the distribution in phase space is approximately homogeneous and so

$$\langle 0|e^{im_1\hat{\theta}(0)}e^{im_2\hat{n}(0)}|0\rangle = \delta_{m_1,0}\,\delta_{m_2,0},$$

we obtaine that the first three correlations are the same as in the classical case upon the substitution $K \to K_q = 2k\sin(T/2)$. The expression for $C_q(4)$ contains an additional term which is usually as small as $C(4)$:

$$C_q(4) = \frac{1}{2}\Big\{ J_2^2(K_q) + \sum_{m\neq -2}\Big[J_m(K_q)J_{m+4}(-K_q)\times$$

$$\times J_{2m+4}\Big(2k\sin\Big(\frac{T}{2}(m+2)\Big)\Big) - J_m^2(K_q)J_{2m+2}\Big(2k\sin\Big(\frac{T}{2}(m+2)\Big)\Big)\Big]\Big\}.$$

Therefore, for $T \gg 1$ the quantum diffusion rate is approximately given by the expression (6) with $K \Rightarrow K_q = 2k\sin(T/2)$. The numerical data for localization length at $k=30$, $0 < T < 2\pi$ demonstrate satisfactory agreement with the theoretical value $l = D_q/2$ (see Fig. 3). In the region $T > 1$ the averaged ratio $\langle l/D_q\rangle \approx 0.6$ is slightly larger than 1/2; this is apparently due to the not very small value of the ratio $k/l \approx 0.1$. From (6) it follows that the period of oscillations in $D(K)$ is equal to $\delta K_q \approx 2\pi$ (see Fig. 3). Therefore, at fixed T the period in the dependence of l/D_{ql} on k is equal to $\delta k \approx \frac{\pi}{\sin(T/2)}$. Such oscillations for $T = 1;\ 2$ have been observed in Ref. [23] with the period in satisfactory agreement with the above value.

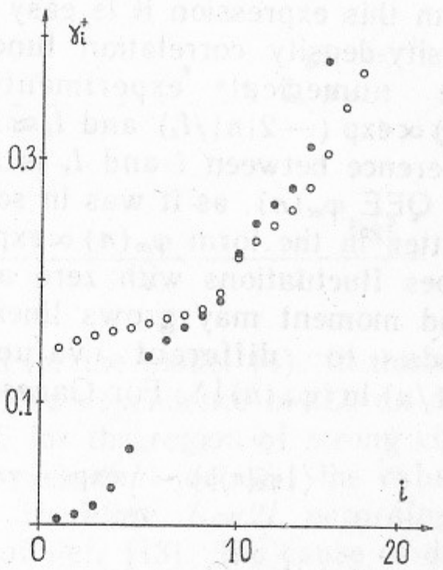

Fig. 4. Spectrum of Lyapunov exponents in model (4): quantum chaos $k=30$, $K=5$ (points); stable motion $k=30$, $K=0.003$ (circles).

Up to now we have discussed the properties of the minimal LEX which determine the asymptotic behaviour of QEE. However, there is a spectrum of LEX γ_i^+ in the system (4). A typical picture of this spectrum is shown in Fig. 4 for both the stable and chaotic regimes. Each γ_i is related to an eigenvector which is cross-ortogonal to all the other vectors. We conjecture that QEE is a linear superposition of all these vectors and that the probability of each vector is γ_i/N (γ_i appears from the normalization condition).

4. STEADY-STATE DISTRIBUTION

If we initially excite one level then after some time the localization will lead to the steady-state distribution of the probabilities over unperturbed levels

$$\bar{f}(n)=\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau d\tau\, f(n,\tau).$$

The expression for this distribution function may be obtained by using the QEEs with quasi-energies ε_m. If initially only the $n=0$ level was excited, then

$$\bar{f}(n)=\sum_m |\varphi_m(0)\,\varphi_m(n)|^2. \qquad (15)$$

From this expression it is easy to see that $\bar{f}(n)$ is analogous to the density-density correlation function in solid state problems [19]. The numerical experiments [13, 20] have shown that $\bar{f}(n)\propto\exp(-2|n|/l_s)$ and $l_s\approx D=2l$ (see Fig. 1). The cause of the difference between l and l_s is apparently related to fluctuations of the QEE $\varphi_m(n)$, as it was in solid-state problems. The QEE may be written in the form $\varphi_m(n)\propto\exp(-\gamma|n-m|+\xi_{mn})$ where ξ_{mn} describes fluctuations with zero average $\langle\xi_{nm}\rangle=0$. However, the second moment may grows linearly with n: $\langle(\Delta\xi_{mn})^2\rangle=D_\xi\Delta n$, that leads to different values of $(1/n)\ln\langle|\varphi_m(n)|\rangle$ and $\langle(1/n)\ln|\varphi_m(n)|\rangle$. For Gaussian fluctuations we obtain:

$$\langle|\varphi_0(n)|\rangle\sim\int_{\gamma n}^{\infty}\exp(-\gamma n+\xi)\exp\left(-\frac{\xi^2}{2D_\xi n}\right)d\xi\sim e^{-\gamma_s n}.$$

Then $\gamma_s=1/l_s$ and

$$\gamma_s=\gamma-\frac{D_\xi}{2},\quad \gamma\geqslant D_\xi;\qquad \gamma_s=\frac{\gamma^2}{2D_\xi},\quad \gamma\leqslant D_\xi. \qquad (16)$$

The value of D_ξ may be determined by the LEX method from $\ln\|u_n\|$. To this end, the whole interval n_m was divided into Δn_1 parts. The value of $(\Delta\xi_{mn})^2$ was computed on each part and then the average $\langle(\Delta\xi_{mn})^2\rangle$ of these values was determined. The γ value

was computed in the whole interval n_m. The results of numerical experiments have shown that $\langle(\Delta\xi_{mn})^2\rangle$ indeed grows linearly with n and allowed to find D_ξ in the dynamical Lloyd model and in the quantum standard map. In the former the whole interval $n_m\approx5\cdot10^6$ was divided into $\Delta n_1=500$ and 1000 parts. The parameter k was changed in the interval $2\div100$. No dependence on T was observed. The results obtained (Fig. 5) show that $lD_\xi=2$ for $l\gg1$ and therefore $l_s=4l$. This ratio is the same as the theoretical one for the case of one-dimensional random potential when the localization length is much larger than the distance between sites [19, 35].

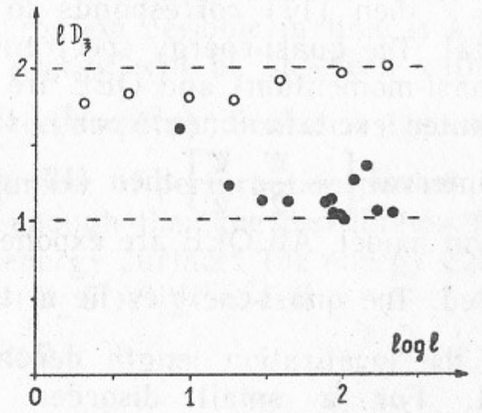

Fig. 5. The product of diffusion rate D_ξ in QEE by localization length for different l. Circles are for the dynamical Lloyd model ($T=1$), points for the quantum standard map.

The situation is more complicated for the model (4). In this case we used $n_m\approx10^6$, $\Delta n_1=200\div1000$. The dependence of lD_ξ on l obtained from $\|u_n^{(1)}\|$ is shown in Fig. 5 for the region of strong chaos ($5\leqslant K\leqslant10$, $5\leqslant k\leqslant20$, $T\leqslant1$). It is seen that for $l\gg1$ the value of lD_ξ is close to one ($\langle lD_\xi\rangle=1.14$). Therefore, $l_s=2l$, according to (16), in agreement with the results of Ref. [13]. The cause of different values of the ratio l_s/l for these two models is again related to the fact that in the Lloyd model there are only two interacting sites, while in the other this number of sites is $N\sim k/2\gg1$. A special check has shown that $lD_\xi=1$ also in the model (4) with random phases χ_n. However, the value of lD_ξ decreases with K for $K\lesssim1$ and for $K\ll1$ and $k\gg1$ reaches the limit $lD_\xi\approx1/4$. No theory exists so far to explain this behaviour.

5. THE AKULIN—DYKHNE MODEL

An interesting model of molecular excitation was introduced in Ref. [17]: in the vicinity of the energy ε_{nm} there is a zone of N nearly levels labelled by index m, and a monochromatic field $\mathscr{E}$ produces dipole transitions with $\Delta n = \pm 1$, and matrix element μ. The case of a small zone width $W \ll \omega$ and small perturbation $V = \mathscr{E}\mu \ll \omega$ is considered. Then one can neglect nonresonant terms and obtain the stationary equation for QEE [17]:

$$\varepsilon \psi_{nm} = \Delta_{nm} \psi_{nm} + \sum_{n',m'} V_{nm}^{n'm'} \psi_{n'm'} \qquad (17)$$

where $\Delta_{nm} = \varepsilon_{nm} - n\omega$.

Now we consider several different cases. If $N=1$, $\Delta_{nm}=0$, $V_n^{n'} = V$ then (17) corresponds to the case of a one-dimensional crystal. The quasi-energy spectrum is continuous $\varepsilon(p) = 2V\cos p$ (p is quasi-momentum) and QEE are delocalized thereby leading to an unlimited excitation ($\langle n^2 \rangle \propto \tau^2$). If Δ_n are randomly distributed on the interval $\left[-\frac{W}{2}, \frac{W}{2}\right]$ then (17) represents a one-dimensional Anderson model. All QEE are exponentially localized, and excitation is limited. The quasi-energies lie in the interval $\left[-\frac{W}{2} - 2V, \frac{W}{2} + 2V\right]$ and the localization length depends on ε. For $W \gg V$ the length $l \ll 1$. For a small disorder $W \ll V$ and $\varepsilon \sim W$ the length $l \approx 100(V/W)^2$ (see, for example, [36]).

For a zone with an unlimited number of levels, randomly distributed Δ_{nm} and

$$V_{nm}^{n'm'} = V[\delta_{n,n'-1}(\delta_{m,m'-1} + \delta_{m,m'+1}) + \delta_{n,n'+1}(\delta_{m,m'-1} + \delta_{m,m'+1})]$$

the model (17) corresponds to the two-dimensional Anderson model. According to [22, 31] all states in the model are localized and for $\varepsilon \lesssim W$ the localization length varies from $l \sim 1$ at $V/W \approx 0.07$ up to $l \sim 10^6$ at $V/W = 0.5$. Asymptotically, $\ln l_\infty \sim (V/W)^2$ and QEE decay exponentially with n: $\psi_n \propto \exp(-|n|/l_\infty)$. For $1 \ll N \ll l_\infty$ the localization length is proportional to N: $l \sim N(V/W)^2$. In the case when all $\Delta_{nm} = 0$ and only the matrix element $V_{nm}^{n'm'}$ fluctuates all QEE are also localized.

The typical variant of the Akulin—Dykhne model in which transitions from one to many sublevels of neighbouring zones are allowed is analysed in Section 7.

6. ONE-DIMENSIONAL LOCALIZATION IN MANY-DIMENSIONAL SYSTEMS

At first glance it seems that the localization of chaos takes place only in one-dimensional systems (Sec. 2) and is therefore a rather special phenomenon. However, it may happen that under some appropriate conditions such a localization occurs also in many-dymensional systems. Consider a conservative system with d degrees of freedom, energy E and density of levels $\rho(E)$. We assume that the motion of the system is chaotic and that energy is the only motion integral. Then the motion on the energy surface is diffusive and may be characterized by a diffusion rate $D_\parallel$. If $D_\parallel$ is nearly independent of the direction then the time of spreading of a narrow quasi-classical distribution function throughout the suface is equal to $\tau_\parallel \sim E^2/D_\parallel$. If an external perturbation periodic in time is added, then also a diffusive growth of energy will be observed with the rate $D_\perp = \frac{(\Delta E)^2}{\Delta\tau}$. This diffusion proceeds perpendicular to the energy surface, and τ is measured in a number of perturbation periods.

If the latter diffusion is slow enough then the distribution function has time to cover the whole energy surface. The energy change in time $\tau_\parallel$ is relatively small $\Delta E = (D_\perp E^2/D_\parallel)^{1/2} \ll E$ if

$$D_\perp \ll D_\parallel. \qquad (18)$$

Under this condition we conjecture that the number of excited levels grows as it were a one-dimensional diffusion: $\Delta n = \rho \Delta E \sim \rho\sqrt{D_\perp \tau}$. Here we assume that in the interval ΔE all levels are excited with approximately equal probabilities. This condition is satisfied if there are no special selection rules for the matrix elements. Besides that, for high frequency $\omega \gg 1/\rho$ it is necessary to have $\mu\mathscr{E} \gtrsim \omega$, otherwise the excitation of nonresonant levels will be small. If these conditions are satisfied then the expression for localization length can be obtained, as in the one-dimensional case, from equation (7):

$$l = \frac{1}{2}\rho^2 D_\perp \sim \Delta n \sim \tau_D, \quad l_E = \frac{l}{\rho} \qquad (19)$$

where l_E is the localization length in energy scale. The conditions of applicability for this equation are a large number of absorbed photons $N_\Phi = l_E/\omega \gg 1$, and excited levels $\Delta n \sim l \gg 1$. For a monochro-

matic perturbation the diffusion goes on only for $\rho\omega \gtrsim 1$ and the first condition is decisive. Also, it is necessary to have $\tau_D \gg \tau_\parallel$, otherwise the localization will take place faster than the spreading of the wave function on the whole energy surface, and the l value will be smaller than in (19).

The expression (19) holds in the case of inhomogeneous localization (when l_E depends on E) only if $l_E \sim \rho D_\perp \ll E$. Instead, for $l_E \gtrsim E$ delocalization takes place. For example, if $\rho D_\perp = \rho_0 E^\beta$ then delocalization occurs for $\beta \geqslant 1$ and $\rho_0 \gtrsim E^{1-\beta}$ (see, also [13, 16]).

Assuming the above picture is true, we discuss the motion of a particle in a two-dimensional billiard under the influence of a periodic perturbation as an example of the described effect. Let the Hamiltonian of the system be

$$H = \frac{\hat{p}_1^2}{2} + \frac{\hat{p}_2^2}{2} + k \sin x_1 \sin\left(\frac{x_2}{\sigma}\right) \delta_T(t) \qquad (20)$$

and the wave function is equal to zero on the border of the billiard, $\sigma \approx 1$. In the case of right-angled billiard with $0 \leqslant x_1 \leqslant \pi$, $0 \leqslant x_2 \leqslant \pi\sigma$ the unperturbed spectrum is $E_{n_1 n_2} = (n_1^2 + n_2^2/\sigma^2)/2$. The average density of levels is constant: $\rho \sim 1$. For $kT \gg 1$ the motion can be shown to be chaotic and diffusive excitation takes place in each degree of freedom: $(\Delta n_1)^2 \approx (\Delta n_2)^2 = D\tau$ with $D \sim k^2$. For $D \ll 1$ the localization length $l \ll 1$ (quantum border, see Sec. 2, Ref. [6]) and almost the whole probability is concentrated on the initial level. For $D \gg 1$ the localization length is exponentially large: $\ln l \sim D$ (see (8) and [22, 31]). Let now the border be deformed inwards, which corresponds to the case of Sinai's billiard (see, for example, [10]). Then the dynamics would be chaotic even in the absence of the external perturbation. After one collision with a border $\Delta p \sim p$ (the case of strong deformation). Taking into account that the time between collisions $\sim E^{-1/2}$ we obtain $D_\parallel \sim E^{5/2} T$. The diffusion rate is equal to $D_\perp \sim Ek^2$. For high energies the conditions (18) and $\tau_\parallel \ll \tau_D$ are satisfied and we predict one-dimensional localization (19) with $l_E \sim Ek^2$. Due to the dependence of l_E on E localization takes place only for $k \lesssim 1$. For $k \gtrsim 1$ the QEEs are delocalized due to increase in diffusion rate. The significant difference from the integrable case (right-angled billiard) consists in the fact that even for small perturbation ($k \ll 1$) the localization length may be large enough $l \sim l_E \gg 1$ if $E \gg k^{-2}$.

7. LOCALIZATION OF PHOTON TRANSITIONS

In the previous section we calculated $D_\perp$ as the diffusion rate in the classical system. For a monochromatic field there is another way of calculating $D_\perp$. It is based on the expression for the probability of one-photon transition per unit time:

$$w = \frac{\pi}{2} |\mu(E, E+\omega)|^2 \mathscr{E}^2 \rho .$$

In such an approach absorption and reabsorption of photons leads to diffusive excitation of the system. At $\omega\rho \gtrsim 1$, according to [14, 17, 37],

$$D_\perp = \frac{(\Delta E)^2}{\Delta\tau} = 2w\omega^2 \frac{2\pi}{\omega} = 2\pi^2 \mu^2 \mathscr{E}^2 \rho\omega . \qquad (21)$$

From (19) and (21) we obtain the localization length of QEEs which is conveniently measured in the number of absorbed photons:

$$l_\Phi = \frac{l_E}{\omega} = \pi^2 \mu^2 \mathscr{E}^2 \rho^2 = \pi D_\Phi \rho , \qquad (22)$$

where $D_\Phi = \pi\mu^2\mathscr{E}^2\rho$ is the diffusion rate in the number of photons per unit time. This result does not depend on the field frequency and therefore it is natural to think that the assumed condition $\mu\mathscr{E} \gtrsim \omega$ may not be necessary at all for the derivation of (22). In order to confirm this conjecture let us consider the situation when $\rho^{-1} \ll \mu\mathscr{E} \ll \omega$ that corresponds to the typical variant of the Akulin — Dykhne model (see Section 4 in Ref. [17]). In this case, an effective excitation takes place only for the levels close to resonance, lying in a zone of width $\Delta E \sim \mu\mathscr{E}$ near the energies of one-photon transitions $n\omega$ (levels with $\Delta_{nm} \lesssim \mu\mathscr{E}$ in (17)) [17]. According to (21) and [17], the number of absorbed photons (number of zones) grows diffusively with time: $N_\Phi = \sqrt{\pi\mu^2\mathscr{E}^2\rho t}$. In each zone the number of excited levels is of the order of $\mu\mathscr{E}\rho$. Then the whole number of excited levels is $N \sim \sqrt{\mu^2\mathscr{E}^2\rho t}\,\mu\mathscr{E}\rho$. Since the excited levels lie in narrow zones of width $\mu\mathscr{E} \ll \omega$ near the energies $n\omega$, then all quasi-energies also lie in an interval of width $\sim\mu\mathscr{E}$. Their average spacing is then $\Delta\varepsilon \sim \frac{\mu\varepsilon}{N}$. In the same way as for (7), from the uncertainly relation, we obtain an estimate for the time after which diffusion limitation occurs:

$$t_D \sim \frac{1}{\Delta\varepsilon} \sim N_\Phi \rho \sim \rho \sqrt{\mu^2 \mathscr{E}^2 \rho t_D}$$

and for the localization length $l_\Phi \sim N_\Phi \sim \mu^2 \mathscr{E}^2 \rho^2$. The obtained result is applicable in the quasi-classical region when $l_\Phi \gg 1$. For $\mu \mathscr{E} \ll \omega$ the steady-state distribution looks like a chain of equally spaced (in energy) peaks which maxima are exponentially decaying. Since, as in the case of the quantum standard map, the number of interacting sites is large the localization length of steady-state is equal to $l_{\Phi s} = 2 l_\Phi$. It is important to note that the localization is homogeneous when $\mu \mathscr{E} \rho = \text{const}$. Therefore, quantum effects lead to localization of photon transitions and to limitation of the system excitation.

The estimates obtained for localization length of QEEs allow to investigate the excitation of different multilevel systems by periodic field when the standard perturbation theory is inapplicable.

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Localization of Diffusive Excitation in Multi-Level Systems

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