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The Kicked Rotator as a Limit of the Kicked Top.

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Abstract. – We compare the classical and quantum behaviour of the kicked rotator and a kicked three-dimensional top. The most prominent classical difference lies in the topology of the respective phase spaces which is cylindrical for the rotator and spherical for the top. Quantum mechanically, localization of the angular momentum is possible for the rotator but not for the top. We show, however, that in a very special limit the top goes over into the rotator. Incidentally, that limit illustrates the nonuniqueness of quantization.

For nearly a decade the periodically kicked rotator has yielded an important testing ground for the study of classical chaos and its modifications by quantum effects. Its classical stroboscopic description, Chirikov's «standard» map [1], yields a dominantly regular behaviour in the cylindrical phase space for small kicking strengths; at a certain critical kicking strength the last KAM trajectory disappears and global diffusion sets in. Quantum mechanically, a different behaviour arises depending on whether the driving frequency and the natural quantum frequency of the free rotator have a rational or an irrational ratio γ [2]. In the irrational case, there is convincing evidence for dynamical localization of quasi-energy eigenfunctions in the momentum representation (analogous to Anderson localization in a random lattice), a pure-point spectrum of quasi-energies, level clustering even for kicking strong enough to produce classical chaos, and quasi-periodic (*i.e.* recurrent) temporal behaviour of expectation values. In the rational case (when the ratio in question can be represented as $\gamma = M/N$ in terms of integers M and N without common divisors; it is important to note that M/N is a dimensionless measure of Planck's constant, $M/N \ll 1$ implying the semi-classical limit) the eigenvalue problem for the quasi-energies is equivalent to the Schrödinger equation for a one-dimensional tight-binding model of a particle in a periodic potential. The quasi-energy eigenfunctions are, therefore, not localized and the quasi-energies continuously fill N bands [3]. Within each band, different quasi-energies are continuous functions of a Bloch wave number α [4]. Even though in the rational case the quasi-energy eigenfunctions are of the Bloch type rather than localized, the concept of the localization length can still be useful in a slightly modified sense; if the localization length $l/\hbar$ for typical irrational γ close to M/N is much less than N (with $N, M \gg 1$, $M/N = \mathrm{const}$), the quasi-energy eigenfunctions have moduli of the form, in the momentum representation, of

N-periodic chains of localized exponentials. The N quasi-energies at fixed α lend themselves to statistical analysis if $N \gg 1$. In the semi-classical limit $M/N \ll 1$ with $l/\hbar \ll N$ the N quasi-energies in question turn out to be uncorrelated among themselves and thus to have a Poisson distribution of their nearest-neighbour spacings; in the regime $l/\hbar > N$, on the other hand, level repulsion prevails according to a Wigner distribution of the spacings [5]. As regards the time evolution of expectation values, the rational case does not imply quasi-periodicity, due to the presence of continua of quasi-energies. As a further complication of the already intricate quantum behaviour just sketched, the possibility of nonrecurrent (*i.e.* not quasi-periodic) time evolution even for certain irrational values of the frequency ratio mentioned has been demonstrated recently [6].

Quite different and, in fact, a lot simpler is the behaviour of the periodically kicked three-dimensional top [7]. Here, the classical phase space has the topology of a sphere, since the squared angular momentum is conserved (see below). Correspondingly, the quantum-mechanical Hilbert space has the finite number of dimensions $2j+1$ with the quantum number j integer or half-integer. The classical limit is approached as j grows large. As the kicking strength is increased there is again, classically, a transition from dominantly regular to dominantly chaotic motion. That transition is paralleled by a change in the statistics of the $(2j+1)$ quasi-energies if $j \gg 1$. Level clustering according to a Poissonian spacing distribution is concomitant with mostly regular classical motion; level repulsion *à la* Wigner accompanies the classical predominance of chaos. The temporal evolution of expectation values is quasi-periodic in all cases, but takes significantly different forms for weak and strong kicking: a nearly periodic sequence of collapse and revival with a quasi-period of order j is typical of the weak kicking leading to near-integrable classical dynamics, while erratic recurrencies arise together with classical chaos.

To reveal the difference but also certain similarities between the rotator and the top, we need to look at the Hamiltonians [1]

$$\begin{cases} H_{\mathrm{R}} = \dfrac{1}{2I} p^2 + k \cos\varphi \displaystyle\sum_{n=-\infty}^{+\infty} \delta(t-n)\,, \\[2ex] H_{\mathrm{T}} = \dfrac{\tau}{2\hbar j} J_z^2 + \alpha J_x \displaystyle\sum_{n=-\infty}^{+\infty} \delta(t-n)\,, \end{cases} \tag{1}$$

in which p and φ are canonical variables for the rotator, while $\boldsymbol{J}$ is the angular-momentum vector for the top. The respective quantum commutation relations read

$$[p, \varphi] = \hbar/i \qquad \text{and} \qquad [J_i, J_j] = i\hbar\varepsilon_{ijk} J_k \tag{2}$$

and the corresponding Poisson brackets arise as usual as $[\,,] \to (\hbar/i)\{\,,\}$. Due to the periodicity of the kicking potential, the phase space of the rotator is the cylinder $-\infty < p < \infty$, $0 \leqslant \varphi < 2\pi$. As regards the phase space of the top, we must first note that the obvious conservation law

$$\boldsymbol{J}^2 = \hbar^2 j(j+1) \tag{3}$$

restricts the classical motion of $\boldsymbol{J}$ to a sphere. That sphere is revealed as a phase space by

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[1] Note that we work with a dimensionless time. The Hamiltonians (1) thus have the dimension of an action.

introducing the canonical pair P, φ as

$$(J_x, J_y) = \sqrt{\hbar^2 j(j+1) - p^2}\,(\cos\varphi, \sin\varphi),\ J_z = P\,. \tag{4}$$

There is a certain similarity between the two Hamiltonians given in (1). Each contains a quadratic piece which by itself is integrable. For the rotator that piece describes free rotation with constant momentum p; the classical phase space trajectory is a circle around the cylinder. Similarly, the motion of the top generated by the quadratic term in H_{T} is a «nonlinear» precession of $\boldsymbol{J}$ around the z-axis with angular velocity $\tau J_z/\hbar j$, J_z being conserved; in the classical phase space, $\boldsymbol{J}$ moves uniformly around the circle along which the sphere (3) intersects the plane $J_z = \text{const}$. In a somewhat pictorial language the parameter τ might be called a torsion strength, since points on the sphere with $J_z > 0$ and with $J_z < 0$ rotate in opposite senses. By analogy with the parameter I of the rotator $\hbar j/\tau$ may also be looked upon as a moment of inertia of the top.

The second terms in the Hamiltonians reveal the different topologies of cylinder and sphere somewhat more ostensibly. If we consider them by themselves, *i.e.* set $\tau = 0$ and $I = \infty$, we obtain a sequence of impulsive precessions around the x-axis for the top, the angular increment α arising at each kick; for the rotator, on the other hand, we get a stepwise advance of p by $\varkappa \sin\varphi$ at each kick with a constant angle φ. However, even these second pieces are not without resemblence: as long as $J_z^2 \ll \hbar^2 j(j+1)$, the transformation (4) shows the two Hamiltonians to be essentially equivalent. We shall come back to this important point below.

Due to the modulation of the Hamiltonians (1) by a periodic train of delta-functions a stroboscopic description of both dynamics is indicated and easily accessible. Denoting the momentum of the rotator right after and right before the n-th kick by p_{n+1} and p_n, respectively, and the angle at the n-th kick as φ_n, we have the stroboscopic equations of motion

$$p_{n+1} = p_n + k \sin\varphi_n, \qquad \varphi_{n+1} = \varphi_n + p_{n+1}/I\,, \tag{5}$$

which hold both classically (as Hamilton's equations) and quantum mechanically (as Heisenberg's equations). Similarly, with $\boldsymbol{X}_n = \boldsymbol{J}_n/\hbar j$ as a rescaled angular-momentum vector for the top and with the classical limit $j \to \infty$ performed, we obtain the classical map

$$\begin{cases} X_{n+1} = X_n \cos\tau\tilde{Z} - \tilde{Y}\sin\tau\tilde{Z}\ , \\ Y_{n+1} = X_n \sin\tau\tilde{Z} + \tilde{Y}\cos\tau\tilde{Z}\ , \\ Z_{n+1} = \tilde{Z}\ , \\ \tilde{X} = X_n\ , \\ \tilde{Y} = Y_n \cos\alpha - Z_n \sin\alpha\ , \\ \tilde{Z} = Y_n \sin\alpha + Z_n \cos a\ , \end{cases} \tag{6}$$

which clearly displays the sequence $\boldsymbol{X}_n \to \tilde{\boldsymbol{X}}$, a linear precession around the x-direction, and $\tilde{\boldsymbol{X}} \to \boldsymbol{X}_{n+1}$, a nonlinear precession around the z-direction. The quantum version of the stroboscopic motion of $\boldsymbol{J}$ looks slightly more complicated than (6), due to the occurrence of products of the noncommuting components of $\boldsymbol{J}$ [7*d*]; we shall not need the explicit form of the quantum map here.

Having presented the Hamiltonians as well as the stroboscopic equations of motion for the rotator and the top, it is well to come back, for a moment, to the general discussion given at the beginning of this paper. That discussion rests on the assumption that the parameters I and k for the rotator and, more importantly for the argument to follow, the torsion strength τ and the precession angle α for the top are kept constant when semi-classical, and eventually, classical behaviour is enforced by letting $j \to \infty$. Indeed, the transition from mostly regular to globally chaotic behaviour with increasing, for instance, k for the rotator and τ for the top, takes place when all of the parameters k/I, α and τ are of order unity. In spite of being globally chaotic right above the respective thresholds, both systems still display an interesting difference classically. The momentum of the rotator cannot change, through a single kick, by more than k and that increment is a vanishing small fraction of the infinite length of the cylindrical phase space. In its diffusive motion the phase point covers a long distance Δp in a very long time $\sim (\Delta p)^2$ only. In contrast, the phase point of the top is capable of going around the sphere in a single kick as soon as global chaos has set in. A quantum analogue of the distinction in question is the possibility and impossibility of localization for the rotator and the top, respectively.

We finally turn to the main goal of this paper, proposing the rotator as a very special limit of the top. The nature of that limit has already been touched upon in our qualitative comparison of the two Hamiltonians above. We must scale the torsion strength τ and the precession angle α so as to confine the classical trajectory of the top to a narrow equatorial waist band, $|J_z| \ll \hbar j$, of the sphere. Indeed, from within such a band the sphere cannot be told apart from a cylinder. Formally, the confinement required is achieved by $\alpha \to 0$ and $\tau \to \infty$; it is most intuitive to combine that limit with letting the radius of the sphere grow large and set

$$\alpha = k/\hbar j\,, \qquad \tau = \hbar j/I\,, \qquad j \to \infty\,. \tag{7}$$

In the same spirit we represent the components of the classical vector $\boldsymbol{X} = \boldsymbol{J}/\hbar j$ as

$$X = \cos\Phi\,, \qquad Y = \sin\Phi\,, \qquad Z = P/j\hbar\,. \tag{8}$$

It is a trivial matter to verify that with the transformation (7), (8) the stroboscopic equations of motion (6) of the top go over into those of the rotator (5). The Hamiltonians (1), taken as classical Hamiltonian functions, assume equal appearance as well.

Similarly, the semi-classical version of the quantum top is turned into the quantum rotator. Indeed, if we rescale our *operators* as

$$J_x = \hbar j\hat{X}\,, \qquad J_y = \hbar j\hat{Y}\,, \qquad J_z = \hat{P} \tag{9}$$

the angular-momentum commutators given in (2) read

$$[\hat{X}, \hat{Y}] = i\hat{P}/\hbar j^2\,, \qquad [\hat{Y}, \hat{P}] = i\hbar\hat{X}\,, \qquad [\hat{P}, \hat{X}] = i\hbar\hat{Y}\,. \tag{10}$$

By dropping the $1/j^2$ term in the first of these, we obtain «semi-classical» commutators which can be realized by

$$\hat{X} = \cos\Phi\,, \qquad \hat{Y} = \sin\Phi, \qquad \hat{P} = \frac{\hbar}{i}\frac{\partial}{\partial\Phi}\,. \tag{11}$$

Moreover, the transformation (7), (9), (11) gives an identical form to the quantum Hamiltonians H in (1).

An equivalent way of reducing the (semi-classical) quantum top to the quantum rotator goes via the unitary evolution operators which carry the respective quantum states from immediately after one kick to right after the next,

$$\begin{cases} U_{\mathrm{R}} = \exp\left[-ip^2/2I\hbar\right] \exp\left[-i(k/\hbar)\cos\varphi\right], \\ U_{\mathrm{T}} = \exp\left[-i\tau J_z^2/2\hbar j\right] \exp\left[-i\alpha J_x/\hbar\right]. \end{cases} \tag{12}$$

The left factors in these operators are diagonal in the eigenrepresentations of p and J_z

$$\begin{cases} p|m\rangle = \hbar m|m\rangle, & m = 0,\ \pm 1.0,\ \pm 2.0,\ \ldots \\ J_z|m\langle = \hbar m|m\rangle, & -j \leqslant m \leqslant j. \end{cases} \tag{13}$$

In fact, their diagonal matrix elements coincide once we set $\tau = j/I$ according to (7). The exponents in the right-hand factors in (1) read, in representations (13) and with $\alpha = k/\hbar j$,

$$\begin{cases} \langle m|k\cos\varphi|m'\rangle = \dfrac{1}{2}k\left\{\delta_{m',m+1} + \delta_{m',m-1}\right\}, \\ \langle m|\alpha J_x|m'\rangle = \dfrac{1}{2j}k\left\{\sqrt{(j-m)(j+m+1)}\,\delta_{m',m+1} + \sqrt{(j+m)(j-m+1)}\,\delta_{m',m-1}\right\}. \end{cases} \tag{14}$$

The matrices (14) become identical for $m/j \to 0$ which limit implements the semi-classical approximation and also is the quantum analogue of looking at a narrow equatorial waist band of the classical sphere.

It is instructive to check the consistency of localization with the limit (7). To that end, we first recall the order-of-magnitude estimate for the classical diffusion constant D of the rotator [8], $D \sim k^2$. From a quantum point of view, the diffusive behaviour $\langle(\Delta p)^2\rangle \sim Dt$ can prevail only until the momentum spread has reached the localization length l; this consideration yields a break time t^* through $Dt^* \sim l^2$. An independent estimate is the time needed to resolve the discreteness of the Floquet spectrum; the number of Floquet eigenstates supporting a wave packet at the break between diffusion and quasi-periodic behaviour is $\sim l/\hbar$; the mean separation of the corresponding quasi-energies is $2\pi\hbar/l$ and gives, as its inverse, the break time $t^* \sim l/\hbar$. The two independent estimates for t^* imply a relation between the kicking strength k and the localization length [9]

$$l \sim k^2/\hbar. \tag{15}$$

For localization to arise this length must be small compared to $\hbar j$, *i.e.*

$$k \ll \hbar\sqrt{j}. \tag{16}$$

The latter condition is certainly fulfilled in the limit (7), *i.e.* for $j \to 0$ with k fixed.

We would like to conclude by remarking that the classical Chirikov map (5) may be quantized in many different ways. The canonical one employs the first of the commutators in (2) and leads to the Floquet operator U_{R} in (12). An equally legitimate quantization uses the angular-momentum commutators given in (2) or (10), leads to the Floquet operator U_{T} in (12), and is undone by the limit (7).

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