Physica A 163 (1990) 205–214
North-Holland

CLASSICAL CHAOS, QUANTUM LOCALIZATION AND FLUCTUATIONS: A UNIFIED VIEW

G. CASATI* and I. GUARNERI**
Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA

D.L. SHEPELYANSKY
Institute of Nuclear Physics, 630090 Novosibirsk, USSR

The localization phenomenon plays an essential role in the excitation of hydrogen atoms in microwave fields. The similarity of this phenomenon to Anderson localization, which has been established and discussed in previous papers, is here demonstrated in a transparent way by reducing the quantum dynamics to a one-dimensional pseudo-Anderson model. The effect of slight changes in the driving frequency on the ionization probability is numerically investigated on the quantum Kepler map; huge fluctuations are found, which persist down to a very fine frequency scale and have a qualitatively random nature. It is argued that such fluctuations are a counterpart of the mesoscopic fluctuations of solid-state physics, and that similar fluctuations should be expected any time, when some classical chaotic diffusive process is quantum-mechanically suppressed by dynamical localization.

1. After some ten years of investigations on the quantum dynamics of classically chaotic systems, the basic question whether in quantum mechanics anything survives of the impressive manifestations of classical chaos has not yet been answered clearly. If a general indication is to be drawn from the analytical, numerical and experimental results obtained up to now, it is that classical chaos is suppressed or at least strongly inhibited by quantization; as a consequence, "quantum chaos" – a widespread denomination for this research area – is still considered a questionable concept. In any case, the state of affairs is quite far from being satisfactory; should the above indication be confirmed, then this would imply that chaos is absent in our microscopic world and the different fluctuation phenomena observed in quantum systems would have no relation with deterministic chaotic motion.

As a matter of fact, theoretical analysis and numerical simulations have confirmed an important feature of quantum motion which was discovered many

* Permanent address: Dipartimento di Fisica dell'Universita', Via Celoria, 16-20133 Milano, Italy.
** Permanent address: Dipartimento di Fisica Teorica e Nucleare dell'Universita', 27100 Pavia, Italy.

0378-4371/90/$03.50

years ago [1]; namely, quantum interference sets severe restrictions on the diffusive excitation which would be started in a classical system upon entering the chaotic regime. This phenomenon is similar to the Anderson localization in solid-state physics [2, 3] and is, therefore, called 'dynamical localization'.

Besides that, strong empirical evidence has been obtained that the quantum evolution is very stable, in sharp contrast to the extreme sensitivity to initial conditions which is the very essence of classical chaos [4, 5]. This fact demonstrates that, at least insofar as the measurement process is not taken into account, quantum dynamics is much more predictable than classical dynamics.

The above picture has been confirmed by the analysis of the microwave ionization of highly excited hydrogen atoms. A theoretical analysis based on the concept of dynamical localization and on extensive numerical simulations has provided a fairly good understanding of that problem; more than that, it has yielded some *predictions* about the effect of dynamical localization in this particular system. Unexpected though these predictions may have been at their first appearance [6–10], they were confirmed by recent experimental results on the microwave ionization of hydrogen atoms [11, 12]. In ref. [12], a comparison was made between experimental results, numerical results obtained from the solution of the Schrödinger equation and theoretical predictions from the dynamical localization theory. It was found that experimental and numerical data agree fairly well with the localization theory and at the same time appreciably deviate from classical predictions. The experiments described in ref. [12] were precisely designed for the purpose of checking localization theory; as a matter of fact, special care was taken so that numerical computations could simulate as closely as possible the experimental conditions. Therefore, they provide experimental evidence of the quantum suppression of the classically chaotic diffusion due to the localization phenomenon.

In fig. 1 a comparison of the theory with the experimental data obtained by Galvez et al. [11] is presented. The circles represent the experimentally observed threshold values of the peak–field intensity for 10% ionization. Here, the microwave frequency $\omega/2\pi = 36.02$ GHz, $\epsilon_0 = \epsilon n_0^4$ is the rescaled peak-field intensity, $\omega_0 = \omega n_0^3$ is the rescaled microwave frequency and n_0 is the principal quantum number of the initially excited state. The dotted curve is the classical-chaos border while the dashed line in fig. 1a is the theoretical prediction of localization theory for the 10% threshold value which was derived as shown in the appendix. Both in the experiment and in the quantum numerical computations, the ionization probability is defined as the total probability above a cutoff level n_c. Unlike the previous case of ref. [11], numerical data here (full circles) were obtained from the numerical simulations of the "quantum Kepler map". In such simulations the interaction time, including the switching on and off of the microwave field, was chosen as in

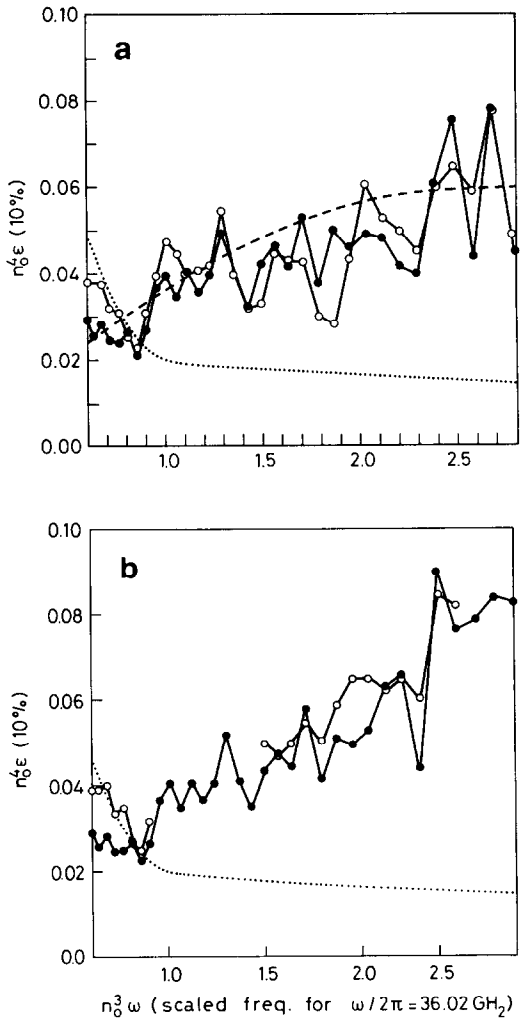

Fig. 1a, b. Scaled 10% threshold fields from experimental results (taken from fig. 2a from ref. [12]), (circles), and from numerical integration of the quantum Kepler map (full circles). Curves have been drawn to guide the eye. The dashed line is the quantum theoretical prediction according to localization theory (see appendix). The dotted curve is the classical-chaos border.

actual experiments. Each numerical point in figs. 1 was obtained by averaging over 11 values of the frequency in an interval $\Delta\nu \simeq 10^{-3}$ around the given value of ω_0. Indeed, due to strong fluctuations (see below) such an averaging is necessary in order to simulate more closely the experimental conditions.

The agreement between experimental and numerical data is the more remarkable, in that the quantum Kepler map is but a crude approximation for

the actual quantum dynamics. In particular, from fig. 1, it is seen that when the principal quantum number n_0 of the initial state is increased, the data follow the predictions of localization theory.

Though the numerical model was one-dimensional, in actual experiments the initially excited state corresponds to a microcanonical distribution over the shell with a given principal quantum number. The classical counterpart for this would be a microcanonical ensemble of orbits. Nevertheless, the experimental data fairly well agree with the predictions of the one-dimensional quantum Kepler map. The reason of this agreement was found in ref. [10]: due to the existence of an approximate integral of the motion, the main contribution to excitation turns out to be given by orbits which are extended along the direction of the (linearly polarized) external field. For such orbits, the use of the one-dimensional model is fully justified (see, e.g., fig. 18b in ref. [10]).

2. We recall that the quantum Kepler map, which was introduced in refs. [9, 10] as a convenient approximation of the actual quantum dynamics of the H-atom in a microwave field, is closely related to the quantum-kicked rotor, i.e., to the very model where dynamical localization has been first identified and unambiguously related to Anderson localization; therefore, the quantum Kepler map is just the theoretical link between Anderson localization and the suppression of chaotic diffusion which takes place in the H-atom.

Nevertheless, the agreement with localization theory holds only "in the average". Indeed, it is seen from fig. 1a that both the numerical and the experimental data exhibit more or less sensible deviations from the average prediction. The same was found in ref. [12]. This is hardly surprising, because the dynamical localization theory was just meant to yield a gross description of the quantum dynamics; actually, as it was pointed out in ref. [10], one would expect even stronger fluctuations than were actually observed. Indeed, according to theory, the quantum distribution is strongly fluctuating around its average exponential shape, and "... *these fluctuations will affect also the ionization rate which will have a very irregular fine structure*" (ref. [10], p. 1429). Similar predictions were also put forth in ref. 19.

The discussion of these fluctuations is the main object of the present paper. We shall demonstrate on numerical data (from the Kepler map) that they are actually much more erratic than one could guess by just inspecting fig. 1. As a matter of fact, they persist on much finer scales. Because of them, the ionization probability (at fixed time and field intensity) depends in such a complicated way on the microwave frequency, that it could be even called a chaotic function of the latter; of course, the word "chaotic' is not a technical term for the time being. We shall show that localization theory provides a natural and very general setting for understanding these fluctuations. Indeed,

they represent a peculiar kind of sensitive dependence on parameters that should be expected to appear any time, when some classical chaotic diffusion is suppressed by quantum localization.

In the following we will briefly recall the connection between the localization in the hydrogen atom problem and the localization phenomenon for one-dimensional disordered lattices. This will lead to assimilate the fluctuations in the hydrogen atom problem to the mesoscopic fluctuations which are attracting much attention in solid state physics.

In the Kepler map formulation, the excitation of a one-dimensional hydrogen atom in a microwave field is described by a quantum map which acts on suitable wave functions. In an appropriate representation, these wavefunctions are defined on a one-dimensional lattice, where the nth site corresponds to the absorption of n photons by the atom. The problem of finding eigenvectors and eigenvalues of the Kepler map can be reduced, as shown in ref. [18], to the solution of

$$\sum_r \Big[\sum_l W_{n-l} \tan\Big(\frac{\nu - H_0(l)}{2} \Big) W_{l-n+r} - \frac{1}{2} \tilde{W}_r \Big] \phi_{n-r} = 0 , \tag{1}$$

where ϕ_n is the wavefunction at site n, ν is an eigenvalue of the map, and the function H_0 is defined as

$$H_0(l) = 2\pi[-2\omega(l - n_I)]^{-1/2}$$

with n_I the number of photons required to go to the continuum from the initial hydrogenic state. Finally, W_r, $\tilde{W}_r$ are Fourier coefficients of $\cos(\frac{1}{2}k\cos\theta)$ and of $\sin(k\cos\theta)$, respectively; therefore, $W_r \sim J_r(k/2)$ and $\tilde{W}_r \sim J_r(k)$. This implies that in eq. (1) a number of sites $\sim k$ are coupled. On the other hand, $H_0(l)$ is not defined for $l > n_I$. This means that the approximation leading to (1) is only justified when $k \ll n_I$.

Eq. (1) has a definite resemblance to the equation for eigenfunctions of a particle in a one-dimensional disordered lattice. The disorder is here associated with the pseudo-random nature of the "potential" $\tan((\nu - H_0(l))/2)$. Since, as remarked above, only sites $l < n_I$ can be considered, eq. (1) establishes a definite connection between the problem of ionization of hydrogen atoms in microwave fields and the problem of localization in one-dimensional *finite* samples.

An important remark is that a change of the microwave frequency ω will modify the pseudo-random potential in (1) so that different values of ω for *one* given atom correspond to *different* samples in the solid-state model. In other words, even a slight change in ω will produce a completely different realization of the pseudo-random potential.

3. The above formalism demonstrates that the problem of microwave excitation of H-atoms is essentially a localization problem (at least, in the case where the frequency is larger than the level spacing). The essential feature of this problem is that the localization problem is actually a problem for a *finite* lattice, because just a limited number of photons can be absorbed before going into the continuum. We have, therefore, two characteristic lengths (in number of photons); one is just the maximum number of photons, and the other the photonic localization length. The theory we have developed [9, 10] rests on the basic assumption that the excitation process and the ionization probability are essentially determined by the ratio of these two characteristic lengths. When this ratio is large, the localization effect will preclude ionization even when the classical dynamics is totally chaotic; in the opposite case, one should expect strong ionization.

When compared to the theory of Anderson localization, this assumption is just the basic *ansatz* of the scaling theory for localization in finite samples [17]. That theory aims at providing a description of how the conductance of a finite sample depends on the size of the sample at very low temperature, and it was found to give a satisfactory *average* picture. Nevertheless, the actual behaviour of a given sample was found to exhibit wild fluctuations around the average scaling behaviour. For example, the dependence of the conductance of a given sample on the number of electrons in the sample (which determines the Fermi energy) is so wild that it has been described as "reproducible chaos" [14]. These fluctuations are qualitatively understood as the result of quantum resonances which take place when the Fermi energy happens to coincide with the energy of some eigenstates of the sample, which behaves like a macroscopic quantum object, with well-defined eigenstates, as soon as the temperature is low enough. The point is, of course, that while these eigenstates are individual and reproducible, they are, nevertheless, random in structure – in other words, they exhibit random fluctuations around their average exponentially localized shape. Since the conductance is determined by the transmission coefficient across the finite sample, it is also affected by analogous fluctuations. In the hydrogen-atom problem, the ionization probability is determined by the rate of exponential decay of those quasi-energy eigenfunctions which have a significant overlap with the initially excited states, over a distance (in number of photons) determined by the experimental conditions. This rate is determined by the localization length. For the *average* of the latter we have a theoretical estimate [6]; the theoretical line in fig. 1 was just obtained from that estimate, as shown in the appendix. Nevertheless, the actual decay rate of q.e. eigenfunctions will fluctuate around the average. Then the Kepler map formalism, suggests that the ionization probability at fixed interaction time and field intensity should display the same kind of fluctuations as conductance does.

We tested this prediction by numerical iteration of the quantum Kepler map under the same conditions as in fig. 1. We investigated the dependence on the scaled frequency ω_0 of the ionization probability; the latter was identified with the total probability over the level $n_c = 92$. Whereas, when producing numerical data for fig. 1, the change in ω_0 was obtained by changing the starting level n_0 at fixed microwave frequency, here n_0 was fixed at 63 and ω was changed instead. Fig. 2a shows the results of 1000 such computations of the ionization

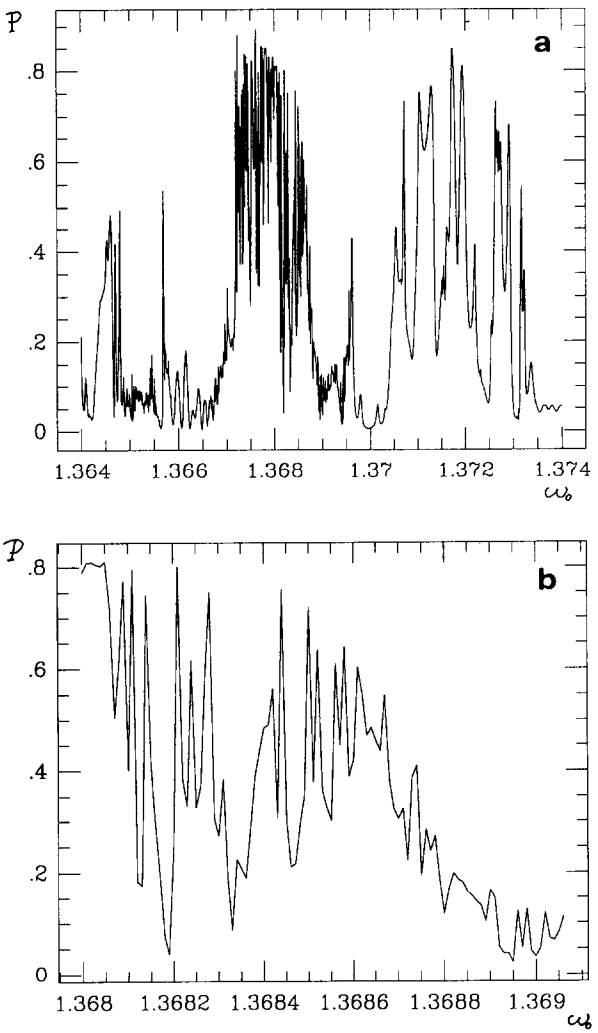

Fig. 2. Ionization probability P as a function of the scaled frequency ω_0 for fixed $n_0 = 63$, $\epsilon_0 = 0.043$. P was obtained by iterating the quantum Kepler map. Fig. 2a shows 1000 values in the interval $1.364 < \omega_0 < 1.374$; fig. 2b shows 100 values in the interval $1.368 < \omega_0 < 1.369$.

probability P, for 1000 different values of ω_0 in the range 1.364–1.374. Small variations in ω_0 caused P to change by two orders of magnitude; the overall picture displays very sharp and narrow peaks. Such a structure persists even on changing ω_0 on much finer scales; fig. 2b shows P for 100 values of ω_0 in the interval 1.368–1.369. These fluctuations can be numerically observed down to a scale $\Delta\nu \sim 10^{-6}$; on smaller scales they disappear. On purely qualitative grounds, the dependence of P looks random; however, whether this qualification would survive a technical analysis is a question which requires more numerical data and which we are currently investigating. The cases of figs. 2 correspond to an average localization length $l \sim 8$ and to a maximum number of photons $N \sim 12$.

4. The above described results point at a peculiar type of quantum fluctuations, which seem intimately connected with the localization phenomenon. However, whereas in the Anderson case the randomness of the fluctuations can be ultimately traced back to the "external" randomness of the potential, in the hydrogen-atom case the only possible source of such fluctuations can be just classical chaos. It is interesting to remark that we have here a kind of "structural" instability, i.e., one which shows up upon changing some external parameter; in the classical case, no such instability can appear (at least, on such small scales), because of the smoothing effect of phase-averaging. One may be tempted to say that the quantum suppression of chaotic diffusion which is produced by quantum interference is being paid at the price of such intrinsically quantum instabilities.

It will be also interesting to ascertain whether and to what extent the fluctuations produced by these instabilities are random; the concept of "mild chaos" introduced by Gutzwiller in connection with fluctuations in the phase shift in quantum scattering problems [15] may be relevant here. In the same vein, it would be interesting to know whether also the Ericsson fluctuations of nuclear physics, for which a relationship to classical "irregular" scattering has been suggested [16], can be reduced within a common, possibly broader, formulation, together with the above discussed ones.

Appendix

We shall here describe the theoretical derivation of the dashed line in fig. 1a, which was first presented in ref. [13]. According to localization theory [9, 10, 13] the quantum excitation process evolves in time towards a (quasi-) stationary distribution which is given in the average by:

$$\bar{f}(N) = \frac{1}{2l}\left(1 + \frac{2|N|}{l}\right) \exp\left(-\frac{2|N|}{l}\right), \tag{A.1}$$

where N is the number of absorbed photons (starting from the initially excited state of principal quantum number n_0) and l is the localization length, given by

$$l \approx 3.33\varepsilon^2\omega^{-10/3}. \tag{A.2}$$

The condition for 10 percent ionization can be identified with the condition for 10 percent probability above the cutoff level n_c; therefore, it reads

$$\int_{N_c}^{\infty} \bar{f}(N)\,\mathrm{d}N = 0.01, \tag{A.3}$$

where

$$N_c = \frac{1}{2\omega}\left(\frac{1}{n_0^2} - \frac{1}{n_c^2}\right).$$

From eqs. (A.1–3) we get

$$\varepsilon_0 \simeq \left[\frac{\omega^{1/6}}{\sqrt{8}}\sqrt{1 - \frac{n_0^2}{n_c^2}}\right]\omega_0. \tag{A.4}$$

The dashed line in fig. 1a is just eq. (A.4) with $n_c = 90$. In experiments, n_c was in the interval 86–92 for the case of fig. 1a and in the interval 160–190 for the case of fig. 1b. In the latter case a higher field is required, the excitation is larger and the localization picture underlying eq. (A.1) is no more valid, because a non-negligible amount of probability flows into the continuum. In that case we have no theoretical prediction for the 10 percent threshold.

References

[1] G. Casati, B.V. Chirikov, J. Ford and F.M. Izrailev, in: Stochastic Behaviour in Classical and Quantum Hamiltonian Systems, Como, June 1977, Lecture Notes in Physics 93 (Springer, Berlin, 1979) p. 334.
[2] S. Fishman, D.R. Grempel and R.E. Prange, Phys. Rev. Lett. 49 (1982) 509; Phys. Rev. A 29 (1984) 1639.
[3] G. Brivio, G. Casati, I. Guarneri and L. Perotti, Physica D 33 (1988) 51.
[4] G. Casati, B.V. Chirikov, I. Guarneri and D.L. Shepelyansky, Phys. Rev. Lett. 56 (1986) 2437.
[5] D.L. Shepelyansky, Physica D 8 (1983) 208.

[6] G. Casati, B.V. Chirikov and D.L. Shepelyansky, Phys. Rev. Lett. 53 (1984) 2525.
[7] G. Casati, B.V. Chirikov, I. Guarneri and D.L. Shepelyansky, Phys. Rev. Lett. 57 (1986) 823.
[8] G. Casati, B.V. Chirikov, I. Guarneri and D.L. Shepelyansky, Phys. Rep. 154 (1987) 77.
[9] G. Casati, I. Guarneri and D.L. Shepelyansky, Phys. Rev. A 36 (1987) 3501.
[10] G. Casati, I. Guarneri and D.L. Shepelyansky, IEEE J. Quantum Electron. 24 (1988) 1420.
[11] E.J. Galvez, B.E. Sauer, L. Moorman, P.M. Koch and D. Richards, Phys. Rev. Lett. 61 (1988) 2011.
[12] J.E. Bayfield, G. Casati, I. Guarneri and D.W. Sokol, Phys. Rev. Lett. 63 (1989) 364.
[13] G.P. Brivio, G. Casati, I. Guarneri and L. Perotti, Physica D 33 (1988) 51.
[14] M.Y. Azbel', A. Hartstein and D.P. Di Vincenzo, Europhys. News 15 (1984) 3.
[15] M.C. Gutzwiller, Physica D 7 (1983) 341.
[16] R. Blümel and U. Smilansky, Phys. Rev. Lett. 60 (1988) 477.
[17] J.L. Pichard, J. Phys. C 19 (1986) 1519.
[18] D.L. Shepelyansky, Physica D 28 (1987) 103.
[19] R. Blümel, S. Fishman and U. Smilansky, J. Chem. Phys. 84 (1986) 2604.
R. Blümel and U. Smilansky, Physica Scripta 35 (1987) 15.