

Diffusion over localized adiabatic states in a modulated quantum kicked rotator

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The introduction of a slow, time-dependent modulation in the kicked rotator problem is shown to produce diffusive excitation, with the diffusion coefficient depending on the small modulating frequency according to a power law. The exponent of this dependence is related to the quasi-energy level statistics.

It is now well established that deterministic classical systems subjected to time periodic perturbations show qualitatively different behaviour depending on the strength of the perturbing field. If the perturbation is larger than a critical value called the chaos border, the trajectories display a very irregular chaotic behaviour, leading to diffusion in the action space.

During the last decade, a great deal of investigations have been devoted to the quantum dynamics of such systems. A most interesting and drastic difference has been observed between the classical and quantum behaviour, namely, the classical chaotic diffusion is suppressed by quantization [1]. This quantum suppression has been christened *dynamical localization*, thus stressing both its analogy and its diversity from the Anderson localization of solid state physics [2]. As a matter of fact, no external random element is responsible for the dynamical localization; in a sense, the randomness is self generated in the system during the time evolution.

Although the theoretical picture of dynamical localization of classical chaos was originally outlined for the kicked rotator model [1-3], the localization phenomenon turns out to be quite general for systems under time periodic perturbations. A well known physical example exhibiting this effect is the microwave excitation of hydrogen atoms [4,5].

It is highly interesting to study how and under what conditions the localization effect can be destroyed, because in this way some important physical processes may be started, such as, e.g., the "diffusive" ionization of hydrogen atoms in the example referred to above. In this line of thought, we study here what happens when the amplitude of the external perturbation is slowly modulated in time. The idea is that the slow modulation may produce a slow diffusive excitation, due to the occurrence of avoided crossing between the instantaneous quasi-energy levels. Such an effect of slow modulation on localization has a distinct physical interest; for example, as will be recalled in the conclusive part of the present paper, the effect of a small constant electric field superimposed on the microwave field in the above quoted problem of microwave excitation of hydrogen atoms can be approximately reduced to a slow modulation in a one-dimensional localization problem.

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The basic object of our investigation is described by the following Hamiltonian,

$$H = \frac{1}{2}p^2 + K(t) \cos \theta \delta_T(t), \quad (1)$$

where

$$\delta_T(t) = \sum_m \delta(t - mT),$$

$$K(t) = k(1 + \epsilon \cos \theta_1), \quad \theta_1 = \Omega t / T.$$

The corresponding classical map is

$$\begin{aligned} p_{m+1} &= p_m + k(1 + \epsilon \cos m\Omega) \sin \theta_m, \\ \theta_{m+1} &= \theta_m + p_{m+1} T, \end{aligned} \quad (2)$$

where p_{m+1} and θ_{m+1} are the values of the momentum and the angle variables immediately after the m th kick. Eq. (2) is the classical map for the modulated kicked rotator. The original kicked rotator corresponds to the case of $K(t) = \text{const}$ in eq. (1), i.e., to $\epsilon = 0$. In this case, if $kT \gg 1$, the classical motion is chaotic and exhibits diffusive behaviour.

Instead, the quantum motion is localized around the initially unperturbed excited state n_0 and, after a while, it settles (on the average) to a steady state probability distribution over the unperturbed levels:

$$\begin{aligned} f(n) &\approx \frac{1}{2l} (1 + 2|n - n_0|/l) \\ &\times \exp(1 - 2|n - n_0|/l), \end{aligned} \quad (3)$$

where l is the localization length. It has been shown [3] that $l \approx \frac{1}{2}k^2$ and that the quasi-energy eigenfunctions are exponentially localized with a localization length $l_E = \frac{1}{2}l$.

This picture, which presents important analogies with the one-dimensional Anderson localization, is sharply modified by the introduction of a modulation. For $\Omega \approx 1$ and $\epsilon \sim 1$, the behaviour of the quantum model (1) is best understood by assimilating it to a two-dimensional solid state model. The possibility of such an identification is easily realized if one considers the extended phase space with additional conjugate variables p_1, θ_1 . Upon introducing such variables, the modulated kicked rotator can be identified with a two-dimensional rotator of the unperturbed Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{\Omega}{T} p_1,$$

subjected to strictly periodic kicks. Then the analogy with the theory of two-dimensional localization in solid state dictates the following picture [7]: the quantum dynamics is still localized in contrast to the unbounded classical diffusion, with the localization length exponentially growing with the diffusion rate:

$$\ln l \propto D_0 \approx \frac{1}{2}k^2(1 + \frac{1}{2}\epsilon^2).$$

However, in the quasi-classical region $k^2 \gg 1$ and l is very large, much larger than any reasonable basis size one could take in numerical simulations. Therefore, such simulations are expected to show a diffusive spread of the wave packet and a Gaussian distribution over the unperturbed levels.

In the present case of slow modulation, namely $\Omega \ll 1$, the perturbation may induce transitions between instantaneous quasi-energy eigenstates over a distance $\sqrt{\Omega}$ (see below and also ref. [6]). If Ω is very small, this distance is much less than the average distance between quasi-energy levels, which is of the order of $1/l$ where l is the localization length for $\Omega = 0$. It thus follows that it is possible to jump from one level to the next only thanks to any fluctuations which may bring the levels closer.

Quantum numerical simulations of model (1) when $\Omega \ll 1$ show an average linear increase of the kinetic energy (fig. 1a):

$$\langle p^2(t) \rangle \approx Dt/T, \quad (4)$$

with superimposed almost regular oscillations. Fig. 1b shows that the distribution over unperturbed levels after 40000 kicks agrees with a Gaussian distribution:

$$f(n, t) = \frac{1}{\sqrt{2\pi Dt/T}} \exp\left(-\frac{(n - n_0)^2}{2Dt/T}\right). \quad (5)$$

Therefore figs. 1a and 1b provide a good empirical evidence that the slow modulation destroys the one-dimensional localization and that the quantum motion is diffusive over the inspected time interval. We shall now sketch a theoretical analysis of this diffusion, based on the statistics of avoided crossings of quasi-energy levels. This will lead to a prediction about the diffusion coefficient D , that will be found to agree with numerical data.

Let the wave function at (integer) time τ be given by $|\psi(\tau)\rangle = S_\tau |\psi(\tau-1)\rangle$ where S_τ is the quantum map over one period of the kicked rotator with the

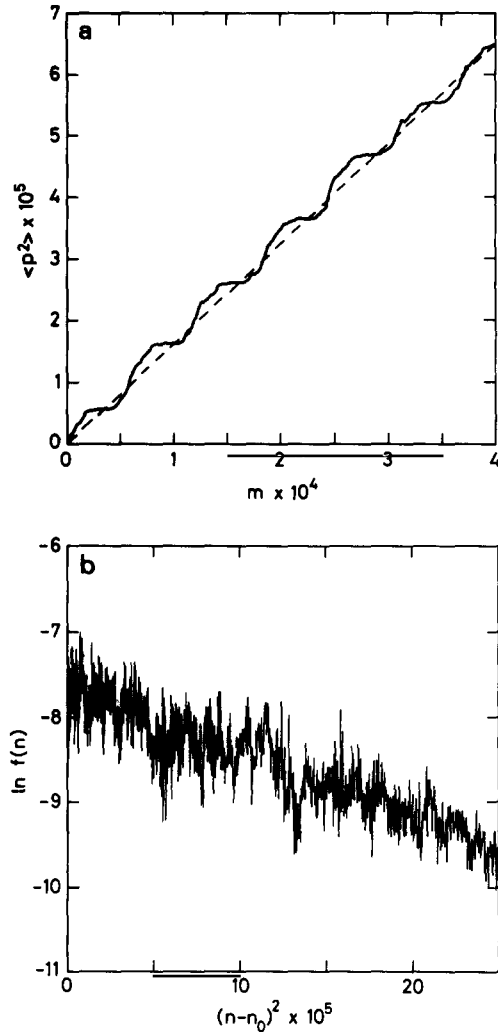


Fig. 1. The average unperturbed energy $\langle p^2 \rangle$ as a function of time measured in number of mapping iterations (a). Probability distribution over the unperturbed momentum states after $m=40000$ mapping iterations (b). Here $k=10$, $\epsilon=0.5$, $T=5/3.5$, $\Omega=0.001$.

instantaneous value $K(\tau)$ of the kick strength. Instantaneous quasi-energy eigenfunctions $\phi_{\lambda\tau}$ ("adiabatic" states) are defined by

$$S_\tau |\phi_{\lambda\tau}\rangle = \exp(i\lambda\tau) |\phi_{\lambda\tau}\rangle. \quad (6)$$

We shall now use a well-known method for the analysis of the adiabatic regime. First we expand the wave function over the instantaneous quasi-energy base according to

$$|\psi(\tau)\rangle = \sum_{\lambda\tau} c_{\lambda\tau}(\tau) \exp\left(i \sum_{\sigma=1}^{\tau} \lambda_{\sigma}\right) |\phi_{\lambda\tau}\rangle.$$

A standard procedure based on first-order perturbation theory can now be applied, the only difference from textbook derivations [6] being that unitary propagators are here involved instead of time-dependent Hamiltonians. As a result we find that if the initial wave packet coincides with ϕ_{λ} , then for small Ω :

$$c_{\mu}(\tau) \sim ik\epsilon\Omega V_{\lambda\mu}(\mu-\lambda)^{-1} \times \{\exp[-i(\lambda-\mu)\tau] - 1\}^{-1}. \quad (7)$$

The subscripts of the quasi-energy eigenvalues have been neglected here, on account of the slight dependence on time of the eigenvalues. $V_{\lambda\mu}$ is the matrix element

$$V_{\lambda\mu} = \langle \phi_{\lambda} | \cos \theta | \phi_{\mu} \rangle = \frac{1}{2} \sum_n \phi_{\lambda}^*(n) [\phi_{\mu}(n+1) + \phi_{\mu}(n-1)], \quad (8)$$

where n labels the momentum eigenstates of the unperturbed rotator. Eq. (7) yields the following condition for significant transitions between the adiabatic quasi-energy states corresponding to quasi-energy eigenvalues λ, μ :

$$(\lambda-\mu)^2 < \Delta_0^2 \sim k\epsilon\Omega V_{\lambda\mu}. \quad (9)$$

We now recall that any eigenfunction $\phi_{\lambda}(n)$ is exponentially localized around some site n_0 , with a localization length $l_E \approx \frac{1}{2}D_0$. Therefore the matrix element given by (8), which depends on the overlap of the quasi-energy eigenfunctions $\phi_{\lambda}, \phi_{\mu}$ will be negligibly small, unless these eigenfunctions are localized not too far from each other, within some "effective distance" of the order of the localization length.

Transitions between adiabatic quasi-energy eigenstates occur when the spacing of the corresponding quasi-energies satisfies the estimate (9). On the other hand, only quasi-energy eigenstates which are localized within an effective distance of order l should be considered, for otherwise the matrix element (8) would be exponentially small. Therefore, the frequency of such transitions is determined by some "effective level spacing statistics" of quasi-energy eigenvalues, constructed by taking into account only

those pairs of eigenstates which lie (roughly) within one effective distance from each other.

Let this statistics be described by a probability density $P(X)$ where, as usual, the spacings X are given in units of the mean level spacing, which in turn is given by $d \approx l^{-1} \approx D_0^{-1}$. The probability of transition will be, in order of magnitude,

$$W \sim \int_0^{x_0} P(X) dX, \quad (10)$$

where X_0 is given by eq. (9):

$$X_0 \sim \Delta_0/d \sim \Delta_0 D_0 \sim D_0 \sqrt{\epsilon \Omega}. \quad (11)$$

In deriving the above estimate, we considered the matrix element (8) as a sum of $\sim l$ random variables, each of order $\sim l^{-1}$, which yields a rough estimate $V \sim l^{-1/2} \approx D_0^{-1/2}$.

Though the level spacing statistics for *all* the spacings of quasi-energy levels of the kicked rotator is Poissonian, the "effective" statistics, which is taken over a finite effective distance, exhibits some degree of repulsion, due to the overlap of localized states. Therefore, the effective statistics $P(X)$ will behave as X^γ for small X . This yields (from (10) and (11))

$$W \sim D_0^{\gamma+1} (\epsilon \Omega)^{(\gamma+1)/2}.$$

It is then reasonable to assume that the diffusion coefficient depends on Ω according to the same law, i.e. (putting $\epsilon \sim 1$) that

$$D \sim W D_0 \sim (D_0^2 \Omega)^\alpha D_0, \quad (12)$$

with

$$\alpha = \frac{1}{2}(\gamma+1).$$

From the above estimate it turns out that, for fixed D_0 and sufficiently small Ω , the diffusion is smaller when the level repulsion is increased.

The above prediction (12) was checked by numerically computing the diffusion coefficient D . In our simulations we initially excited one momentum eigenfunction, and we computed the evolution for different parameter values k , T , Ω . In all cases, D could be computed either from fig. 1a and formula (4), or from fig. 1b and formula (5), with approximately the same results. Fig. 2 shows the values of D thus obtained, for different values of Ω and D_0 and the same value of $\epsilon=0.5$. The power law (12) agrees

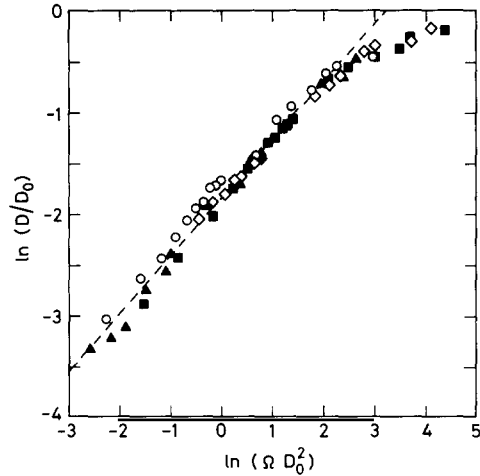


Fig. 2. The diffusion coefficient as a function of the modulating frequency Ω and of the classical (quasilinear) diffusion coefficient D_0 . (\blacktriangle) $k=5$, $T=2$; (\circ) $k=7$, $T=5/3.5$; (\blacksquare) $k=10$, $T=5/3.5$; (\diamond) $k=12$, $T=0.35$. In all cases, $\epsilon=0.5$. $D_0 = \frac{1}{2}k^2(1 + \frac{1}{2}\epsilon^2)$ is the classical diffusion coefficient. The slope of the interpolating dashed line is $\alpha \approx 0.6$.

with numerical data. A best fit of numerical data gives $\alpha \approx 0.6$ which leads to $\gamma \approx 0.2$. The latter is the value of the characteristic exponent for the repulsion of neighboring quasi-energy levels which are approximately inside one localization length. The obtained numerical value $\gamma \approx 0.2$ shows that the repulsion between such levels is relatively small (in comparison with the Wigner distribution). The theoretical derivation of the value of this exponent requires further investigations.

Finally, let us briefly recall how the above described results have a relevance for the problem of microwave ionization of hydrogen atoms. It has been shown [4,5] that the quantum dynamics of a H atom in a microwave field is localized if the microwave intensity is less than a threshold value

$$\epsilon_q \approx \frac{\omega_0^{7/6}}{\sqrt{6.6n_0}}, \quad (13)$$

which is called the *quantum delocalization border*. Here n_0 is the principal quantum number of the initially excited state, $\epsilon_0 = \epsilon n_0^4$ and $\omega_0 = \omega n_0^3$ are the rescaled field and frequency respectively. Although the border (13) was first established for a one-dimensional model, it holds for a two-dimensional atom, too [5].

It was argued [5] that localization in the two-dimensional atom in a microwave field might be possibly destroyed by the introduction of a static electric field, which would eliminate the Coulomb degeneracy. The classical excitation process in the presence of such a field is approximately described by the mapping [5]

$$\bar{N} = N - k[1 + 2(n_{20}/n_0) \cos \omega_s t] \sin \theta, \quad (14a)$$

$$\bar{\theta} = \theta - 2\pi\omega(-2\omega\bar{N})^{-3/2}, \quad (14b)$$

where $N = E/\omega = (-2\omega n^2)^{-1}$ and n_{20} is a parabolic quantum number of the initial state; moreover, $k = 0.82\pi\epsilon\omega^{-5/3}$, $\omega_s \approx 6\pi\epsilon_s n_0^4$ (ϵ_s is the static field strength).

On linearizing eq. (14b) the following map is obtained,

$$\bar{N} = N - k[1 + 2(n_{20}/n_0) \cos \omega_s t] \sin \theta, \quad (15)$$

$$\bar{\theta} = \theta + T\bar{N},$$

where $T = 6\pi\omega^2 n_0^5$. Eq. (15) is a standard map with a modulated kick strength, just like the one which has been discussed in this paper. Therefore, our results about the slowly modulated quantum kicked rotator yield information about the effectiveness of

the static electric field as a means for producing strong excitation in the H atom in a microwave field.

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References

- [1] G. Casati, B.V. Chirikov, J. Ford and F.M. Izrailev, in: Springer lecture notes in physics, Vol. 93. Stochastic behaviour in classical and quantum Hamiltonian systems (Springer, Berlin, 1979) p. 334.
- [2] S. Fishman, D.R. Grempel and R.E. Prange, Phys. Rev. Lett. 49 (1982) 509; Phys. Rev. A 29 (1984) 1639.
- [3] B.V. Chirikov and D.L. Shepelyansky, Radiofizika 29 (1986) 1041.
- [4] G. Casati, B.V. Chirikov, I. Guarneri and D.L. Shepelyansky, Phys. Rep. 154 (1987) 77.
- [5] G. Casati, I. Guarneri and D.L. Shepelyansky, IEEE J. Quantum Electron. 24 (1988) 1420.
- [6] L. Schiff, Quantum mechanics (New York, 1949) p. 207.
- [7] P.A. Lee and T.V. Ramakrishnan, Rev. Mod. Phys. 57 (1985) 287.