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Breakdown of Universality in Renormalization Dynamics for Critical Invariant Torus.

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Abstract. – We consider properties of critical invariant tori with two fixed winding numbers in volume-preserving maps. We present numerical evidence for the existence of *different* renormalization dynamics on small scales which corresponds to breakdown of universality.

Two-dimensional area-preserving maps provide a convenient tool to study the classical chaotic dynamics of Hamiltonian systems. One of the most important problems is the transition from regular to chaotic motion which takes place at some critical value of the perturbation. Near the critical perturbation the structure of the motion is very complicated and consists of a hierarchical mixture of regular and chaotic components. Nonetheless this complicated critical structure obeys a remarkable universal scaling property which imposes a surprising order in this complicated motion [1].

The simplest case of such critical motion [2] corresponds to the destruction of the invariant curve with golden-mean winding number. The structure of such a motion revealed a remarkable universality in the sense that all characteristics depend asymptotically only on the winding number and not on the particular map considered. In this case the scaling structure of the critical golden invariant curve is known to be related to a fixed point in renormalization dynamics [1-3]. A similar fixed-point structure characterizes other well-known routes to chaotic behaviour in parameter space, like period doubling [4] or transition to chaos for circle maps with golden-mean winding number [5, 6]. More complicated universal structures have been invoked in the analysis of critical maps with general winding number [6-9].

A natural question arises whether this universality will persist in higher-dimensional systems (or higher-dimensional maps). To this end, in the present paper we study the destruction of invariant surfaces with two winding numbers (two-torus). From a physical viewpoint such problems arise for example in the classical model of hydrogen atoms in a microwave field [10]. Also, the knowledge of the most stable region in systems with two-frequencies perturbations can find applications in accelerators where usually a certain skill is required to find a region of stability on the two-dimensional frequency diagram of resonances [11].

The investigation of such a model opens the possibility to find a much richer and interesting renormalization group dynamics which appears on small scales of phase space. In particular we found that scaling properties are not determined uniquely by the winding number, which means a breakdown of universality on small scales.

One of the first attempts to study the destruction of tori with two winding numbers was made, for dissipative mappings, by Kim and Ostlund [12, 13]. In particular these authors have discussed in detail the problem of simultaneous rational approximations to a pair of mutually irrational numbers; we will maintain their scheme in what follows.

In order to study the destruction of two-torus, one needs to find periodic orbits in four-dimensional area-preserving maps, which involves remarkable numerical problems, practically unsolvable, especially if long periods have to be considered. To avoid these difficulties, we consider the case in which one of the actions is an integral of motion and the evolution of the corresponding phase is just a rotation with fixed winding number. An interesting model is provided by the standard map with amplitude modulations:

$$\begin{pmatrix} y_{n+1} \\ x_{n+1} \\ z_{n+1} \end{pmatrix} = \boldsymbol{T} \begin{pmatrix} y_n \\ x_n \\ z_n \end{pmatrix} = \begin{pmatrix} y_n - (k + \varepsilon \cos z_n) \sin x_n \\ x_n + y_{n+1} \\ z_n + 2\pi r_2 \end{pmatrix}. \tag{1}$$

The motion is characterized by a pair of frequencies given by the x and z winding numbers, that is

$$r_1 = \lim_{n\to\infty} \frac{x_n - x_0}{2\pi n} \quad \text{and} \quad r_2 = \lim_{n\to\infty} \frac{z_n - z_0}{2\pi n}$$

(r_2 accounts for the internal frequency of the modulation and appears as a fixed parameter in (1)). Notice that invariant surfaces divide the phase space in separate regions with no flux between them. Therefore such surfaces strongly influence transport properties. In the following we investigate the destruction of the invariant surface (torus) whose winding numbers are given by $r_1 = 1/\vartheta^2$, $r_2 = 1/\vartheta$, $\vartheta = 1.324718\ldots$ [12] being the real solution of

$$\vartheta^3 - \vartheta - 1 = 0. \tag{2}$$

This particular choice (known as the spiral mean), is characterized by robust geometric scaling of rational approximants and in [12] it is argued that this irrational pair should play a role analogous to the golden-mean winding number for invariant curves.

We will approach the invariant torus via a sequence of periodic orbits, indexed by their x and z rational winding numbers $(p_{1n}/q_n, p_{2n}/q_n)$, generalized Farey-tree rational approximants of $(\vartheta^{-2}, \vartheta^{-1})$. In practice p_{1n}, p_{2n} and q_n all obey the same (Fibonacci-like) recursion relation $\beta_i = \beta_{i-2} + \beta_{i-3} (\beta_n = p_{jn}, q_n)$: with $(q_{-3}, q_{-2}, q_{-1}) = (0, 1, 1)$, while $p_{1n} = q_{n-2}$ and $p_{2n} = q_{n-1}$. The analysis of periodic orbits for the map (1) can be pushed to remarkably high orders once we take into account symmetry properties: as a matter of fact this map can be written as a product of two involutive applications [14]: $\boldsymbol{T} = \boldsymbol{AB}$ ($\boldsymbol{A}^2 = \boldsymbol{B}^2 = \boldsymbol{1}$). As a result there is a line (the dominant symmetry line $x = 0$, $z = 0$) to which at least one point of each cycle belongs [14, 1]: this allows to reduce locating periodic orbits to a one-dimensional problem.

Before turning to the stability analysis of cycles we must understand how periodic orbits are connected to the motion on the invariant torus. The principal scales of the critical motion correspond to minimal detunings δ_n and are indexed by the integers Q_{1n}, Q_{2n}, P_n:

$$Q_{1n} \cdot r_1 + Q_{2n} \cdot r_2 - P_n = \delta_n . \tag{3}$$

The integer pair (Q_{1n}, Q_{2n}) determines the principal resonances which affect most deeply the critical motion. It can be shown that for these resonances the detunings δ_n decay as $(\max\{|Q_{1n}|, |Q_{2n}|\})^{-2}$ (see for example [15]). For the periodic orbit labelled by the same approximation index n these detunings become equal to zero $(Q_{1n} \cdot p_{1n}/q_n + Q_{2n} \cdot p_{2n}/q_n - P_n = 0)$. As a consequence the above Fibonacci-like rational approximants are indeed the most relevant ones for a given rotation pair r_1, r_2.

The sequences $Q_{1,2n}$ (which label the resonance orders of the cycles under inspection) are determined by recursion relations $Q_{1,2n} = -Q_{1,2n-1} + Q_{1,2n-3}$, whose associate algebraic equation is $\lambda^3 + \lambda^2 - 1 = 0$ (which is obtained from (2) through the substitution $\lambda = \vartheta^{-1}$, and corresponds to the inverse spiral mean). While (2) leads to robust geometric scaling of approximants, as it has a single real root outside the unit circle and two complex-conjugate roots inside, the equation governing the asymptotic behaviour of resonances possesses a real root (ϑ^{-1}) inside the unit circle, while the dominant roots form the complex-coniugate pair $-\sqrt{\vartheta}\exp[\pm i\alpha]$ outside the unit circle. Thus these simple considerations predict oscillations (ruled by the phase α: $\cos\alpha = \vartheta^{3/2}/2$) superimposed to an overall geometric scale, which we expect to be rather weak, being governed by $\sqrt{\vartheta}$.

To study the scaling properties of our invariant torus we follow Green's procedure and determine the residua R_n which characterize the stability properties of resonances and periodic orbits. For orbits with period q_n, $R_n = (1/4)(3 - \mathrm{Tr}\, \boldsymbol{J}_n)$, where $\boldsymbol{J}_n$ is the Jacobian matrix of the map (1).

Theoretical arguments, lying at the basis of the approximate renormalization of the two-dimensional standard map [9], lead to the expectation that the above-described oscillatory behaviour of resonances $(Q_{1,2n})$ will be reflected in the dependence of the residua of the approximants on the order of approximation n. Indeed, according to our numerical results the residua do not have a unique limit in the critical case. However, this does not prevent us

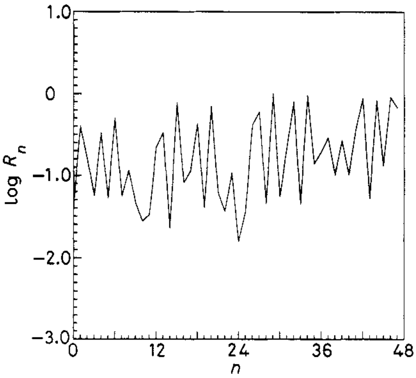

Fig. 1.

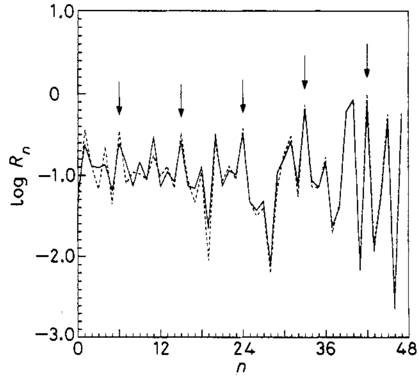

Fig. 2.

Fig. 1. – Residua *vs.* approximants order of renormalization time n for $k_{cr} = 0$ ($\varepsilon_{cr} = 0.40628$).

Fig. 2. – Same as fig. 1, for $k_{cr} = 0.05$ ($\varepsilon_{cr} = 0.34594$; dashed line) and $k_{cr} = 0.3$ ($\varepsilon_{cr} = 0.18$; full line). We found close agreement in the oscillations also for intermediate values of k_{cr}. The approximate 9-periodicity is stressed by the arrows.

from determining critical parameter values: if we fix k in (1) and look at the behaviour of the residua of periodic trajectories R_n *vs.* n for different ε (taking, as in the usual two-dimensional case «elliptic orbits»), we can determine ε_{cr} as the value of ε such that $R_n(\varepsilon) \mapsto \infty$ for $\varepsilon > \varepsilon_{cr}$, and $R_n(\varepsilon) \mapsto 0$ for $\varepsilon < \varepsilon_{cr}$, even though this overall behaviour is modulated by strong oscillations. We consider (for each fixed value of k) periodic orbits up to period 1221537 which correspond to $n = 47$ in the sequence of generalized. Farey rational approximants to $(\vartheta^{-2}, \vartheta^{-1})$. In fig. 1-3 we show the behaviour of residua *vs.* order n of rational approximants of different critical parameter pairs $(k_{cr}, \varepsilon_{cr})$. According to our numerical results there is a region (fig. 2)—seemingly connected in parameter space—in which residua oscillations display universal behaviour. Moreover these oscillations apparently follow a period-9 pattern, and this can be understood in terms of resonances: in fact the phase α characterizing the leading roots of the equation for resonances is very closely approximated by $2\pi/9$ ($\alpha = 2\pi \cdot 0.1120224\ldots$). Similar features appear if we fix k and look at the sequence of $\{\varepsilon_n\}$ values corresponding to periodic orbits (at the n-th step in the rational approximation scheme) with a fixed residue $R = 0.25$. We found that the $\{\varepsilon_n\}$ sequence converges geometrically with superimposed oscillations to the critical value (which depends on k). In the range of critical parameters $0.05 \leq k_{cr} \leq 0.3$ (corresponding to $0.18 \leq \varepsilon_{cr} \leq 0.3$), where residua oscillations are universal, we observed that the oscillations of the convergent sequence $\{\varepsilon_n\}$ were also the same. The scale factor for the geometric convergence rate, as anticipated above, is rather small. Therefore even going to orders as high as $n = 40$ and more allows only to determine this factor up to a few percent. Within this rather high error bar all values we obtained are compatible with a geometric rate $\sqrt{\vartheta}$. Such a rate corresponds to convergence $|\varepsilon_{n+1} - \varepsilon_n| \approx 1/(Q_{1n}^2 + Q_{2n}^2)^{1/2}$ as it is for the one-torus case [2, 9]. This is shown, for example, in fig. 4, where we plot $\log|\varepsilon_n - \varepsilon_{cr}|$ *vs.* n for $k_{cr} = 0.3$.

However the most striking result is that the critical dynamics outside the critical parameters range discussed above exhibits a quite different behaviour of residua as a

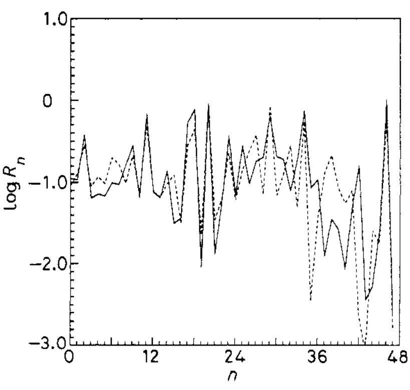

Fig. 3.

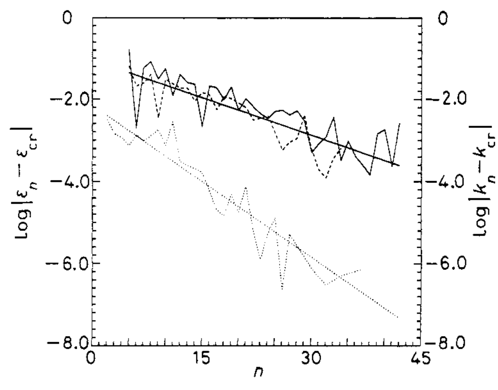

Fig. 4.

Fig. 3. – Same as fig. 1, for $k_{cr} = 0.55$ ($\varepsilon_{cr} = 0.543$; dashed line) and $k_{cr} = 0.65$ ($\varepsilon_{cr} = 0.0179$; full line).

Fig. 4. – Convergence rate (eq. (4)) to the critical parameter value *vs.* n for three different cases: *a*) $k_{cr} = 0.3$ ($\varepsilon_{cr} = 0.18$; full curve), *b*) $k_{cr} = 0.6$ ($\varepsilon_{cr} = 0.0349$; dashed curve), *c*) $\varepsilon_{cr} = 0$ ($k_{cr} = 0.83247$; dotted curve). The scale on the left corresponds to curves *a*) and *b*), the one on the right to curve *c*). The slope of the straight full line corresponds to theoretical scaling factor $\sqrt{\vartheta} = 1.1509\ldots$ (we checked that for the two-torus case the same scaling factor appears if one fixes ε and studies how $\{k_n\}$ converges to the critical value). The least-square fitting of the two curves *a*) and *b*) gives, respectively, 1.14 and 1.18. The straight dotted line is a least-square fitting of curve *c*) with corresponding scaling factor 1.33.

function of renormalization time n. This is pictorially illustrated in fig. 1, where the residua pattern corresponding to k equal to zero is shown, being completely different from the one corresponding to the universal range in fig. 2. The parameter region displaying a behaviour of the type shown in fig. 1 is relatively small ($k_{cr} < 0.05$). A larger region of renormalization behaviour different from the two cases mentioned above (fig. 1, 2) has been observed for $k \geqslant 0.5$. In this region the behaviour of residua is sensibly dependent on small changes in k_{cr}. This is illustrated in fig. 3, where we show that changing from $k_{cr} = 0.55$ to $k_{cr} = 0.65$ leads to different evolutions in renormalization time.

In this region we also studied the law ruling convergence of sequences $\{\varepsilon_n\}$ (and k_n) to the critical value and we found (see fig. 4) that the convergence rate σ is close to $\sqrt{\vartheta}$, as follows from simple resonance considerations:

$$|\varepsilon_n - \varepsilon_{cr}| \approx \sigma^{-n} \approx \frac{1}{Q_{1,2n}}. \tag{4}$$

It is interesting to push the analysis to the limiting case $\varepsilon = 0$, corresponding to the destruction of the invariant curve with one winding number. One of the first attempts to investigate the destruction of such an invariant curve, with rotation number $r = 1/\vartheta$, was made in ref. [16]. This case is similar to ours with $r_1 = 1/\vartheta^2 = 1/(1 + 1/\vartheta) = [1, 1, 3, 12, 1, 1, 3, 2, 3, 2, 4, 2, 141, 80, 2, 5, 1, \ldots]$ and $\varepsilon = 0$. We considered convergence of the sequence $|k_n - k_{cr}|$ (determined by Fibonacci-like approximants) and found that the convergence law is $|k_n - k_{cr}| \approx 1/Q_{1,2n}^2$ with corresponding scaling exponent $\sigma \approx 1.33 \approx \vartheta$ which is different from the value previously obtained for the destruction of two-torus (see fig. 4). Therefore scaling properties of one-torus renormalization dynamics differ totally from the two-torus case (eq. (4)). We also checked the convergence to the critical k when the irrational rotation number r_1 is approximated by periodic orbits (of period q_n) determined by the periodic fraction expansion. Since the entries in the continued-fraction expansion are not bounded and grow with n, we could not reach very high n ($n_{max} = 11$). For these approximants we found that the convergence is approximately described by the relation $|k_n - k_{cr}| \approx 1/q_n^s$ with $s \approx 1.1$. This values is slightly different from the results of ref. [16] ($s \approx 1.44$), where apparently $n_{max} = 7$. However, to determine the limiting value of s one needs to go to higher values of n. Further investigation is also required to find how small ε_{cr} has to be to have two-frequencies scaling close to one-frequency behaviour.

In this paper we have provided empirical evidence for the existence of different basins of attraction for renormalization dynamics on small scales in phase space. Numerical evidence shows that in one set of attraction, corresponding to $0.05 \leqslant k_{cr} \leqslant 0.3$, small changes in critical parameters do not change the resulting renormalization dynamics, so this set can be considered as stable and universal. However, on another set of attraction a small change in critical parameters leads to different behaviours on small scales and therefore renormalization dynamics in this set is probably unstable and chaotic. We conjecture that this region corresponds to a chaotic attractor in the space of 3-dimensional maps with two rotation numbers coinciding with the spiral mean, while the parameter region of fig. 2 probably corresponds to a limiting stable renormalization cycle.

In conclusion we have shown that 3-dimensional, volume-preserving maps obtained by adding modulation to the standard map have a renormalization group behaviour much more rich than the scenario of the 2D case: different critical parameter values lead to quite different behaviour on small scales for a fixed pair of rotation numbers. Instead in the 2D case the asymptotic oscillations of residua for critical invariant curves with a fixed typical irrational winding number are always universal and the same for different maps.

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