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Statistics of quantum lifetimes in a classically chaotic system

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The statistical distribution of lifetimes in a simple quantum model with absorption is numerically investigated. It displays a number of characteristic features that can be related to the classical diffusive mechanism of absorption, described by a Fokker-Planck equation.

The onset of diffusive dynamics in purely deterministic dynamical systems is one of the important physical consequences of classical chaos. Quite often the appearance of chaotic instabilities enforces some sort of relaxation or decay processes which take the form of a deterministic diffusion and can be described by equations of the Fokker-Planck type. A typical "diffusive" decay process is the following: a classical system which possesses "bound" states as well as "continuum" ones is subjected to some external perturbation, which triggers a chaotic diffusion inside the bound-state part of the phase space until the continuum is eventually reached. In this case simple approximate estimates for the decay rates can be gained from a Fokker-Planck description of the chaotic diffusion. A well-known example is the widely studied problem of microwave ionization of hydrogen atoms.[1] All these decay problems can also be formulated as scattering problems with formation of intermediate metastable states and are therefore amenable to the conceptual framework of irregular scattering;[2] nevertheless, the quantum case in which the decay of the compound intermediate state is classically determined by a chaotic *diffusion* appears to have been scarcely studied up to now. This case is the subject of the present work.

Generally speaking, the classical chaotic diffusion is quantum-mechanically limited by the quantum localization phenomenon.[1] Nevertheless, under suitable conditions (localization length comparable to the size of the system), a mechanism of quantum excitation appears which looks more or less like the classical chaotic diffusion, and this leads to the decay of the system into continuum. Then the problem arises of determining the quantum and classical decay rates in the presence of chaotic diffusion.

The model used in order to study this problem is that of a kicked rotator with absorption at some value of the action. The classical phase space is $\{(n,\theta), -N/2 < n < N/2, 0 \le \theta < 2\pi\}$ and the discrete time dynamics is defined by the map

$$\bar{n} = n + k \sin\left(\theta + \frac{Tn}{2}\right), \quad \bar{\theta} = \theta + \frac{T}{2}(n + \bar{n}) \tag{1}$$

which is a symmetrized form of the standard map.[3] When the orbits reach the boundaries of the finite box of size N in action they are absorbed. In the chaotic regime, when $K = kT \gg 1$, the classical motion can be described by the diffusion equation

$$\frac{\partial f(n)}{\partial t} = \frac{D}{2} \frac{\partial^2 f(n)}{\partial n^2} , \tag{2}$$

where t is the time measured in the number of iterates of the map (1). $D = \beta D_{QL}$ is the diffusion coefficient with $D_{QL} = k^2/2$ the quasilinear diffusion rate. The coefficient β depends on the chaos parameter K.[4] The classical ionization process is described by Eq. (2) supplemented with boundary conditions at the absorption border. These conditions easily follow from (1) by requiring that the flux across the boundary should be equal to the probability of ionization in one kick:

$$-\frac{D}{2} \frac{\partial f}{\partial n}\bigg|_{n=\pm N/2} = \pm \frac{k}{\pi} f(\pm N/2) . \tag{3}$$

This boundary value problem is easily solved. It possesses a complete set of eigenfunctions u_m, $m = 1, 2, \ldots$ with eigenvalues γ_m, monotonically increasing with m. The mth eigenfunction has the parity of $m+1$ (under reflection in n), and decays in time according to the exponential law $\exp(-\gamma_m \tau)$. Eigenvalues γ_m are given by

$$\gamma_m = \frac{2D}{N^2} \nu_m^2 , \tag{4}$$

where ν_m is a root of the equation

$$\tan(\nu_m) = \frac{kN}{2\nu_m \pi D}[1-(-1)^m] - \frac{\nu_m \pi D}{2kN}[1+(-1)^m] .$$

Therefore, in the classical case we have just a discrete spectrum of decays rates.

The corresponding quantum problem is described by a quantum map:

$$\bar{\psi}=\hat{U}\psi=\hat{P}e^{-iT\hat{n}^2/4}e^{-ik\cos\theta}e^{-iT\hat{n}^2/4}\,, \quad (5)$$

where $\hat{P}$ is the projector over states $|n|\leq N/2$. Here $\hbar=1$; the classical limit corresponds to $k\rightarrow\infty$, $K=kT=$const, $k/N=$const. The operators to the right of $\hat{P}$ give the evolution of the quantum kicked rotator,[5] from one half-period before a kick to one half-period after the kick. Thanks to this formulation of the rotator dynamics, the matrix $\hat{U}$ is symmetric. Due to the absorption the probability is not conserved by $\hat{U}$, which has therefore complex eigenvalues $\lambda=e^{-\gamma/2+i\phi}$ lying within the unit circle ($\gamma>-0$).

For a "big" box, the absorption should not significantly affect the evolution of states lying near the center. Indeed, according to well-known results on the quantum kicked rotator,[5] these states are exponentially localized with a localization length $l\approx D$, and if $D\ll N$, most eigenstates will have exponentially small γ. In this paper we are mainly interested in the opposite case $D\gg N$, in which localization is not expected to play a significant role. In that case the classical diffusive picture may be expected to emerge under an additional condition $D\ll N^2$, ensuring that many kicks are required to reach the border.

We numerically computed the spectrum of the matrix $\hat{U}$. First, we investigated the statistical distribution of the "level widths" γ. Figure 1 shows the distribution $P(\gamma)$ of γ's for three different matrix sizes: $N=800$, 1600, 2000 with fixed ratio $N/k=10$ and fixed chaos parameter $K=kT=7$. For this calue, no islands of stability survive in the classical case and the diffusive picture should be valid. The position of the maximum of the distribution, as well as the shape of the distribution to the right of the maximum, appear to be substantially stable and can therefore be assumed to describe the actual semiclassical behavior. The position of the maximum coincides fairly well with the first classical eigenvalue γ_1 (4), for a value of $\beta=2$. This is at variance with analytical results[4] and numerical simulations,[5] indicating that the value of β for $K=7$ should be approximately 2.8. This discrepancy may be due to the fact that in our case we have a limited diffusion time; the value $\beta=2.8$ is instead related to diffusion observed over an infinite time. Computations with different values of K also yielded a maximum of $P(\gamma)$ in the position of the corresponding classical γ_1.

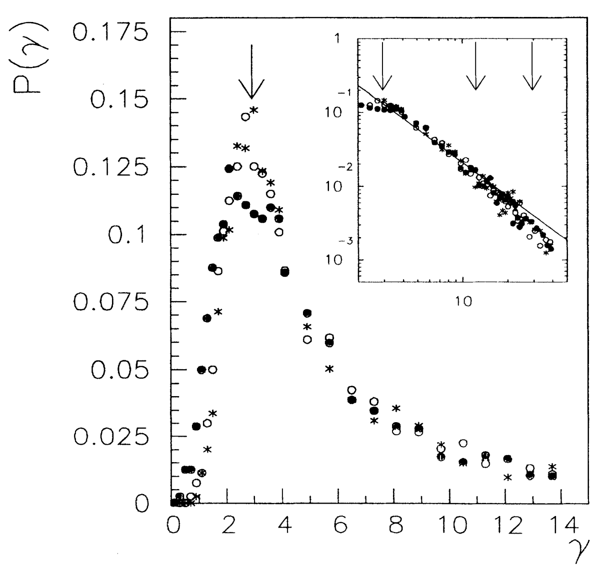

FIG. 1. Probability distribution of level widths for fixed ratio $N/k=10$ and $K=7$, on the x axis; $\bar{\gamma}=N^2\gamma/k^2$. Solid circles, $N=800$; open circles, $N=1600$; stars, $N=2000$. The arrows show the positions of the three lowest classical eigenvalues. Inset: decay of the distribution for large γ in log-log scale, the line is the theoretical $\bar{\gamma}^{(-3/2)}$ law.

To the right of the maximum, the distribution $P(\gamma)$ follows a power law with an exponent which in this case is close to 1.5. In other cases, in which the diffusive condition $N^2\gg D$ was not so well satisfied, the exponent was found to range between 1.5 and 2. Notice that in all cases a number $\sim 2k$ of eigenfunctions would yield anomalously large γ. These functions correspond to states lying close to the border, which classically would be ionized in just one kick. Such states were not taken into account.

The following argument provides some understanding of this power-law decay. The typical time needed to reach the absorption border leaving from a distance n_i from the border is $t\simeq D^{-1}n_i^2$. For such trajectories the typical ionization rate is $\gamma\simeq 1/t\simeq D/n_i^2$. Therefore, the density of states having ionization rate γ is $P(\gamma)=dn_i/d\gamma\simeq D^{1/2}\gamma^{-3/2}$.

One would naturally expect some higher classical eigenvalues γ_m, $m=2,\ldots$ to appear in the quantum $P(\gamma)$. In no computation of ours could this be observed. On the other hand, the quantum problem has some nontrivial features—e.g., the eigenstates of $\hat{U}$ are not mutually orthogonal—and it may be the case that in order to reproduce the classical spectrum some appropriate weighting of the eigenvalues λ is needed.

The part of the distribution $P(\gamma)$ to the left of the maximum seems determined by purely quantum effects. Indeed, in the classical case, the probability of survival

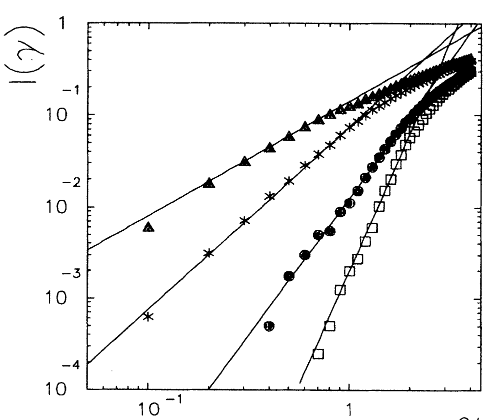

FIG. 2. Log-log plot of the integrated distribution of levels widths for $N/k=10$, $K=7$. Triangles, $N=200$; stars, $N=400$; solid circles, $N=800$; squares, $N=1600$. The straight lines indicate power laws with the exponents 1.25, 2, 3, 5.

inside the box for times much higher than $1/\gamma_1$ is exponentially small, which would suggest that $P(\gamma)$ should vanish exponentially fast for $\gamma \to 0$. Instead, we observed a power-law decay of $P(\gamma)$ for small γ. This is shown in Fig. 2. The exponent in this power law increases with the increase of k, i.e., going towards the classical limit (we remember that the ratio k/N and the classical chaos parameter $K = kT$ were kept constant); this increase is consistent with the classical exponential behavior. According to the numerical results in Fig. 3, the dependence of the exponent α (for the integrated distribution) on the parameters should be of the form

$$\alpha \simeq c \left[\frac{D}{N} \right]^{1/2} \frac{\mu}{2} + \Delta(\mu) , \tag{6}$$

where $\mu = kN/\pi D$. From the three cases in Fig. 3, $c = 0.91 \pm 0.04$. The dependence of $\Delta(\mu)$ could not be exactly determined from the data; anyway, since for $D \sim N$ one enters the localized regime, in which $P(\gamma)$ does not vanish as $\gamma \to 0$, then $\Delta(\mu)$ cannot be far from $2 - \mu/2$ (which would be consistent with numerical data).

The reason of this quantum behavior near $\gamma = 0$ could be clarified by analyzing the structure of the eigenfunctions corresponding to the lowest values of γ. The Wigner functions corresponding to these eigenfunctions[6] display scars which appear to be responsible for the slow decay. One particularly impressive such example is given by Fig. 4 which shows the Wigner representation of the eigenfunction with the lowest γ, for $k = 80$, $K = kT = 7$, $N = 1600$. The pattern of the maxima appears to follow the unstable periodic orbits (periods 1,2,3, . . .), in agreement with current views,[7] though most of them do not exactly coincide.

An interesting remark on the structure of Fig. 4 is about its symmetries. The form of the classical map was chosen so as to have time-reversal invariance (in the absence of absorption). With the absorption, one can identify three sets of classical orbits that may influence the structure of the quantum eigenstates: (a) the set of never-ionizing orbits (neither in the future, nor in the past), (b) the set of orbits which will never ionize in the future, but would have ionized in the past, (c) the set obtained by the previous one by exchanging past and future. (a) contains, e.g., all periodic orbits contained in the box; (b) contains the stable manifolds of orbits belonging to (a), and so on. The set (a) is obviously invariant under time reversal, i.e., under the change $n \to -n$. The quantum state of Fig. 4 mainly reflects the classical structure (a), but its slight defects of symmetry suggest that it is also affected by (b).

Finally, we investigated the statistics of spacings for the complex eigenvalues λ. Previous results[8] have shown

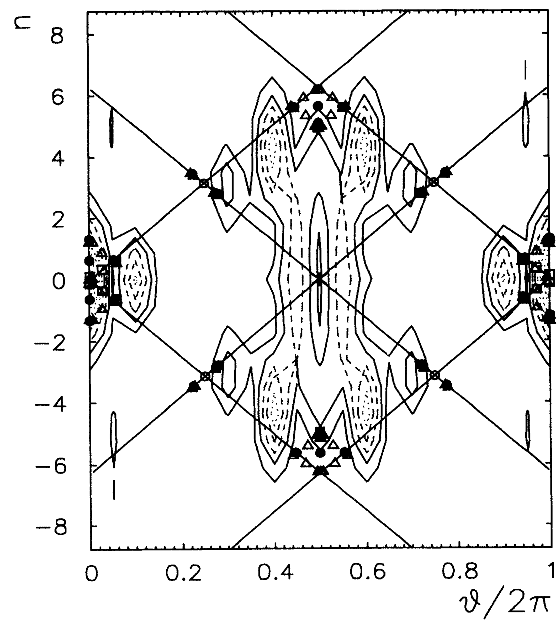

FIG. 4. Contour plot of the Wigner function corresponding to the eigenfunction with the smallest level width for $N = 1600$, $k = 80$, $K = 7$. Points are some classical orbits of period 1 (stars), 2 (open circles), 3 (closed circles), 4 (squares), 5 (triangles). The symmetry lines of the symmetric standard map (Ref. 3) are also shown.

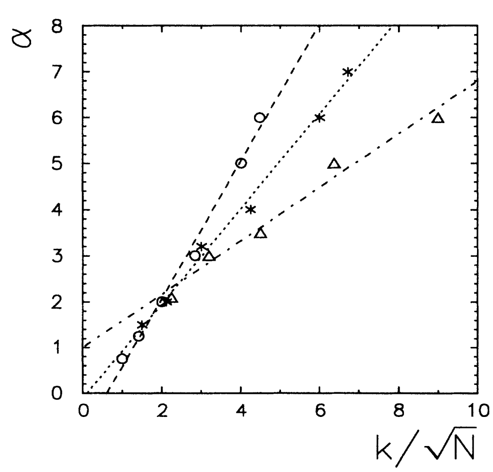

FIG. 3. The exponent of the power laws illustrated in Fig. 2, as a function of $kN^{-1/2}$. Open circles, $N/k = 10$; stars, $N/k = 6.67$; triangles, $N/k = 4.4$. The lines give the least-square linear fitting.

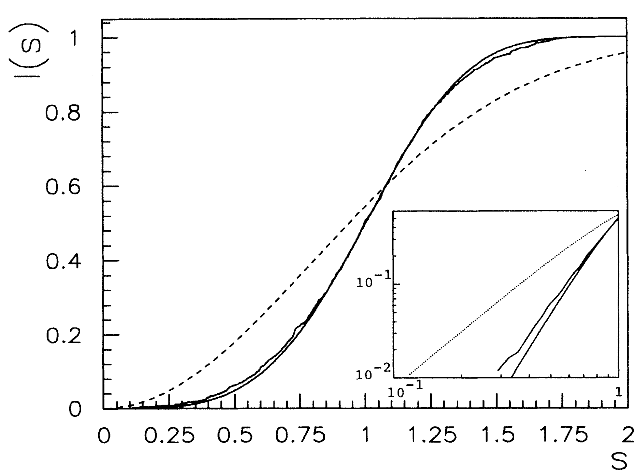

FIG. 5. Integrated distribution of nearest-neighbor spacings in the complex plane for $N = 1600$, $k = 160$, $K = 7$. Dashed line and solid line are the regular distribution and the chaotic one (both from Ref. 8).

that the statistics of nearest-neighbor spacings for quantum dissipative maps describing the evolution of density matrices display a universal cubic repulsion in the presence of classical chaos. Our case is somewhat different because (i) our quantum map describes the evolution of ψ functions and (ii) instead of dissipation here we have absorption (as a model for the physical ionization process). Nevertheless, we could observe the same universal spacing statistics (Fig. 5). In order to extract the fluctuation properties of the spectrum the eigenvalues were renormalized so as to have a homogeneous distribution, following the same procedure as in Ref. 8.

The repulsion of complex eigenvalues would be expected to become a two-dimensional Poissonian (i.e., linear) in a regime of integrable motion. However, our model is not a convenient one for investigating this regime, because in the integrable case the distribution becomes very inhomogeneous: part of the eigenvalues falls close to the center and the rest are very close to the unit circle $|\lambda|=1$. The same situation occurs in the localized case. It is very difficult in such cases to properly analyze the spectrum.

The above-described results show that the classical diffusive nature of the motion allows for an understanding of the quantum statistics of lifetimes without any statistical assumptions. This statistics is very different from the Porter-Thomas distribution which is a common reference in this type of problem.[9]

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