Break-up of the Spiral Mean Torus in a Volume-preserving Map

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Abstract—We consider a modulated standard map, and study the break-up of the spiral mean torus: though the critical line seems smooth, the behaviour is not universal, some discussion on scaling features of the model is included.

One of the most remarkable features of low-dimensional transitions to chaos consists of their universality properties; in particular this suggests that complicated sets of equations and, hopefully, the real world, may share (both qualitatively and quantitatively) properties exhibited by extremely simplified models. This is indeed the case for the period-doubling route in 1-D maps and the quasiperiodic transition for circle maps [1–3], where critical exponents have been measured in several experiments, with results consistent with theoretical predictions.

In the afore-mentioned examples a renormalization group analysis can be carried through (thus reinforcing the parallelism between critical systems and low-dimensional chaos), and the universal features are connected to the existence of a non-trivial fixed point for the proper composition operator. Renormalization theory has been also applied to the problem of (strongly) irrational circles break-up in a class of area-preserving maps [4] (whose non-linearity is driven by a single parameter). Critical behaviour is here uniquely determined by the winding number, each case being characterized by a fixed point of a composition operator naturally induced by the expansion into continued fraction of the winding number itself. However, in this context (and also in the circle maps case), more complicated asymptotic structures in renormalization dynamics have been invoked when generic winding numbers are considered [3, 5–7] (typically conventional renormalization techniques are fitted to deal with quadratically irrational frequencies only). It has also been pointed out that for other classes of maps (ruled by two parameters) even noble frequencies exhibit novel features in the transition to chaos [8, 9]: the critical line contains a Cantor set of cusps, and the critical exponents depend on this structure on all scales.

The interest in area-preserving maps is tied to their representing a convenient tool in

investigating typical Hamiltonian systems with two degrees of freedom, a considerable
interest lies in examining what happens in higher dimensions. This extension is crucial,
different from the theory of critical phenomena, low-dimensional chaos is still far from a
deep understanding of the role of dimensionality in universality theory; moreover three-fre-
quency systems (like the one we will deal with) have been observed in fluid experiments
[10], and are conjectured to play a relevant role in studying hydrogen atoms in a
microwave field [11], or accelerator dynamics [12].

A straightforward extension would require investigating four-dimensional symplectic
maps, while an intermediate step is obtained by freezing one of the actions (and thus
having a free-rotating angular variable): we also stress that general volume-preserving maps
have an interest of their own (even if we lose canonical structure) as they model passive
scalars in a general time-periodic incompressible fluid [13].

The dissipative analogue of this line of research involves analysis of torus maps. In this
context one of the first attempts to study the destruction of tori with two winding numbers
was made by Kim and Ostlund [14, 15]. In particular these authors have discussed in detail
the problem of simultaneous rational approximations to a pair of mutually irrational
numbers; we will maintain their scheme in what follows.

The model we will consider is provided by the standard map with amplitude modu-
lations:

$$\begin{pmatrix} y_{n+1} \\ x_{n+1} \\ z_{n+1} \end{pmatrix} = \mathbf{T} \begin{pmatrix} y_n \\ x_n \\ z_n \end{pmatrix} = \begin{pmatrix} y_n - (k + \varepsilon \cos z_n) \sin x_n \\ x_n + y_{n+1} \\ z_n + 2\pi r_2 \end{pmatrix}. \tag{1}$$

This map contains a pure rotation in z (the conjugate action is a constant of motion and
does not appear explicitly in the map), and reduced to the usual standard map when $\varepsilon = 0$:
in the terminology of [13] it is a one-action volume-preserving map, and KAM like
behaviour is expected. We checked this by looking at phase portraits corresponding to
increasing values of one of the parameters (usually the way of visualizing the resulting
overall behaviour is by considering almost two dimensional 'slices' of the three-dimensional
phase space; Fig. 1 shows a typical sliced phase portrait for critical parameter pairs: there is
a close similarity to phase portrait of area-preserving maps). A relevant feature is that
invariant surfaces divide phase space (like in the area-preserving case), so the problem of
torus break-up is relevant to study transport properties. Each orbit can be labelled by a
pair of frequencies (x and z winding numbers) $r_1 = \lim_{n \to \infty}(x_n - x_0)/2\pi n$ and $r_2 = \lim_{n \to \infty}$
$(z_n - z_0)/2\pi n$. For area-preserving maps, persistence of invariant circles is dependent on
irrationality properties of the winding number labelling the circle: this is intuitively grasped
by thinking in terms of overlapping resonances [16] and is supported by a detailed
numerical investigation [17]. We note that the notion of good or bad irrationals is based on
metric properties of rationals that approximate them [18]. The extension to pairs of
(mutually irrational) irrational numbers is by no means straightforward. We will adopt the
scheme proposed in [14]: the winding numbers we will consider are given by $r_1 = 1/\vartheta^2$,
$r_2 = 1/\vartheta$, $\vartheta = 1.324718 \ldots$ [14] being the real solution of

$$\vartheta^3 - \vartheta - 1 = 0. \tag{2}$$

This particular choice (known as the spiral mean), is characterized by robust geometric
scaling of rational approximations and in [14] it is argued that this irrational pair should
play a role analogous to the golden-mean winding number for invariant curves. A
somewhat similar choice has been investigated by Guckenheimer, Hu and Rudnick [19]
(see also [20]): while the use of roots of cubic equations seems natural, there are some
number-theoretic arguments [21] that seem to question the parallel between these choices
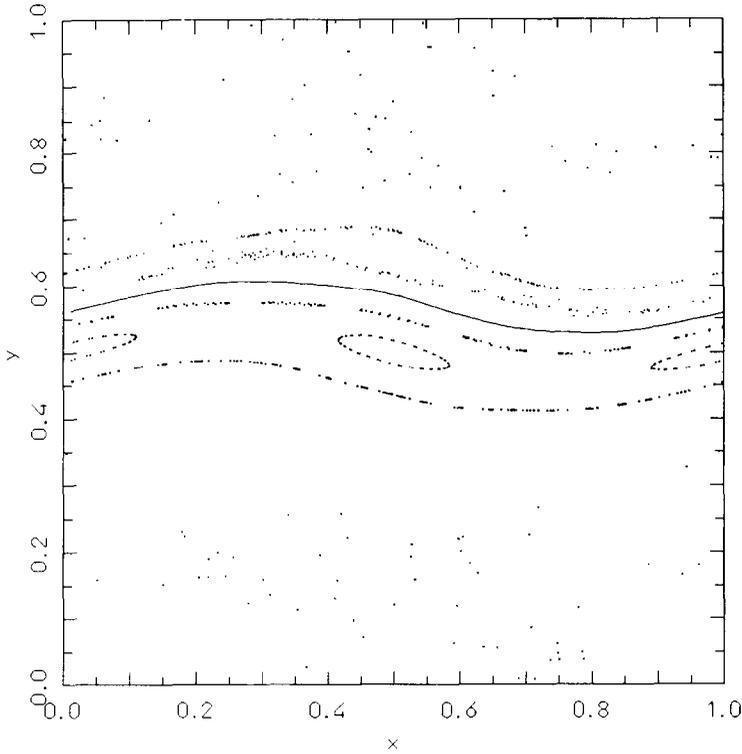and golden-mean in one dimension.

Fig. 1. Sliced phase space $z \in [0, 0.01]$, for $k = 0.3$, $\varepsilon = 0.18$ (critical case): the solid line refers to spiral mean torus, the points are obtained by taking eight different initial conditions.

We will follow Greene's criterion [17, 22] to locate critical parameter pairs; that is we will approach the invariant torus via a sequence of periodic orbits, indexed by their x and z rational winding numbers $(p_{1n}/q_n, p_{2n}/q_n)$, generalized Farey-tree rational approximants of $(\vartheta^{-2}, \vartheta^{-1})$. In practice p_{1n}, p_{2n} and q_n all obey the same (Fibonacci-like) recursion relation $\beta_i = \beta_{i-2} + \beta_{i-3} (\beta_n = p_{jn}, q_n)$, with $(q_{-3}, q_{-2}, q_{-1}) = (0, 1, 1)$, while $p_{1n} = q_{n-2}$ and $p_{2n} = q_{n-1}$.

The map can be written as a product of two involutive applications $(T = AB, (A^2 = B^2 = 1))$: this symmetry property implies the existence of a dominant symmetry line [23] to which at least one point of each cycle belongs: thus finding cycles numerically amounts to a one-dimensional search.

Together with rational approximations to the pair of winding numbers, we introduce another number-theoretic sequence of pairs, which plays a major role in gauging our intuition on what we expect to get from numerical investigations. These indices (Q_{1n}, Q_{2n}) label detunings in the following sense: they give the resonance order of each periodic approximant:

$$Q_{1n} \cdot \frac{p_{1n}}{q_n} + Q_{2n} \cdot \frac{p_{2n}}{q_n} = 0|_{\text{mod1}}$$

and correspond to minimal detunings δ_n along the quasiperiodic structure

$$Q_{1n} \cdot r_1 + Q_{2n} \cdot r_2 = \delta_n|_{\text{mod1}}. \tag{3}$$

General number-theoretical reasoning leads to the expectation of a detuning decay of the form $\delta_n \approx (\max \{|Q_{1n}|, |Q_{2n}|\})^{-2}$, and indeed this can be checked explicitly.

Obviously the scheme adopted here is not *a priori* physically meaningful (even if the resonance picture suggests that the particular choice of rational approximations is motivated, in the sense that we must check afterwards that the sequence of approximations that we select is able to catch the relevant scales of critical motion: we will comment at the end how Fourier analysis reveals which rational sequence correctly describes scaling of critical motion (a more detailed analysis will be presented in [24]).

The sequences $Q_{1,2n}$ (which label the resonance orders of the cycles under inspection) are determined by recursion relations $Q_{1,2n} = -Q_{1,2n-1} + Q_{1,2n-3}$, whose associate algebraic equation is $\lambda^3 + \lambda^2 - 1 = 0$ (which is obtained from (2) through the substitution $\lambda = \vartheta^{-1}$, and corresponds to the inverse spiral mean). While (2) leads to robust geometric scaling of approximations, as it has a single real root outside the unit circle and two complex-conjugate roots inside, the equation governing the asymptotic behaviour of resonances possesses a real root (ϑ^{-1}) inside the unit circle, while the dominant roots form the complex-conjugate pair $-\sqrt{\vartheta}e^{\pm i\alpha}$ outside the unit circle. Thus these simple considerations predict oscillations (ruled by the phase α: $\cos\alpha = \vartheta^{3/2}/2$) superimposed on an overall geometric scale, which we expect to be rather weak, being governed by $\sqrt{\vartheta}$.

We now apply Greene's criterion. For each sequence of rational approximations we analyse the behaviour of residua $R_n = 1/4(3 - \mathrm{Tr}\mathbf{J}_n)$ where $\mathbf{J}_n$ is the Jacobian matrix of the map (1), critical parameter pairs are chosen as the values for which the $R_n \to \infty$ behaviour changes, in the sense that lower parameter values result in $R_n \to 0$, while upper values lead to $R_n \to \infty$. These limits are to be understood in an average sense, as each sequence is characterized by strong oscillations superimposed on an overall behaviour.

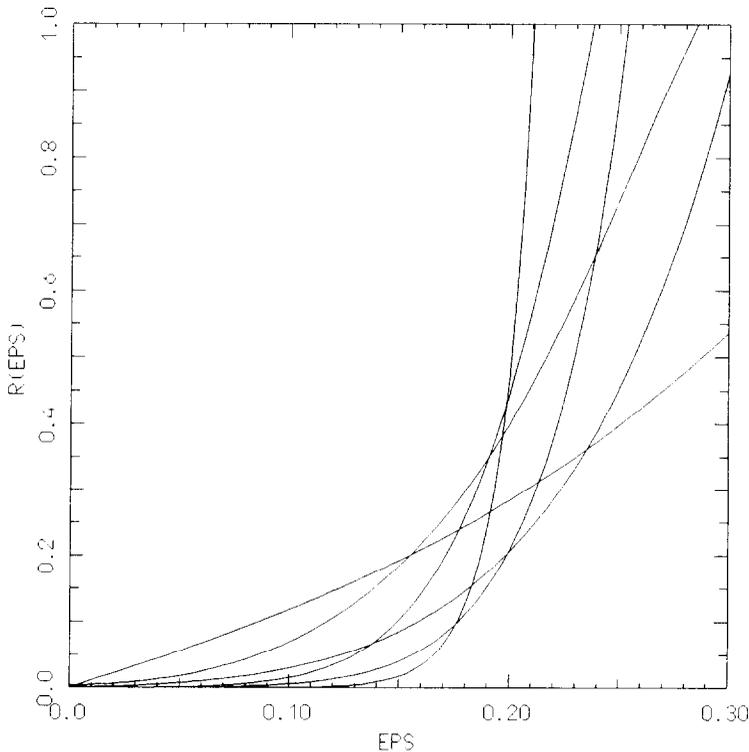These oscillations have a dramatic role if one looks for straightforward extensions of

Fig. 2. Residua of periodic orbits vs ε, for $k = 0.3$: the different lines refer to the $n = 6, 9, 11, 13, 15, 18$ order approximations.

Greene's criterion [25]. For instance if one looks at residua of periodic orbits vs ε (for k fixed) (see Fig. 2) oscillations rule out the possibility of singling out a critical ε, as is the case with the standard map.

Though it is almost impossible to visualize the changes in small scale behaviour of the approximant orbits we used sliced phase-space portraits to check that parameter pairs so chosen single out the transition point: for supercritical parameter values there is numerical evidence of single orbits hopping through the partition induced by the spiral mean set of points (see Fig. 3), so Greene's criterion is still valid in the 3-D case. We now want to motivate, on a number-theoretic basis, the nature of the oscillations found in the residua pattern. Already in the area-preserving case, resonances have been claimed to be the meaningful label for the approximating periodic orbits [7]. This leads to the conjecture that the oscillations have to be related to the phase appearing in the resonance behaviour. This is indeed the case for a relevant portion of the critical parameter pairs that we have chosen. The phase α characterizing the leading roots of the resonance equation is very closely approximated by $2\pi/9$ and in Fig. 4 this approximate pattern is stressed by arrows.

Before turning to a more systematic account of the parameter space analysis it is worth stressing that maybe the results in [19] and [20] can be given a similar interpretation: their scheme also involves a cubic equation with a pair of non-leading complex conjugate roots whose phase accounts for the observed absence of smooth scaling. Also recent results by Mao and Helleman [26] are strongly supportive of wild oscillations superimposed on an overall convergence in the behaviour of symplectic maps in higher dimensions.

We now sketch our previous work [22]. With the method described above we located a number of critical parameter pairs: even if this set is too small to draw definite conclusions
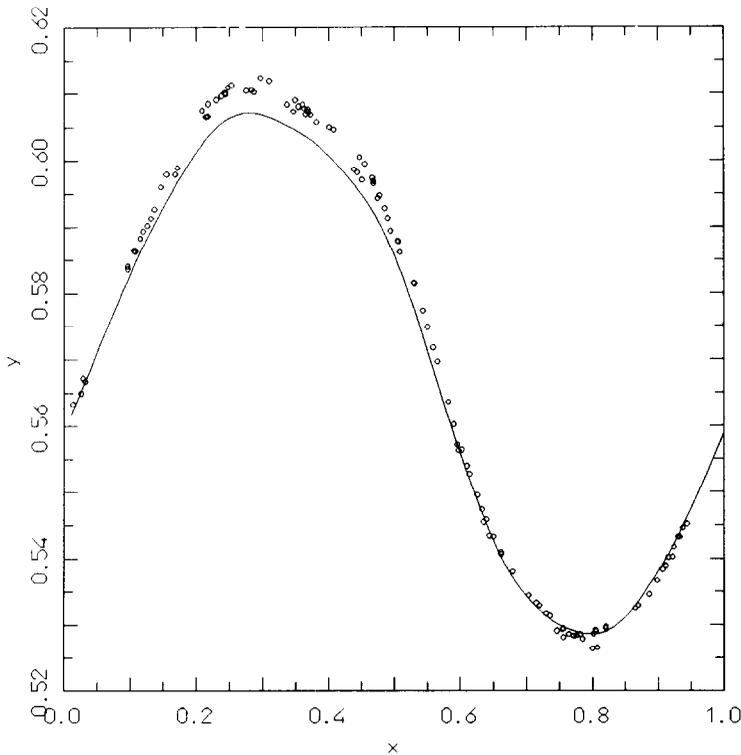

Fig. 3. Slices phase space $z \in [0, 0.01]$, for $k = 0.3$, $\varepsilon = 0.21$ (supercritical case): the solid line is obtained by joining points of the spiral mean orbit, the circles refer to a single initial condition.
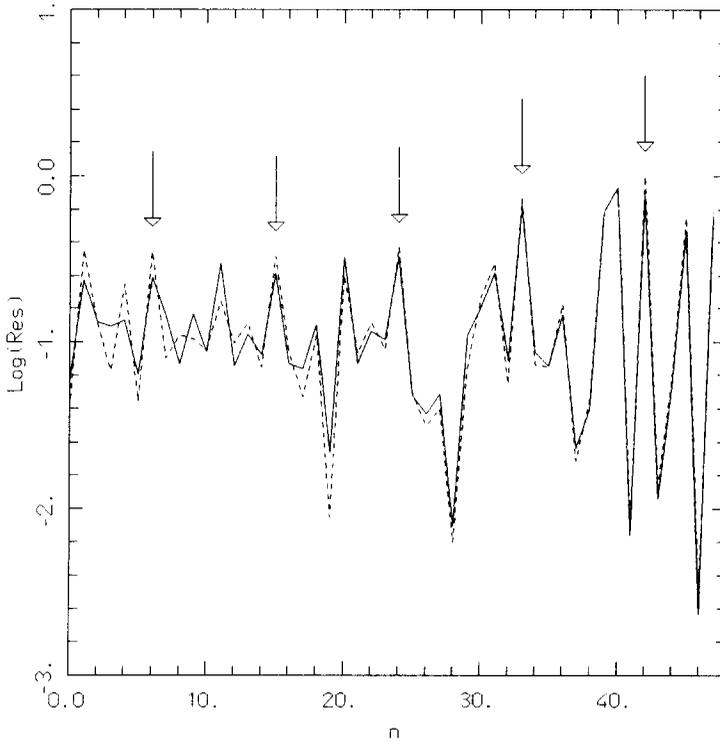
Fig. 4. Residua vs approximants order of renormalization time n for $k_{cr} = 0.05$ ($\varepsilon_{cr} = 0.34594$; dashed line) and $k_{cr} = 0.3$ ($\varepsilon_{cr} = 0.18$; full line). We also found close agreement in the oscillations for intermediate values of k_{cr}. The approximate 9-periodicity is stressed by the arrows.

all the points are consistent with an hypothesis of a smooth critical line in the (k, ε) plane bounding the region of existence of a smooth spiral mean torus (see Fig. 5). This view is reinforced by the universality in residua patterns which we found in a connected region along the critical pairs, see Fig. 4. The oscillations are consistent with the number-theoretic phase commented on before.

The most striking result is that the critical dynamics outside the critical parameters range discussed above exhibit a quite different behaviour of residua as a function of renormalization time n. For instance, when $k \geqslant 0.5$, the behaviour of residua is sensibly dependent on small changes in k_{cr}. This is illustrated in Fig. 6, where we show that changing from $k_{cr} = 0.55$ to $k_{cr} = 0.65$ leads to different evolutions in renormalization time.

In [22] other scaling sequences are considered, in particular the standard map limit can also be considered. We found evidence of a totally different renormalization behaviour in this limit.

Many of these features are better understood if we consider a Fourier analysis of high periodic approximations to the critical torus: for a $p_{j,M}$, q_M periodic approximation we calculate the Fourier spectrum by

$$x_n - x_n^0 = \sum_{j=0}^{q_M-1} A_j \cos\left(\frac{2\pi nj}{q_M}\right)$$

where $\{x_n^0\}$ is the unperturbed pure rotation. In particular we consider the case $M = 44$, which corresponds to $q_M = 525456 = 2^4 \cdot 3^2 \cdot 41 \cdot 89$, so that modified FFT routines are still reasonably fast.
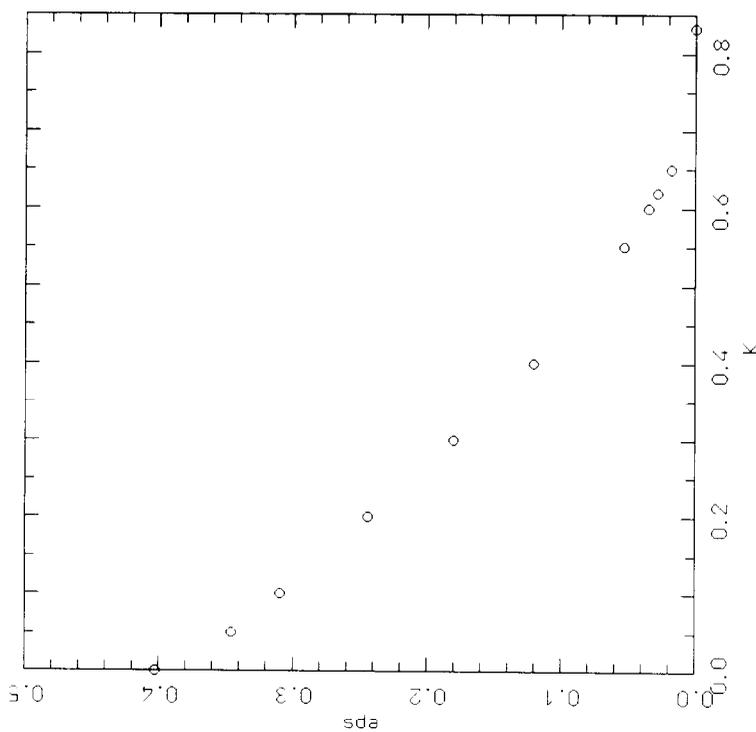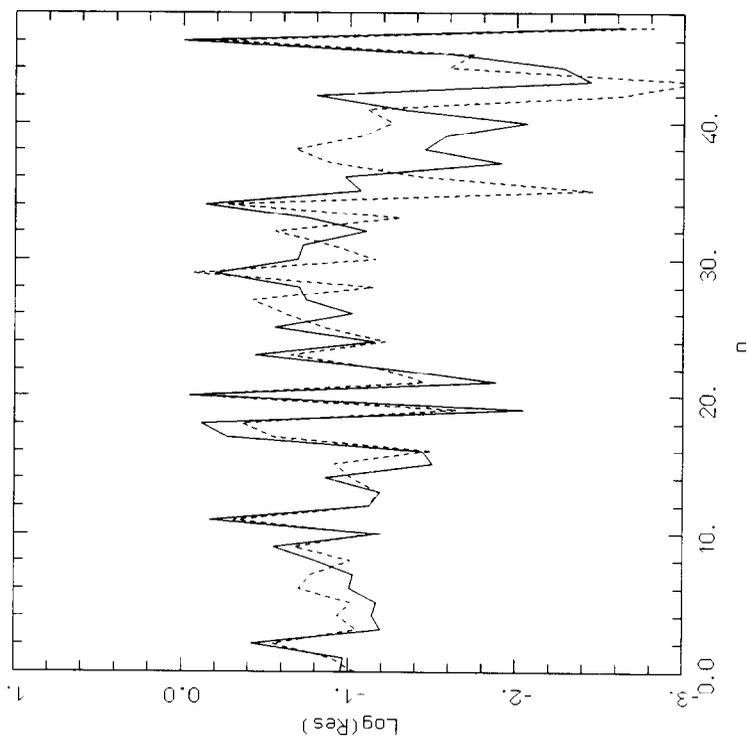
Fig. 6. Same as Fig. 4, for $k_{cr} = 0.55$ ($\varepsilon_{cr} = 0.053$; dashed line) and $k_{cr} = 0.65$ ($\varepsilon_{cr} = 0.0179$; full line).

Fig. 5. Critical pairs in the positive quadrant in parameter space.

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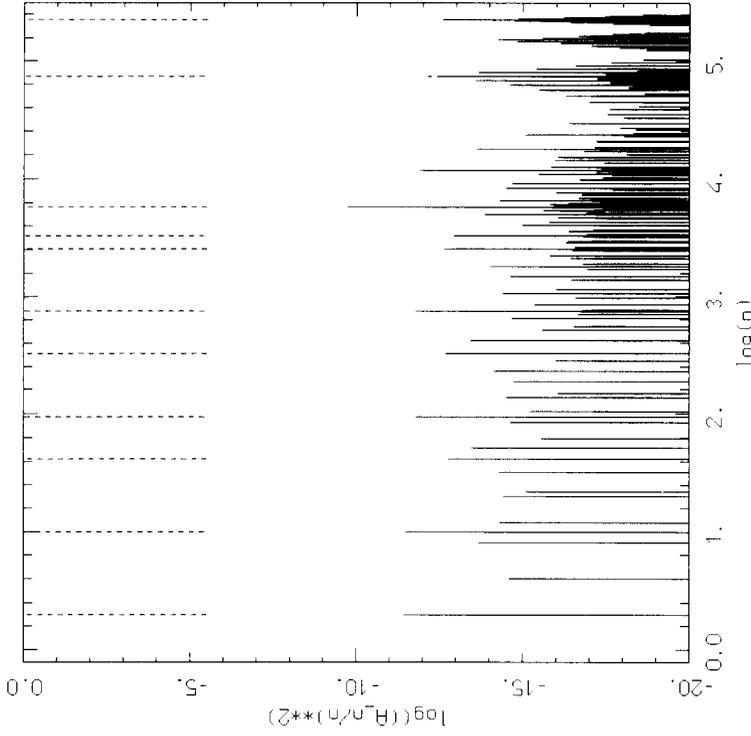
Fig. 8. Scaled Fourier spectrum $\log(|A_n|^2/n^2)$ vs $\log n$ for the $n = 45$ periodic approximation to the spiral mean torus ($\varepsilon_{crit} = 0$); dashed lines mark the position of components related to continued fraction Q_l.
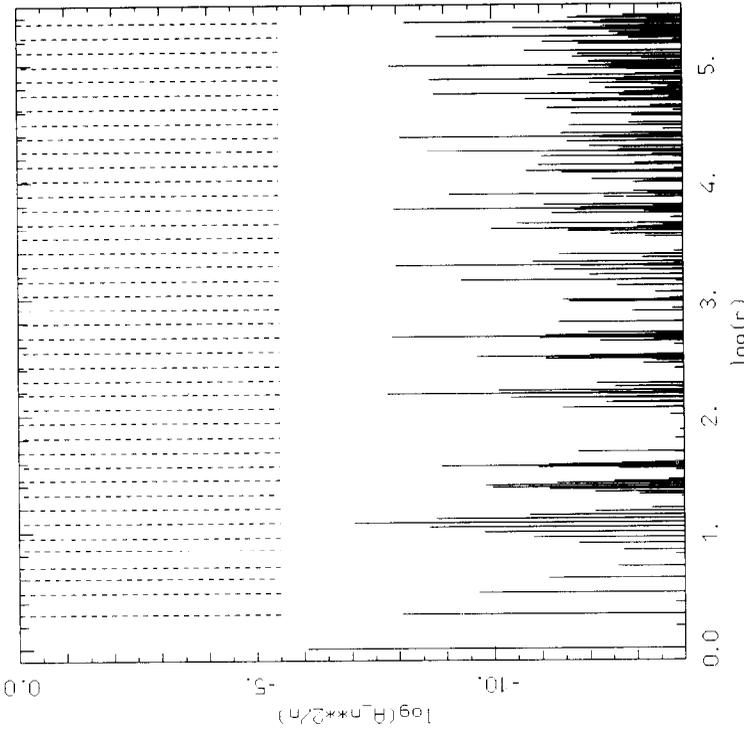

Fig. 7. Scaled Fourier spectrum $\log(|A_n|^2/n)$ vs $\log n$ for the $n = 45$ periodic approximation to the spiral mean torus ($k_{crit} = 0.3$); dashed lines mark the position of Farey-triangle q_l.

The first indication is scaling behaviour of Fourier peaks: that is $|A_j|/\sqrt{j} \approx C$ (scaling behaviour is thus completely different from noble circles in standard-like maps [27]), with oscillations of the same nature as observed in residua patterns. This scaling can be explained by a resonance analysis. From the resonance picture we have that Fourier components A_j of $x(t)$ scale like $A_j \sim 1/Q_{1,2(n)}$ (this estimate was also correct for one winding number [7]). On the other side the physical detunings δ_n [see equation (3)] determine the Fourier frequencies $j/q_M = \delta_n$. Since $\delta_n \sim 1/|Q_{1,2}|^2$ we get the estimate $A_j \sim \sqrt{j}$. This result is in agreement with an argument given by Chirikov for the general case in which N frequencies are present [28]. Thus the critical torus in a sense exhibits structure at all length scales.

In Figs 7 and 8 some of the Fourier plots are shown. We refer to [24] for a full analysis. We just summarize here the major features: the same kind of peak scaling is observed for all parameter pairs with $\varepsilon_{\text{crit}} \neq 0$, the peaks are clearly related to denominators of Farey-triangle approximations q_j (this confirms the relevance of the renormalization scheme), and the modulation of the rescaled peaks is of the very same nature as the residua patterns. The intermediate k_{crit} region is characterized by universal Fourier spectra, and the A_{q_l} oscillate with the same phase as residua. A totally different behaviour is observed in the standard map limit: here peaks are related to continued-fraction approximants of the x winding number, and the form of scaling appears quite different (see Fig. 8); we remark that the different forms of scaling are compatible with an interpretation in terms of resonances [6, 7, 22].

We conclude by remarking a few facts: first of all the applicability of Greene's procedure to the study of cubic torus breakup: the extension we presented finds numerical support both from a phase-space analysis and Fourier components scaling; the existence of different renormalization regimes along the critical line, and the possibility of understanding part of the observed behaviour on the basis of number-theoretic properties of resonance indices.

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